A CHARACTERIZATION THEOREM FOR FAMILIES

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Let P be a finite ordered set and $a \in P$. Then we define the following numerical functions(Rhee [1], [2]):

$f_1(a) = |\{ x \in P | x > a \}|$, the number of *descendants* of a,

$f_2(a) = |\{ x \in P | x < a \}|$, the number of *ancestors* of a.

We call P a *family* if both $f_1(a) > f_1(b)$ and $f_2(a) < f_2(b)$ imply $a < b$ for any elements a and b in P.

In this paper we prove the following theorem which characterizes the class of all families. The notations and terminlogy in the theorem will be defined below. We assume throughout that all ordered sets are finite.

THEOREM. *The following conditions on an ordered set P are equivalent:*

(i) P *is a family.*

(ii) *If* $(a)_P \subset (b)_P$ *and* $\langle a \rangle_P \supset \langle b \rangle_P$, *then* $a > b$ *in* P.

(iii) P *belongs to* $\mathcal{F}$.

(iv) $P = C + D$, *where* C *is connected, series-parallel, A-free and* A^d*-free and* D *is an antichain or empty.*

(v) P *is a series-parallel interval order.*

For an element a of an ordered set P, let $(a)_P = \{ x \in P | x > a \}$ and $\langle a \rangle_P = \{ x \in P | x < a \}$. Here $\subset$ and $\supset$ mean proper inclusions. For ordered sets P and Q, we denote as usual by $P + Q$ and $P \oplus Q$ the *cardinal* (or *disjoint*) *sum* and the *ordinal* (or *linear*) *sum* of P and Q, respectively. An ordered set is said to be *series-parallel* if it can be decomposed into singletons using $+$ and $\oplus$. It is well known(cf. Rival [3]) that an ordered set is series-parallel if and only if it contains no subset isomorphic to the ordered set N (Figure 1). In particular, an ordered set is called a *tower* (or *P-graph*, or *weak order*) if it can

be decomposed into antichains using only $\oplus$. An ordered set is called an *interval order* if it contains no subset isomorphic to $\mathbf{2} + \mathbf{2}$, where $\mathbf{2}$ is the two-element chain. However, series-parallel ordered sets and interval orders are known to be two very important classes of ordered sets.
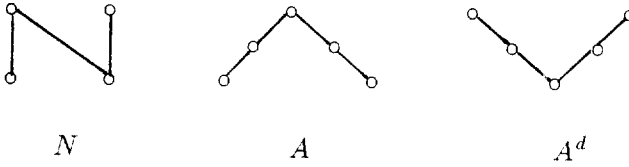

$$N \qquad\qquad A \qquad\qquad A^d$$

Figure 1

A class $\mathcal{F}$ of ordered sets is constructed inductively as follows:
(1) Every antichain belongs to $\mathcal{F}$.
(2) If P and Q belong to $\mathcal{F}$ and D is an antichain, then $P \oplus Q$ and $P + D$ also belong to $\mathcal{F}$.

Let P be an ordered set. For $a > b$ in P, we say that a *covers* b (b is *covered by* a, a is an *upper cover* of b, b is a *lower cover* of a), – in symbols, $a \succ b$ or $b \prec a$ – if there is no element x in P such that $a > x > b$. A subset Q of P is said to be *cover-preserving* if $a \prec b$ in Q implies $a \prec b$ in P. We then say that P is X-*free* if P contains no cover-preserving subset isomorphic to an ordered set X. Here X^d denotes the dual of an ordered set X.

A subset C of an ordered set P is said to be *connected* if $C = X + Y$ for some subsets X and Y of P implies that X or Y is empty. Every maximal connected subset of P is called a (*connected*) *component* of P.

