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PENALIZED APPROACH AND ANALYSIS OF AN OPTIMAL SHAPE CONTROL PROBLEM FOR THE STATIONARY NAVIER–STOKES EQUATIONS

HONGCHUL KIM

ABSTRACT. This paper is concerned with an optimal shape control problem for the stationary Navier–Stokes system. A two–dimensional channel flow of an incompressible, viscous fluid is examined to determine the shape of a bump on a part of the boundary that minimizes the viscous drag. By introducing an artificial compressibility term to relax the incompressibility constraints, we take the penalty method. The existence of optimal solutions for the penalized problem will be shown. Next, by employing Lagrange multipliers method and the material derivatives, we derive the shape gradient for the minimization problem of the shape functional which represents the viscous drag.

1. Introduction

We deal with a specific drag minimization problem in two–dimensions. We consider a channel flow with a bump to be determined according to the scheme to minimize the drag profile. Existence results for this problem were given in [7], where one may also find a derivation of the model problem and motivation for the study. In this paper, we are concerned with a penalized approach to the state equations to the problem. In [8], we dealt with the penalized stationary incompressible Navier–Stokes system with the inhomogeneous Dirichlet boundary condition imposed on the part of the boundary. By employing the parameter-dependent nonlinear functional settings as in Brezzi-Rappaz-Raviart framework([2]), the existence and convergence results for the penalized solutions of the stationary

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Navier–Stokes systems were shown in [8]. Combining those results, we will show the existence of optimal solutions to the minimization problem.

Shape sensitivities are concerned with the relationship between available control parameters and responses of the state variables and shape functionals to variations on those parameters. Such relationship can be embodied essentially by finding the shape gradient. For this purpose, we establish adjoint equations by employing Lagrange multipliers and then apply the material derivative method to compute the variation of the shape parameter. The shape gradient for the shape functional is systematically achieved with the help of Lagrange multipliers. The justification of Lagrange multipliers and numerical analyses for the problem will be discussed in the subsequent papers.

The plan of the rest of the paper is as follows. In the remainder of this section, we describe the model problem and introduce some notations. Then, in §2, we show the existence of optimal solutions. In §3, we employ the Lagrange multipliers technique to induce adjoint equations for the systems. By taking the material derivative method, we derive the shape gradient for the shape functional.

1.1. Description of the problem

We consider the two–dimensional incompressible flow of a viscous fluid passing through a channel having a finite depth; see Figure 1. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be the preset velocities at the inflow Γ_1 and outflow Γ_2 of the channel, respectively. Along the bottom and top sides of the channel the velocity vanishes. The arc $\Gamma_b(\alpha)$, which is part of the bottom boundary, represents the bump, which is to be determined.

Let the boundary shape corresponding to the bump be represented by the graph of the curve $\alpha : [M_1, M_2] \to \mathbf{R}$. The domain Ω_α is composed of two fixed rectangles and a domain with an unknown boundary. Thus, the

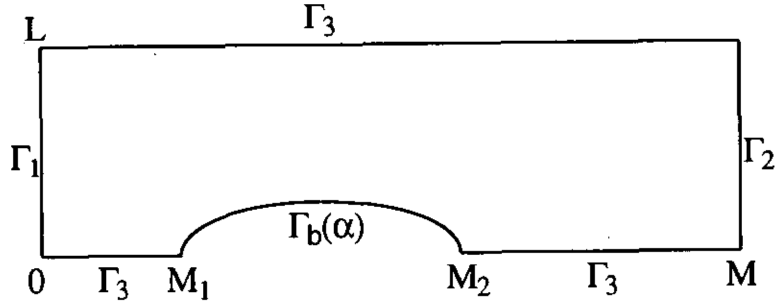

FIG. 1. *The domain Ω_α for flow through a channel with a bump.*

domain Ω_α is determined by the shape of the unknown boundary $\Gamma_b(\alpha)$ which we assume is given by

