Journal of KSNVE, Vol. 10, No. 2, pp. 361~366, 2000.

〈Original Paper〉

Development of a Finite Element for Vibration Analysis of an Annular Plate with Slight Deviation

미소한 비대칭이 존재하는 원판의 진동해석을 위한 유한요소 개발

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(Received January 25, 2000 : Accepted March 6, 2000)

Key Words : Finite Annular Plate Element(환형평판 유한요소), Finite Element Method(유한요소해석), Axisymmetric Annular Plate(축대칭 환형평판), Deviation(편차)

ABSTRACT

In this paper, a new finite annular plate element is developed, which considers the effects of the slight deviation from a perfect axisymmetry. It is assumed that, when a local deviation is introduced to an axisymmetric plate, the natural modes are separated into the symmetric and asymmetric modes. The proposed method is very efficient because a few elements are demanded and lots of active degrees of freedom are reduced in comparison with commercial numerical analysis programs. In addition, when the deviation is small enough, it is more accurate than the result of using usual plate elements of commercial FEM programs.

요 약

본 연구에서, 축대칭으로부터 약간의 편차가 있는 효과를 고려한 환형평판요소가 개발되었다. 국부 편차가 축대칭 평판에 발생되었을 때 모드형상이 대칭모드와 비대칭모드로 분리된다고 가정한다. 본 논문에서 제안된 방법은 기존의 상용프로그램에 비해 적은 수의 요소를 필요로 하며, 따라서 자유도가 많이 줄어드는 장점을 가진다. 그리고, 편차가 충분히 작을 때, FEM을 이용하는 기존의 상용프로그램의 평판요소를 사용한 결과보다 더욱 정확한 결과를 얻을 수 있다.

1. Introduction

The vibration analysis of structures deviating from perfect axisymmetry is an important topic. Many practical machine elements and structural components have non-axisymmetry such as ribs, grooves and concentrated weights to ensure adequate stiffness, rigidity, balance, or noise control. These are sometimes significant and obviously related to the dynamic characteristics of the structure. The influence of structural imperfection on thin-walled structures was introduced in the recently published book by Godoy[1]. He presented that small imperfections may introduce changes in the stresses due to the loads. Shen and Mote[2] have proposed a perturbation approach to the vibration analysis of finite

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solids containing small elastic imperfections. Hong and Lee[3] have presented an analytical method to predict the effects of local deviation on the free in-plane vibration of nearly axisymmetric rings. They show that, when a local deviation is introduced to an axisymmetric ring, the natural modes are separated into the symmetric and asymmetric modes. Chung and Lee[4] have proposed a finite element method to analyze the natural frequencies and modes of a nearly axisymmetric shell structure with local deviation. In order to analyze complex structures, the finite element method appears to be ideal. However, the general axisymmetric elements cannot be applied to the structures owing to the slight deviation. The structures demand many square or triangular elements; the computation time and costs are required much.

In this study, a new finite annular plate element is developed to analyze dynamic characteristics of the annular plate with slight deviation. The displacements are assumed as a pair of mode shapes: symmetric and asymmetric modes. After using the relations between the displacements and strain, the mass and stiffness matrices are derived. Finally, It is shown that the proposed finite elements are efficient to compute natural frequencies compared to the elements of commercial codes.

2. Annular Plate Element with Slight Deviation

An annular plate with slight deviation is discretized into annular plate elements as shown in Fig. 1. The slight deviation is located along the co-ordinate r. The nodal circles generated by intersection of each plate

