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A NOTE ON SHARMA'S FORMULA

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1. Sharma has proved the formula [1, p.138, equ.(5)]

$$F\left[\begin{matrix}-n,\ \beta+f_1+f_2+n;\ \frac{1}{2}f_1,\ \frac{1}{2}+\frac{1}{2}f_1;\ \frac{1}{2}f_2,\ \frac{1}{2}+\frac{1}{2}f_2;\ 1,\ 1\\ \frac{1}{2}(\beta+f_1+f_2),\ \frac{1}{2}(1+\beta+f_1+f_2);\ 1+f_1;\ 1+f_2;\end{matrix}\right]$$

$$=\frac{(\beta)_n}{(\beta+f_1+f_2)_n}. \tag{1}$$

In (1), the following notation due to Burchnall and Chaundy [3] has been used to represent the hypergeometric series of higher order and of two variables.

$$F\left[\begin{matrix}(a_p);\ (b_q);\ (c_r);\ x,\ y\\ (d_s);\ (e_h);\ (f_k);\end{matrix}\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n x^m y^n}{[(d_s)]_{m+n}[(e_h)]_m[(f_k)]_n m!\,n!} \tag{2}$$

where (a_p) and $[(a_p)]_{m+n}$ will mean the sequence and the product $(a_1)_{m+n}\cdots(a_p)_{m+n}$ respectively.

In the investigation we use the formula due to Whipple [4, p.244. eq.(III.11)]

$${}_4F_3\left[\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ \beta,\ -n;\ -1\\ \frac{1}{2}\alpha,\ 1+\alpha-\beta,\ 1+\alpha+n\end{matrix}\right]=\frac{(1+\alpha)_n}{(1+\alpha-\beta)_n}, \tag{3}$$

and Dougall [4, p.244. eq. (III.13)]

$${}_5F_4\left[\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ \beta,\ \gamma,\ -n;\ 1\\ \frac{1}{2}\alpha,\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\alpha+n;\end{matrix}\right]=\frac{(1+\alpha)_n(1+\alpha-\beta-\gamma)_n}{(1+\alpha-\beta)_n(1+\alpha-\gamma)_n}. \tag{4}$$

2. The first formula to be proved is

$$F\left[\begin{matrix}-n,\ \beta+\alpha_1+\alpha_2+n;\frac{1}{2}\alpha_1,\ \frac{1}{2}+\frac{1}{2}\alpha_1;\ \frac{1}{2}\alpha_2,\ \frac{1}{2}+\frac{1}{2}\alpha_2;\ 1,\ 1\\ \frac{1}{2}(\beta+\alpha_1+\alpha_2),\ \frac{1}{2}(1+\beta+\alpha_1+\alpha_2);\ 1+\alpha_1-\gamma_1;\ 1+\alpha_2-\gamma_2;\end{matrix}\right]$$

$$=\frac{(\beta)_n}{(\beta+\alpha_1+\alpha_2)_n}F\left[\begin{matrix}-n;\alpha_1,\ \gamma_1;\ \alpha_2,\ \gamma_2;\ -1,\ -1\\ 1-\beta-n;\ 1+\alpha_1-\gamma_1;\ 1+\alpha_2-\beta_2;\end{matrix}\right] \tag{5}$$

PROOF. To prove (5), we start with the left side of (5)

$$F\left[\begin{matrix}-n,\ \beta+\alpha_1+\alpha_2+n; \frac{1}{2}\alpha_1,\ \frac{1}{2}+\frac{1}{2}\alpha_1;\ \frac{1}{2}\alpha_2,\ \frac{1}{2}+\frac{1}{2}\alpha_2;\ 1,\ 1\\ \frac{1}{2}(\beta+\alpha_1+\alpha_2),\ \frac{1}{2}(1+\beta+\alpha_1+\alpha_2);\ 1+\alpha_1-\gamma_1;\ 1+\alpha_2-\gamma_2;\end{matrix}\right]$$