$$\Gamma_b(\alpha) = \{(x_1, x_2) \in [M_1, M_2] \times [0, L] \mid x_2 = \alpha(x_1)\},$$

where $\alpha(x_1)$ is a function to be determined by the optimization process. Assume that $\Gamma_b(\alpha) \subset [M_1, M_2] \times [0, L]$ and that both end points of $\Gamma_b(\alpha)$ are fixed (at $x_1 = M_1, x_2 = 0$ and $x_1 = M_2, x_2 = 0$) for all admissible domains. Since the domain Ω_α is determined by the shape of $\Gamma_b(\alpha)$, one may define the admissible family of curves defining $\Gamma_b(\alpha)$ as follows:

$$\begin{aligned}\mathcal{U}_{ad} = \{\, \alpha \in C^{0,1}([M_1, M_2]) \mid 0 \le \alpha(x_1) \le L \text{ and } \\ |\alpha(x_1) - \alpha(\overline{x}_1)| \le \beta |x_1 - \overline{x}_1| \ \forall x_1, \overline{x}_1 \in [M_1, M_2], \\ \alpha(M_1) = \alpha(M_2) = 0 \,\},\end{aligned}$$

where the positive constant β is chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. We have denoted the set of Lipschitz continuous functions in $[M_1, M_2]$ by the symbol $C^{0,1}([M_1, M_2])$. The condition $|\alpha(x_1) - \alpha(\overline{x}_1)| \le \beta|x_1 - \overline{x}_1|$ is invoked to prevent the "blow-up" of the boundary, i.e., to suppress excessive oscillations of $\Gamma_b(\alpha)$ (c.f. [9]).

We consider, for each $\alpha \in \mathcal{U}_{ad}$, the penalized stationary incompressible Navier–Stokes equations

$$-\nu \Delta \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f} \quad \text{in} \quad \Omega_\alpha \tag{1.1}$$

and

$$\nabla \cdot \mathbf{u}_\epsilon = -\epsilon p_\epsilon \quad \text{in} \quad \Omega_\alpha \tag{1.2}$$

along with the Dirichlet boundary conditions

$$\mathbf{u}_\epsilon = \mathbf{g} = \begin{cases} \mathbf{g}_1 & \text{on} \quad \Gamma_1 \\ \mathbf{g}_2 & \text{on} \quad \Gamma_2 \\ \mathbf{0} & \text{on} \quad \Gamma_3 \cup \Gamma_b(\alpha), \end{cases} \tag{1.3}$$

where $\mathbf{f}$ and $\mathbf{g}_i$, $i = 1, 2$, are given functions and $\epsilon > 0$ is a given parameter. Here, ν denotes the kinematic viscosity in the nondimensional form corresponding to the reciprocal of the Reynolds number Re and $\mathbf{f}$ the given external body force. Note that the constant density has been absorbed into the pressure and the body force. For the compatibility and regularity of solutions, we assume

$$\text{(1.4)} \qquad \text{support of } \mathbf{g}_i \subset \Gamma_i \quad \text{and} \quad \int_{\Gamma_1} \mathbf{g}_1 \cdot \mathbf{n}\, d\Gamma + \int_{\Gamma_2} \mathbf{g}_2 \cdot \mathbf{n}\, d\Gamma = 0\,.$$