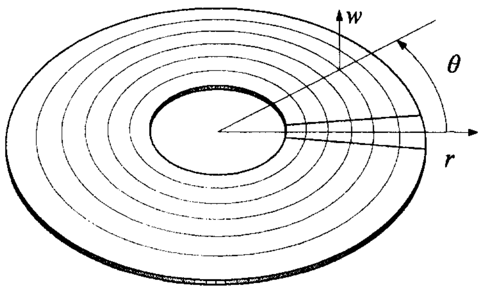

Fig. 1 Example of a discretized annular plate with sight deviation

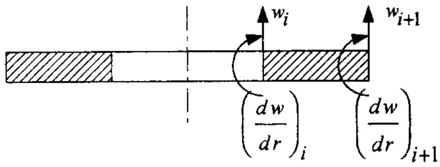

Fig. 2 Displacements in the annular plate element

element, that is, an element is bounded by two adjacent nodal circles: element i is bounded by nodal circles i and $i+1$. The displacements in the annular plate element i are shown in Fig. 2.

3. Determination of the Displacement Function

In this study, the mode shapes of the plate element with slight asymmetry are given according to Hong and Lee[3]. They showed that the natural modes of a ring with local deviation are separated into symmetric and asymmetric ones: local deviation is positioned on an anti-nodal point of the symmetric mode while it is located on a nodal point of the asymmetric mode. This means that the asymmetric mode shape rotates with $\frac{\pi}{2n}$ from the symmetric mode shape for each harmonic number n. For example, the asymmetric mode shape rotates with 45° from the symmetric one for $n=2$. Under this assumption, for motions associated with the n-th circumferential mode number, we may write

$$w(r,\theta)=\cos\left(n\theta-\frac{\pi b}{2}\right)W(r) \tag{1}$$

where n is the circumferential mode number, b is the parameter related to the symmetric or asymmetric modes. $W(r)$ is the magnitude of the deflections and depends on r only. If $W(r)$ is approximated by cubic interpolation function, $w(r,\theta)$ can be described as

$$w=\cos\left(n\theta-\frac{\pi b}{2}\right)(\alpha_1+\alpha_2 s+\alpha_3 s^2+\alpha_4 s^3) \tag{2}$$

or

$$w=\cos\left(n\theta-\frac{\pi b}{2}\right)\{1,s,s^2,s^3\}\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^T \tag{3}$$

where s is the local coordinate along radial direction. From Fig. 2, nodal displacements vector is as follows:

$$W_e=\left\{w_i,\left(\frac{dw}{dr}\right),w_{i+1},\left(\frac{dw}{dr}\right)_{i+1}\right\} \quad (4)$$
$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&L&L^2&L^3\\0&1&2L&3L^2\end{bmatrix}\begin{Bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{Bmatrix}$$

By solving equation (4) for the vector and substituting it into equation (3), we may obtain

$$(n\neq 0):\quad w=C_n(\theta)\,\boldsymbol{N}\boldsymbol{w}_e \quad (5)$$

where

$$C_n(\theta)=\cos\left(n\theta-\frac{\pi b}{2}\right) \quad (6)$$

$$N=\left[\frac{(L-s)^2(L+2s)}{L^3}\ \frac{(L-s)^2 s}{L^2}\ \frac{(3L-2s)s^2}{L^3}\ -\frac{(L-s)s^2}{L^2}\right]. \quad (7)$$

In case the circumferential mode number n is zero, the displacement is a function of r only, and no function related to θ is needed.

$$(n=0):\ w(r,\theta)=W(r) \quad (8)$$

Following the work of the case ($n\neq 0$), we obtain

$$w=\boldsymbol{N}\boldsymbol{w}_e \quad (9)$$

where matrix $\boldsymbol{N}$ is identical as equation (7).

