Kyungpook Math. J.
Volume 21, Number 2
December, 1981.

CHARACTERIZATIONS OF THE EXPONENTIAL AND GEOMETRIC DISTRIBUTIONS BY TRUNCATIONS

By Kong-Ming Chong

1. Introduction

The exponential distribution as a failure model has wide applicability. A well known characterization of an exponential random variable T can be obtained by means of its lack of memory property (Feller [1, *p.*8]), viz.,

$$P(T>t+s)=P(T>t)P(T>s),\ \ s,t\geq 0 \tag{1.1}$$

which also serves to characterize a geometric random variable T, provided that T is positive integer-valued and that s and t in (1.1) are positive integers.

In the study of the exponential or geometric distribution as a failure model for the description of a certain observational phenomenon, it is to be noted that the lack of memory characterization (1.1) assumes the somewhat not readily accessible information regarding the probability distribution. This paper presents the following more easily applicable results which can be used to predict the probability distribution of the failure model on the basis of the more readily available knowledge concerning the expected values of the distribution truncated from above at various points. Our results read as follows.

THEOREM 1. *If T is a nonnegative random variable with finite mean and if $F(t)=P(T\leq t)$, $t\in R$, denotes the distribution function of T, then T is exponentially distributed if and only if, for some constant $\alpha>0$,*

$$E(T\wedge t)=\alpha F(t) \text{ for all } t\geq 0 \tag{1.2}$$

where $T\wedge t$ denotes the infimum of T and t.

THEOREM 2. *A positive nondegenerate random variable T has a geometric distribution (i.e., $P(T=k)=pq^{k-1}$, $k=1,2,3,\cdots$, for some $p>0$ and $q>0$ satisfying $p+q=1$) if and only if, for some constant $\alpha>1$,*

$$E(T\wedge[t])=\alpha P(T\leq t) \text{ for all } t>0 \tag{1.3}$$

where $[t]$ denotes the integral part of t.

2. Proof of theorem 1

We first observe that condition (1.2) entails $E(T)=\alpha$

If T is exponentially distributed with mean α, i.e., $F(t)=1-e^{-t/\alpha}$, $t\geq 0$, and $F(t)=0$, $t<0$, then it is easily seen that (1.2) holds.

Conversely, suppose (1.2) holds. Then it is a direct consequence of Lebesgue's dominated convergence theorem that $E(T\wedge t)$ and hence $F(t)$ are continuous in t. Moreover, it is easily seen using integration by parts [2, Theorem 21.67, p. 419] that

$$E(T\wedge t)=\int_0^\infty s\wedge t\,dF(s)=t-\int_0^t F(s)ds,\ \ t\geq 0.$$

Thus (1.2) reduces to the integral equation

$$\int_0^t F(s)ds+\alpha F(t)=t,\ \ t\geq 0$$

which is equivalent to

$$\int_0^t R(s)ds+\alpha R(t)=\alpha,\ \ t\geq 0 \tag{2.1}$$

where $R(t)=1-F(t)$, $t\geq 0$, which is continuous since $F(t)$ is. Now it is easily seen (using Lebesgue's monotone convergence theorem and integration by parts) that $R(t)$ is Lebesgue integrable with $\int_0^\infty R(t)dt=E(T)=\alpha$. To solve (2.1), let $S(t)=\int_0^t R(s)ds$, $t\geq 0$. Then since $R(t)$ is integrable and continuous for $t\geq 0$, the Fundamental Theorem of Integral Calculus [2, Theorems 18.16—18.18, pp. 285—286] asserts that

$$\frac{dS(t)}{dt}=R(t),\ \ t>0 \tag{2.2}$$

and so (2.2) is equivalent to

$$\alpha\frac{dS(t)}{dt}+S(t)=\alpha,\ \ t>0 \tag{2.3}$$

subject to $S(0)=0$. It is easy to see that (2.3) yields the solution $S(t)=\alpha(1-e^{-t/\alpha})$, $t\geq 0$. Since $R(0)=1-F(0)=1$, it follows from (2.2) that $R(\mathrm{t})=e^{-t/\alpha}$, $t\geq 0$, and so

$$F(t)=1-R(t)=1-e^{-t/\alpha},\ \ t\geq 0.$$

Clearly, $F(t)=0$ whenever $t<0$. Hence T is exponentially distributed.

3. Proof of theorem 2

It is clear that (1.3) requires T to be integer valued. For this reason, it suffices to allow t to assume the values 1, 2, 3, $\cdots$, in (1.3).

If T is a geometric random variable, then, for any integer $n\geq 1$, we have

$$\begin{aligned}E(T\wedge n)&=\sum_{k=1}^{n}kP(T=k)+nP(T>n)\\&=\sum_{k=1}^{n}kpq^{k-1}+nq^{n}\\&=(1-q)^{n}/p\\&=\alpha P(T\leq n),\ \text{ where } \alpha=1/p.\end{aligned}$$

Conversely, suppose that (1.3) holds for $t=n\geq 1$. For $k=0, 1, 2, \cdots, n$, let $u_k=P(T\leq k)$. Then (1.3) is equivalent to

$$\sum_{k=1}^{n}k(u_k-u_{k-1})+n(1-u_n)=\alpha u_n,\ \ n=1, 2, 3, \cdots$$

or

$$n-\sum_{k=1}^{n-1}u_k=\alpha u_n,\ \ n=1, 2, 3, \cdots$$

which can be written as

$$\left[(n-1)-\sum_{k=1}^{n-2}u_k\right]+1-u_{n-1}=\alpha u_n,\ \ n=1, 2, 3, \cdots$$

or

$$\alpha u_{n-1}+1-u_{n-1}=\alpha u_n,\ \ n=1, 2, 3, \cdots$$

Thus, for $n=1, 2, 3, \cdots$, we have

$$\begin{aligned}\alpha u_n&=1+[(\alpha-1)/\alpha]\,(\alpha u_{n-1})\\&=1+[(\alpha-1)/\alpha]\,\{1+[(\alpha-1)/\alpha]\,(\alpha u_{n-2})\}\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&=1+[(\alpha-1)/\alpha]+[(\alpha-1)/\alpha]^2+\cdots+[(\alpha-1)/\alpha]^{n-1}\ (\alpha u_1)\\&=\{1-[(\alpha-1)/\alpha]^n\}/\{1-[(\alpha-1)/\alpha]\},\end{aligned}$$

since $\alpha u_1=1+(\alpha-1)u_0=1$.

Hence, $P(T\leq n)=u_n=1-[(\alpha-1)/\alpha]^n$, $n=1,2,3,\cdots$, i.e., T has a geometric distribution with $P(T=n)=u_n-u_{n-1}=(1/\alpha)[(\alpha-1)/\alpha]^{n-1}$, $n=1, 2, 3, \cdots$

Dept. of Mathematics
University of Malaya
Kuala Lumpur 22—H, Malaysia

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