The penalty method is often introduced to relax the incompressibility constraint with regard to Navier–Stokes system by introducing an artificial compressibility $-\epsilon p_\epsilon$ instead of the incompressibility constraint and to expect the near incompressibility. For the existence and convergence results for the penalized systems (1.1)–(1.3), one may consult [8]. The major advantage of penalized formulation is the elimination of the divergence free constraint and the pressure term. This will reduce the problem size and may facilitate the complicated sensitivity analysis. For $\epsilon > 0$, one can set $p_\epsilon = -\dfrac{\nabla \cdot \mathbf{u}_\epsilon}{\epsilon}$. By eliminating the corresponding pressure term, we can pose (1.1)–(1.3) into the following formulation in which only velocity is involved:

$$\text{(1.5)} \qquad \begin{aligned} -\nu\, \Delta \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon - \frac{1}{\epsilon}\nabla(\nabla \cdot \mathbf{u}_\epsilon) = \mathbf{f} \qquad & \text{in} \quad \Omega_\alpha\,, \\ \mathbf{u}_\epsilon = \mathbf{g} \qquad & \text{on} \quad \partial\Omega_\alpha\,. \end{aligned}$$

After finding $\mathbf{u}_\epsilon$ from (1.5), the appropriate pressure p_ϵ can be easily recovered.

One can examine several objectives for determining the shape of the bump, e.g., the reduction of the drag due to viscosity or the identification of the velocity at a fixed vertical slit downstream of the bump. To fix ideas, we focus on the minimization of the cost functional

$$\text{(1.6)} \qquad \begin{aligned} \mathcal{J}(\alpha) = \mathcal{J}\Big(\Omega_\alpha, \mathbf{u}_\epsilon(\alpha)\Big) &= 2\nu \int_{\Omega_\alpha} D(\mathbf{u}_\epsilon) : D(\mathbf{u}_\epsilon)\, d\Omega \\ &= \frac{\nu}{2} \sum_{i,j=1}^{2} \int_{\Omega_\alpha} \Big(\frac{\partial u_{\epsilon i}}{\partial x_j} + \frac{\partial u_{\epsilon j}}{\partial x_i}\Big)^2 d\Omega\,, \end{aligned}$$

where $\mathbf{u}_\epsilon(\alpha)$ is a solution of (1.1)–(1.3) in Ω_α and $D(\mathbf{u}_\epsilon) = \frac{1}{2}(\nabla\mathbf{u}_\epsilon + (\nabla\mathbf{u}_\epsilon)^T)$ is the deformation tensor for the flow $\mathbf{u}_\epsilon$. This functional represents the rate of energy dissipation due to deformation. Physically, except for an unimportant additive constant whose value depends on the data $\mathbf{f}$, $\mathbf{g}_1$, and $\mathbf{g}_2$, this functional represents the viscous drag of the flow. In (1.6), the colon denotes the scalar product operator between two tensors.

The extremal problem we consider is then given as follows:

$$(1.7)\qquad \begin{aligned} &\min_{\alpha\in\mathcal{U}_{ad}} \mathcal{J}(\Omega_\alpha, \mathbf{u}_\epsilon(\alpha)) \quad \text{such that} \\ &\mathbf{u}_\epsilon(\alpha) \text{ is a solution of (1.5) in } \Omega_\alpha\,. \end{aligned}$$

1.2. Notations

We denote by $H^s(\mathcal{D})$, $s \in \mathbf{R}$, the standard Sobolev space of order s with respect to the set $\mathcal{D}$, which is either the flow domain Ω_α, or its boundary Γ_α, or part of its boundary. Whenever m is a nonnegative integer, the inner product over $H^m(\mathcal{D})$ is given by

$$(f,g)_{m,\mathcal{D}} = (f,g)_{0,\mathcal{D}} + \sum_{0<|\lambda|\le m} (D^\lambda f, D^\lambda g)_{0,\mathcal{D}}\,,$$

where $(f,g)_{0,\mathcal{D}} = \int_{\mathcal{D}} fg\,d\mathcal{D}$ denotes the inner product over $H^0(\mathcal{D}) = L^2(\mathcal{D})$ and λ denotes a multi-index. Hence, we naturally associate the norm on $H^m(\mathcal{D})$ with $\|f\|_{m,\mathcal{D}} = \sqrt{(f,f)_{m,\mathcal{D}}}$. Whenever there is no chance for confusion, we will let $(\cdot,\cdot)_{m,\Omega_\alpha} = (\cdot,\cdot)_m$ and $\|\cdot\|_{m,\Omega_\alpha} = \|\cdot\|_m$ for the flow domain Ω_α. For vector–valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\mathcal{D}) = [H^s(\mathcal{D})]^n$ denotes the space of $\mathbb{R}^n$–valued functions such that each component belongs to $H^s(\mathcal{D})$.