4. Relations between the Displacements and Strain Vectors

The transversal displacement w of a vibrating plate will result in the curvature changes and twist of the middle surface. Assuming that the length of one element is small enough, the radius r can be replaced by the center radius of an element R and the deformation vector is given by

$$\varepsilon=\begin{Bmatrix}\chi_s\\\chi_\theta\\\chi_{s\theta}\end{Bmatrix}=\begin{Bmatrix}-\frac{\partial^2 w}{\partial r^2}\\-\frac{1}{R}\frac{\partial w}{\partial r}-\frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2}\\-2\frac{1}{R}\frac{\partial^2 w}{\partial r\partial\theta}+2\frac{1}{R^2}\frac{\partial w}{\partial\theta}\end{Bmatrix}, \quad (10)$$

where χ_s and χ_θ are the curvature changes, $\chi_{s\theta}$ is the middle surface twist. By expressing the strain vector in terms of the nodal displacements, one obtains the following expressions.

$$(n\neq 0):\ \varepsilon=\ \boldsymbol{T}_B\boldsymbol{B}\boldsymbol{w}_e \quad (11)$$

where

$$\boldsymbol{T}_B=diag[\,C_n(\theta)\ \ C_n(\theta)\ \ S_n(\theta)] \quad (12)$$

$$C_n(\theta)=\cos\left(n\theta-\frac{\pi b}{2}\right) \quad (13)$$

$$S_n(\theta)=\sin\left(n\theta-\frac{\pi b}{2}\right) \quad (14)$$

and the 3 by 4 matrix $\boldsymbol{B}$ is obtained algebraically, which is a function of s. On the other hand,

$$(n=0):\ \varepsilon=\ \boldsymbol{B}_0\boldsymbol{w}_e \quad (15)$$

where the 3 by 4 matrix $\boldsymbol{B}_0$ is obtained algebraically, which is a function of s.

The corresponding stresses for isotropic material may be related to the strains by the elasticity matrix $\boldsymbol{D}$.

$$\sigma=\boldsymbol{D}\varepsilon=\begin{bmatrix}D&\nu D&0\\\nu D&D&0\\0&0&\frac{(1-\nu)D}{2}\end{bmatrix}\varepsilon, \quad (16)$$

where : $D=Eh^3/12(1-\nu^2)$; E is the Young's modulus, h is the thickness of the plate and ν is the Poisson's ratio.

5. Derivation of the Mass and Stiffness Matrix

Following the framework of the finite element method [5], the mass and stiffness matrix may be expressed as

$$(n\neq 0):\ \boldsymbol{M}_e=\int_V \boldsymbol{N}^T C_n(\theta)\rho C_n(\theta)\,\boldsymbol{N}dV \quad (17)$$

$$\boldsymbol{K}_e=\int_A \boldsymbol{B}^T\boldsymbol{T}_B\boldsymbol{D}\boldsymbol{T}_B\boldsymbol{B}dA \quad (18)$$

$$(n=0):\ \boldsymbol{M}_e=\int_V \boldsymbol{N}^T\rho\boldsymbol{N}dV \quad (19)$$

$$\boldsymbol{K}_e=\int_A \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}dA \quad (20)$$

After introducing a parametric coordinate which is defined by $\xi=2s/L-1$ into equations (17)~(20), the results of analytical calculations can be obtained.

$$\boldsymbol{M}_e = \frac{L}{2}\int_{-1}^{1}(\boldsymbol{N}^T\rho_M\boldsymbol{N})\left(R+\frac{L}{2}\xi\right)d\xi \quad (21)$$

$$\boldsymbol{K}_e = \frac{L}{2}\int_{-1}^{1}(\boldsymbol{B}^T\boldsymbol{D}_K\boldsymbol{B})\left(R+\frac{L}{2}\xi\right)d\xi \quad (22)$$

where

$(n \neq 0)$:

$$\rho_M = \pi\rho_0 h_0 + \frac{1}{2}\left(\Delta\theta + \frac{(1-6)^b \sin n\Delta\theta}{n}\right) \times (\rho_a h_a - \rho_0 h_0)$$

$$\boldsymbol{D}_K = \pi\boldsymbol{D}_0 + \frac{1}{2}\left(\Delta\theta\boldsymbol{I} + \frac{(-b)^b \sin n\Delta\theta}{n}\boldsymbol{T}_K\right) \times (\boldsymbol{D}_a - \boldsymbol{D}_0)$$