Proof of Theorem. (i) $\Rightarrow$ (v) If P is not series-parallel or an interval order, then P has a subset P_1 or P_2 in Figure 2. In either case, choose a minimal element $a_0 \in P$ such that $a_0 \leq a$ and $a_0 \nleq d$ and then choose a maximal element $d_0 \in P$ such that $d \leq d_0$ and $a_0 \nleq d_0$. Then $f_1(a_0) > f_1(d_0)$ and $f_2(a_0) < f_2(d_0)$ and so $a_0 < d_0$, a contradiction. Hence P is a series- parallel interval order.
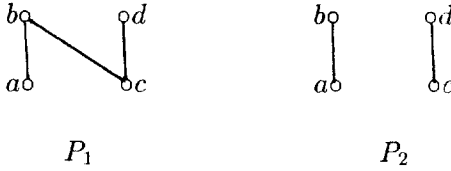
Figure 2

(v) $\Rightarrow$ (iv) Suppose that P is a series-parallel interval order. If P is an antichain, we are done. Otherwise, we choose a connected component C containing a two-element chain. If $P = C + D$ for some D, then obviously C is A-free and A^d-free and D is an antichain since P is an interval order.

(iv) $\Rightarrow$ (iii) Suppose that an ordered set P has subsets C and D as in (iv). Then the proof is proceeded by induction on $|P|$.

Case 1. $D \neq \emptyset$. Trivially, C also satisfies (iv) and so $C \in \mathcal{F}$ by induction hypothesis. Hence $P \in \mathcal{F}$.

Case 2. $D = \emptyset$.

In this case , $P = Q \oplus R$ since P is series-parallel. If Q is connected then $Q \in \mathcal{F}$ as above. If not, $Q = Q_1 + Q_2$. Now Q_1 or Q_2 is an antichain, for otherwise there are $x_1, y_1 \in Q_1$ such that $x_1 \prec y_1$ and y_1 is maximal in Q_1 and there are $x_2, y_2 \in Q_2$ such that $x_2 \prec y_2$ and y_2 is maximal in Q_2 and these elements with a minimal element in R form the ordered set A (Figure 1) as a cover-preserving subset of P. Hence $Q \in \mathcal{F}$ by induction hypothesis. Similary, $R \in \mathcal{F}$, proving that $P \in \mathcal{F}$.

(iii) $\Rightarrow$ (ii) Let an ordered set P belong to $\mathcal{F}$. Then we again use an induction on $|P|$. Suppose that $(a)_P \subset (b)_P$ and $\langle a \rangle_P \supset \langle b \rangle_P$.

Case 1. $P = Q + D$, where $Q \in \mathcal{F}$ and D is an antichain.

If $a \in D$ or $b \in D$, then $\emptyset = \langle a \rangle_P \not\supset \langle b \rangle_P$ or $(a)_P \not\subset (b)_P = \emptyset$, which is a contradiction. If $a, b \in Q$, then $(a)_Q = (a)_P$, $(b)_Q = (b)_P$, $\langle a \rangle_Q = \langle a \rangle_P$ and $\langle b \rangle_Q = \langle b \rangle_P$. Hence $(a)_Q \subset (b)_Q$ and $\langle a \rangle_Q \supset \langle b \rangle_Q$, which implies $a > b$.

Case 2. $P = Q \oplus R$, where $Q, R \in \mathcal{F}$

Since the other cases are treated similarly or trivially, we only consider the case when $a, b \in Q$. Now $(a)_P = (a)_Q \bigcup R$, $(b)_P = (b)_Q \bigcup R$, $\langle a \rangle_P = \langle a \rangle_Q$, $(b)_P = \langle b \rangle_Q$. Hence $(a)_Q \subset (b)_Q$ and $\langle a \rangle_Q \supset \langle b \rangle_Q$, which implies $a > b$.