For each $\alpha \in C^{0,1}([M_1, M_2])$, let $\Gamma_\alpha = \Gamma_3 \cup \Gamma_b(\alpha)$ and $\Gamma_g = \Gamma_1 \cup \Gamma_2$ so that $\partial\Omega_\alpha = \Gamma_\alpha \cup \Gamma_g$. Since $\mathbf{u}_\epsilon = \mathbf{0}$ on Γ_α, we may define a generalized velocity space as

$$\mathbf{V}_\alpha = \mathbf{H}^1_{\Gamma_\alpha}(\Omega_\alpha) = \{\,\mathbf{v} \in \mathbf{H}^1(\Omega_\alpha) \ \mid\ \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma_\alpha\,\}\,.$$

Let $\mathbf{V}^*_\alpha$ be the dual space of $\mathbf{V}_\alpha$. Note that $\mathbf{V}^*_\alpha$ is a subspace of $\mathbf{H}^{-1}(\Omega_\alpha)$, where the latter is the dual space of $\mathbf{H}^1_0(\Omega_\alpha)$. The duality pairing between $\mathbf{V}^*_\alpha$ and $\mathbf{V}_\alpha$ is denoted by $<\cdot,\cdot>_{-1}$.

Let $\mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g) = \{\,\mathbf{s} \in \mathbf{H}^{1/2}(\Gamma) \mid \mathbf{s} = \mathbf{0} \ \text{ on } \Gamma_\alpha\,\}$. Let $\mathbf{W}^*_\alpha$ denote its dual space and let $<\cdot,\cdot>_{-1/2,\Gamma_g}$ denote the duality pairing between $\mathbf{W}^*_\alpha$ and $\mathbf{W}_\alpha$.

Since Γ_g is smooth, the trace mapping $\gamma_g : \mathbf{H}^1(\Omega_\alpha) \to \mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g)$ is well–defined and $\mathbf{W}_\alpha = \gamma_g(\mathbf{H}^1_{\Gamma_\alpha}(\Omega_\alpha)) = \gamma_g(\mathbf{V}_\alpha)$ for each $\alpha \in \mathcal{U}_{ad}$. Now, let $\mathbf{g}$ be an element of $\mathbf{W}_\alpha = \mathbf{H}^{1/2}(\Gamma_g)$. It is well–known that $\mathbf{W}_\alpha$ is a Hilbert space with the norm

$$\|\mathbf{g}\|_{1/2,\Gamma_g} = \inf_{\mathbf{v}\in\mathbf{V}_\alpha,\ \gamma_{\Gamma_g}\mathbf{v}=\mathbf{g}} \|\mathbf{v}\|_{1,\Omega_\alpha} \qquad \forall\, \mathbf{g} \in \mathbf{W}_\alpha\,.$$

Let $\mathbf{s}^*$ belong to $\mathbf{W}_\alpha^*$. By the definition of the dual norm, we note that

$$\|\mathbf{s}^*\|_{-1/2,\Gamma_g} = \sup_{\mathbf{g}\in\mathbf{W}_\alpha,\, \mathbf{g}\neq\mathbf{0}} \frac{<\mathbf{s}^*,\mathbf{g}>_{-1/2,\Gamma_g}}{\|\mathbf{g}\|_{1/2,\Gamma_g}} \qquad \forall\,\mathbf{s}^*\in\mathbf{W}_\alpha^*\,.$$