$$=\sum_{p=0}^{p+q}\sum_{q=0}^{\leq n}\frac{(-n)_{p+q}(\beta+\alpha_1+\alpha_2+n)_{p+q}\left(\frac{1}{2}\alpha_1\right)_p\left(\frac{1}{2}+\frac{1}{2}\alpha_1\right)_p\left(\frac{1}{2}\alpha_2\right)_q\left(\frac{1}{2}+\frac{1}{2}\alpha_2\right)_q}{\left(\frac{1}{2}\beta+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)_{p+q}\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)_{p+q}(1+\alpha_1-\gamma_1)_p\,(1+\alpha_2-\gamma)_q p!\ q!}\quad \text{by}(2)$$

$$=\sum_{p=0}^{p+q}\sum_{q=0}^{\leq n}\frac{(-n)_{p+q}(\beta+\alpha_1+\alpha_2+n)_{p+q}\left(\frac{1}{2}\alpha_1\right)_p\left(\frac{1}{2}+\frac{1}{2}\alpha_1\right)_p\left(\frac{1}{2}\alpha_2\right)_q}{\left(\frac{1}{2}\beta+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)_{p+q}\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)_{p+q}}$$

$$\frac{\left(\frac{1}{2}+\frac{1}{2}\alpha_2\right)_q}{(1+\alpha_1)_p(1+\alpha_2)_q\,p!\,q!}\sum_{r=0}^{p}\frac{(\alpha_1)_r\left(1+\frac{1}{2}\alpha_1\right)_r(\gamma_1)_r(-p)_r(-1)^r}{\left(\frac{1}{2}\alpha_1\right)_r(1+\alpha_1-\gamma_1)_r(1+\alpha_1+p)_r r!}$$

$$\sum_{s=0}^{q}\frac{(\alpha_2)_s\left(1+\frac{1}{2}\alpha_2\right)_s(\gamma_2)_s(-q)_s(-1)^s}{\left(\frac{1}{2}\alpha_2\right)_s(1+\alpha_2-\gamma_2)_s(1+\alpha_2+q)_s\ s!}\qquad \text{by}\quad (3)$$

$$=\sum_{r=0}^{r+s}\sum_{s=0}^{\leq n}\frac{(-n)_{r+s}(\beta+\alpha_1+\alpha_2+h)_{r+s}(\alpha_1)_r(\gamma_1)_r(\alpha_2)_s(\gamma_2)_s}{(\beta+\alpha_1+\alpha_2)_{2r+2s}(1+\alpha_1-\gamma_1)_r(1+\alpha_2-r_2)_s\,r!\ s!}$$

$$F\left[\begin{matrix}-n+r+s,\ \beta+\alpha_1+\alpha_2+n+r+s;\ \frac{1}{2}\alpha_1+r,\ \frac{1}{2}+\frac{1}{2}\alpha_1+r;\ \frac{1}{2}\alpha_2+s,\ \frac{1}{2}+\frac{1}{2}\alpha_2+s;\ 1,\ 1\\ \frac{1}{2}(\beta+\alpha_1+\alpha_2+2r+2s),\ \frac{1}{2}(1+\beta+\alpha_1+\alpha_2+2r+2s);\ 1+\alpha_1+2r;\ 1+\alpha_2+2s;\end{matrix}\right]$$

$$=\frac{(\beta)_n}{(\beta+\alpha_1+\alpha_2)_n}\ F\left[\begin{matrix}-n;\ \alpha_1,\ \gamma_1;\ \alpha_2,\ \gamma_2;\ -1,\ -1\\ 1+\beta-n;\ 1+\alpha_1-\gamma_1;\ 1+\alpha_2-\gamma_2;\end{matrix}\right]\qquad \text{by}\quad (1)$$