(ii) $\Rightarrow$ (i) Assume that an ordered set P satisfies (ii). If P is not a family then there exist elements a and b in P such that $f_1(a) > f_1(b)$ and $f_2(a) < f_2(b)$ but $a \not< b$. Since $b \not\leq a$, a and b are incomparable. Choose an element c which is maximal with respect to the property that $b \leq c$ but $a \not< c$ and then choose an element d which is minimal with respect to the property that $d \leq a$ but $d \not< c$. Now $(d)_P \supseteq (c)_P$ and $\langle d \rangle_P \subseteq \langle c \rangle_P$. Since $f_1(d) \geq f_1(a) > f_1(b) \geq f_1(c)$ and $f_2(d) \leq f_2(a) < f_2(b) \leq f_2(c)$, we have $(d)_P \supset (c)_P$ and $\langle d \rangle_P \subset \langle c \rangle_P$, which implies that $d < c$, a contradiction.

COROLLARY 1. *Every subset of a family is also a family.*

COROLLARY 2. *An ordered set is a cardinal sum of connected families if and only if it is series-parallel, A-free and A^d-free.*

For an ordered set P, the *length* $l(P)$ of P is defined to be the maximum cardinality of a chain in P minus 1 – in symbols, $l(P) = \max\{|C| - 1 \,|\, C \text{ is a chain in } P\}$. An ordered set is called *graded* if every maximal chain between any two elements has the same finite length.

COROLLARY 3. *An ordered set is a tower if and only if it is graded family.*

Proof. Let P be a graded family. If P is not an antichain, then it is connected, i.e., $P = Q \oplus R$ for some $Q, R \in \mathcal{F}$. Then Q and R are also graded. By induction on $|P|$, we can see that P is a tower. The converse is obvious.

For $i = 1, 2$, let $f_i(P) = \{f_i(x) \,|\, x \in P\}$.

COROLLARY 4. *For a family P, $l(P) = |f_i(P)| - 1$ for all $i = 1, 2$.*

Proof. By induction on $|P|$ we only show that $l(P) = |f_1(P)| - 1$. Let C be a chain of the maximum length in P. Since $|C| = l(P) + 1$, we shall show that $|f_1(P)| = |C|$.

Case 1. $P = Q + D$, where $Q \in \mathcal{F}$ and D is an antichain.

Since $f_1(D) = \{0\}$ and $0 \in f_1(Q)$, we have $f_1(P) = f_1(Q)$. However, C is a chain of the maximum length in Q and so $l(Q) = l(P)$. Now $l(P) = l(Q) = |f_1(Q)| - 1 = |f_1(P)| - 1$.

Case 2. $P = Q \oplus R$, where $Q, R \in \mathcal{F}$.

In this case $C_1 = C \bigcap Q$ and $C_2 = C \bigcap R$ are also chains of the maximum length in Q and R, respectively. Now we observe that $f_1(P) = \{n + r \mid n \in f_1(Q)\} \bigcup f_1(R)$, where $r = |R|$. Since $f_1(x) > r$ for any $x \in Q$, $|f_1(P)| = |f_1(Q)| + |f_1(R)| = |C_1| + |C_2| = |C|$.

The (McNeille) *completion* $\underset{\sim}{C}(P)$ of an ordered set P turns out to be the smallest lattice into which P can be embedded.

COROLLARY 5. *Let P be an ordered set. Then $\underset{\sim}{C}(P)$ is a family if and only if so is P.*

Proof. The proof is also by induction on $|P|$.
Case 1. $P = Q + D$, where $Q \in \mathcal{F}$ and D is an antichain.
Observe that $\underset{\sim}{C}(P) = \{0\} \oplus (\underset{\sim}{C}(Q) + D) \oplus \{1\}$.
Case 2. $P = Q \oplus R$, where $Q, R \in \mathcal{F}$.
If Q and R have the largest element and the smallest element, respectively, then $\underset{\sim}{C}(P) = \underset{\sim}{C}(Q) \oplus \underset{\sim}{C}(R)$. Otherwise, $\underset{\sim}{C}(P) = (\underset{\sim}{C}(Q) \ominus \underset{\sim}{C}(R)) - \{x\}$, where x is the largest element of $\underset{\sim}{C}(Q)$ or the smallest element of $\underset{\sim}{C}(R)$.

The converse is immediate.

References

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