$$\boldsymbol{T}_k = diag[1 \;\; 1 \;\; -1]. \quad (23,24)$$

$(n=0)$:

$$\rho_M = 2\pi\rho_0 h_0 + \Delta\theta(\rho_a h_a - \rho_0 h_0), \quad (25)$$

$$\boldsymbol{D}_K = 2\pi\boldsymbol{D}_0 + \Delta\theta(\boldsymbol{D}_a - \boldsymbol{D}_0) \quad (26)$$

in which $\boldsymbol{I}$ is the 3 by 3 identity matrix, $\boldsymbol{D}_a$ and $\boldsymbol{D}_0$ are elasticity matrix of deviated portion and not-deviated portion respectively; $\Delta\theta$ is the angle corresponding to the local deviation.

The Gaussian quadrature is used to compute the element mass and stiffness matrix with respect to ξ from -1 to 1. The global mass and stiffness matrix can be obtained by assembling the element matrices.

6. Numerical Results

Natural frequencies and mode shapes are obtained from the eigenproblem, which is given by

$$(\boldsymbol{K} - \omega_n^2\boldsymbol{M})\boldsymbol{X} = 0 \quad (27)$$

where ω_n is the natural frequency and $\boldsymbol{X}$ is the eigenvector.

6.1 Case Study 1: Convergence of the Method for a Flat Annular Plate

A first set of calculations is undertaken to determine the requisite number of annular plate elements for the determination of natural frequencies. Calculations are made for the uniform free-free annular plate with the number of annular plate elements varying from 5 to 80. The material properties of the plate are ρ =7800 kg/m^3, E=210 GN/m^2 and ν=0.27. The thickness is 1 cm, the outer radius is 0.5 m and the inner radius is 0.15 m.

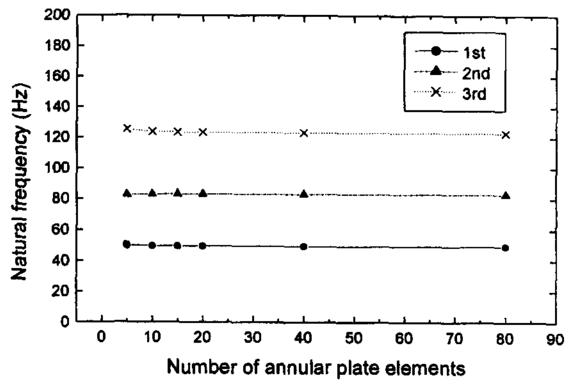

Fig. 3 Natural frequencies of flat annular plate as a function of the number of annular plate elements

Table 1 Natural frequencies of the flat annular plate

Mode number (n, m)	Proposed method	ANSYS		
	80 elements	7920 elements	6480 elements	2520 elements
1 (2,0)	49.311	49.323	49.323	49.370
2 (0,1)	82.976	83.003	83.007	83.081
3 (3,0)	123.22	123.25	123.25	123.38
4 (1,1)	182.12	182.25	182.26	182.60
5 (4,0)	218.60	218.67	218.67	218.96
6 (2,1)	327.04	327.15	327.16	327.47

Figure 3 shows the results for first three natural frequencies of the plate. Figure 4 presents the first natural frequency in detail. The proposed method uses the cosine function for the θ coordinate, which satisfies the governing equation of the annular plate.

Therefore, as may be seen in Table 1, the proposed method demands only few elements to converge compared to the usual finite element analysis program. In addition, all the results by the proposed method are slightly lower than that of ANSYS. It results from the exact solution for the coordinate.