This completes the proof of (5). If we take $\alpha_2=0$ in (5), we obtain

$$ {}_4F_3\left[\begin{matrix}-n,\ \beta+\alpha+n,\ \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ 1\\ \frac{1}{2}(\beta+\alpha),\ \frac{1}{2}(1+\beta+\alpha),\ 1+\alpha-\gamma;\end{matrix}\right]=\frac{(\beta)_n}{(\beta+\alpha)_n}$$

$$ {}_3F_2\left[\begin{matrix}-n,\ \alpha,\ \gamma;\ -1\\ 1-\beta-n,\ 1+\alpha-\gamma;\end{matrix}\right]\qquad (6)$$

(6) is equivalent to the well-known formula

$$
{}_2F_1\,[\alpha,\ \gamma;\ 1+\alpha-\gamma;\ x]=(1+x)^{-\alpha}
$$

$$
{}_2F_1\left[\tfrac{1}{2}\alpha,\ \tfrac{1}{2}+\tfrac{1}{2}\alpha;\ 1+\alpha-\gamma;\ \frac{4x}{(1+x)^2}\right] \tag{7}
$$

3. The second formula to be proved is

$$
F\left[\begin{matrix}-n,\ \beta+\alpha_1+\alpha_2+n;\ \frac{1}{2}\alpha_1,\ \frac{1}{2}+\frac{1}{2}\alpha_1,\ 1+\alpha_1-\rho_1-\delta_1;\ \frac{1}{2}\alpha_2,\ \frac{1}{2}+\frac{1}{2}\alpha_2,\ 1+\alpha_2-\rho_2-\delta_2;\ 1,\ 1\\ \frac{1}{2}(\beta+\alpha_1+\alpha_2),\ \frac{1}{2}(1+\beta+\alpha_1+\alpha_2);\ 1+\alpha_1-\rho_1,\ 1+\alpha_1-\delta_1;\ 1+\alpha_2-\rho_2,\ 1+\alpha_2-\delta_2;\end{matrix}\right]=\frac{(\beta)_n}{(\beta+\alpha_1+\alpha_2)_n}
$$

$$
F\left[\begin{matrix}-n;\ \alpha_1,\ \rho_1,\ \delta_1;\ \alpha_2,\ \rho_2,\ \delta_2;\ 1,\ 1\\ 1-\beta-n;\ 1+\alpha_1-\rho_1,\ 1+\alpha_1-\delta_1;\ 1+\alpha_2-\rho_2,\ 1+\alpha_2-\delta_2\end{matrix}\right] \tag{8}
$$

PROOF. (8) can be proved in the same way as (5) by using (4) instead of (3). On putting $\alpha_2=0$ in (8), we get

$$
{}_5F_4\left[\begin{matrix}-n,\ \beta+\alpha+n,\ \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha,\ 1+\alpha-\rho-\delta;\ 1\\ \frac{1}{2}(\beta+\alpha),\ \frac{1}{2}(1+\beta+\alpha),\ 1+\alpha-\rho,\ 1+\alpha-\delta;\end{matrix}\right]
$$

$$
=\frac{(\beta)_n}{(\beta+\alpha)_n}\ {}_4F_3\left[\begin{matrix}-n,\ \alpha,\ \rho,\ \delta;\ 1\\ 1-\beta-n,\ 1+\alpha-\rho,\ 1+\alpha-\delta;\end{matrix}\right] \tag{9}
$$

(9) is equivalent to the formula due to Whipple [2, p. 90 eq. (1)]

$$
{}_3F_2\left[\begin{matrix}\alpha,\ \rho,\ \delta;\ x\\ 1+\alpha-\rho,\ 1+\alpha-\delta;\end{matrix}\right]=(1-x)^{-\alpha}
$$

$$
{}_3F_2\left[\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha,\ 1+\alpha-\rho-\delta;\ \frac{-4x}{(1-x)^2}\\ 1+\alpha-\rho,\ 1+\alpha-\delta;\end{matrix}\right] \tag{10}
$$

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