6.2 Case Study 2: Mass and Stiffness Effects

Two cases of annular plates with slight deviation have been studied. The plates have slight deviation along the r coordinate, as shown in Fig. 1. The first one is a free-free annular plate which has a slight sector of doubled

Table 2 Natural frequencies of the annular plate with deviation (doubled mass effect)

Mode number (n, m)	Proposed method (80 elements)	ANSYS (7920elements)
1 (2,0)	48.009	49.309
2 (2,0)	49.297	48.093
3 (0,1)	81.847	81.856
4 (3,0)	120.01	120.34
5 (3,0)	123.14	123.18
6 (1,1)	177.28	177.36
7 (1,1)	182.11	182.24
8 (4,0)	212.99	213.85
9 (4,0)	218.36	218.43
10 (2,1)	318.40	318.50
10 (2,1)	326.95	327.06

density and the second, of doubled bending stiffness; the sector is a portion of 10 degree, other material parameters and dimensions of the plates are identical as case study 1. The finite element model of the method is only 160, while that of ANSYS is 24192. As may be seen in Table 2 and 3, the results obtained by plate by ANSYS is shown in Fig. 5. Note that the number of the active degree of freedom of the proposed the proposed method are in good agreement with those of ANSYS.

As well as in Table 1, all the results of the proposed method are slightly lower than those of ANSYS in Table 2, while the results in Table 3 are not. The bending stiffness may influence the assumption that the mode shape of the annular plate is represented by the cosine function. On the other hand, the effect of doubled density to the assumption may seem to be negligible compared to the stiffness effect.

Table 3 Natural frequencies of the annular plate with deviation (doubled stiffness effect)

Mode number (n, m)	Proposed method (80 elements)	ANSYS (7920elements)
1 (2,0)	49.732	49.669
2 (2,0)	50.230	49.851
3 (0,1)	84.120	83.604
4 (3,0)	124.34	124.14
5 (3,0)	125.47	124.59
6 (1,1)	184.46	183.85
7 (1,1)	184.78	184.18
8 (4,0)	220.56	220.26
9 (4,0)	222.62	221.08
10 (2,1)	330.33	329.37
10 (2,1)	332.77	330.74

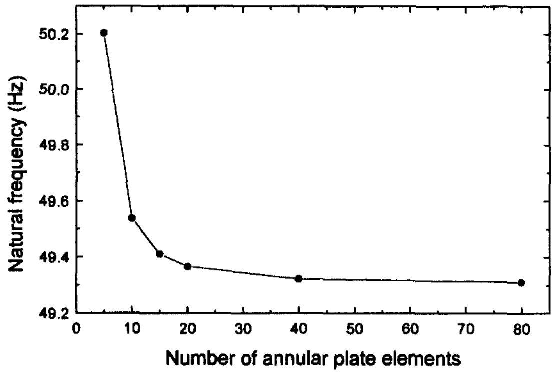

Fig. 4 The first natural frequency of the flat annular plate as a function of the number of annular plate elements

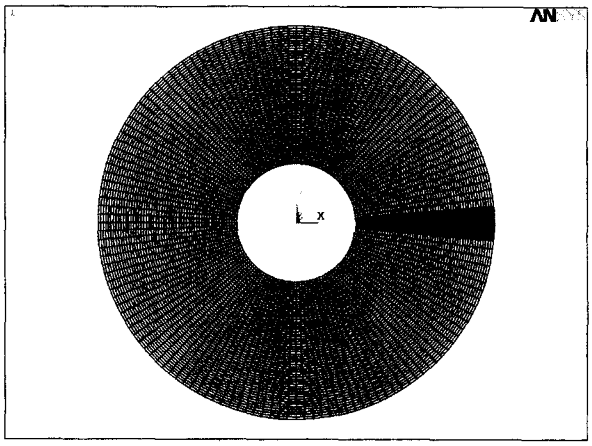

Fig. 5 The finite element model of the annular plate with deviation by ANSYS (7920 elements)

7. Conclusions

A new finite annular plate element is developed, based upon the fact that, for each harmonic number n, the asymmetric mode shape rotates with $\frac{\pi}{2n}$ from the symmetric mode shape. The developed element is applied to the FEM program which computes natural frequencies for the annular plate with slight deviation. The case study shows that the finite element method using the proposed element can save a great deal of computation time and cost.

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