It is shown in [7] that

$$(1.8)\qquad \|\mathbf{s}^*\|_{-1/2,\Gamma_g} = \sup_{\mathbf{v}\in\mathbf{V}_\alpha,\, \mathbf{v}\neq\mathbf{0}} \frac{<\mathbf{s}^*,\gamma_{\Gamma_g}\mathbf{v}>_{-1/2,\Gamma_g}}{\|\mathbf{v}\|_{1,\Omega_\alpha}} \qquad \forall\,\mathbf{s}^*\in\mathbf{W}_\alpha^*$$

provides an alternate and equivalent definition for the dual norm $\|\cdot\|_{-1/2,\Gamma_g}$. Whenever $\mathbf{s}\in\mathbf{W}_\alpha^*$ and $\mathbf{v}\in\mathbf{V}_\alpha$, we will simply write $<\mathbf{s},\mathbf{v}>_{-1/2,\Gamma_g}$ instead of $<\mathbf{s},\gamma_{\Gamma_g}\mathbf{v}>_{-1/2,\Gamma_g}$.

We define the space of generalized pressures to be

$$S_\alpha = L_0^2(\Omega_\alpha) = \left\{ p\in L^2(\Omega_\alpha) \;\Big|\; \int_{\Omega_\alpha} p\,d\Omega = 0 \right\}.$$

Thus, S_α consists of square integrable functions having zero mean over Ω_α.

2. Existence results of optimal solutions

We now show the existence of optimal solutions satisfying (1.7). We first recast this problem into a precise function space setting.

2.1. Weak formulation of the state equations

For the weak variational formulation, we will use the forms

$$a_\alpha(\mathbf{u},\mathbf{v}) = 2\int_{\Omega_\alpha} D(\mathbf{u}):D(\mathbf{v})\,d\Omega \quad \forall\mathbf{u},\,\mathbf{v}\in\mathbf{H}^1(\Omega_\alpha)\,,$$

$$d_\alpha(\mathbf{u},\mathbf{v}) = \int_{\Omega_\alpha} (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\,d\Omega \quad \forall\mathbf{u},\;\mathbf{v}\in\mathbf{H}^1(\Omega_\alpha)\,,$$

$$b_\alpha(\mathbf{v},q) = -\int_{\Omega_\alpha} q\nabla\cdot\mathbf{v}\,d\Omega \quad \forall\mathbf{v}\in\mathbf{H}^1(\Omega_\alpha),\, q\in L^2(\Omega_\alpha)\,,$$

and

$$c_\alpha(\mathbf{w},\mathbf{u},\mathbf{v}) = \int_{\Omega_\alpha} (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega \quad \forall\mathbf{u},\;\mathbf{v},\;\mathbf{w}\in\mathbf{H}^1(\Omega_\alpha)\,.$$

Obviously, $a_\alpha(\cdot,\cdot)$ and $d_\alpha(\cdot,\cdot)$ are continuous bilinear forms on $\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)$, and $b_\alpha(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega_\alpha)\times L^2(\Omega_\alpha)$;

also, $c_\alpha(\cdot,\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)\times\mathbf{H}^1(\Omega_\alpha)$ which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega_\alpha)$ into $\mathbf{L}^4(\Omega_\alpha)$ and Hölder's inequality (c.f. [5], [12]). Moreover, we have the coercivity property

$$a_\alpha(\mathbf{v},\mathbf{v}) \geq C\|\mathbf{v}\|_1^2 \quad \forall\,\mathbf{v}\in\mathbf{V}_\alpha \tag{2.1}$$

and the inf–sup condition (or LBB–condition)

$$\inf_{q\in S_\alpha}\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega_\alpha)}\frac{b_\alpha(\mathbf{v},q)}{\|\mathbf{v}\|_1\|q\|_0}\geq C\,. \tag{2.2}$$

One can show that (1.1)–(1.3) have the following weak formulation: for each $\alpha\in\mathcal{U}_{ad}$, find $\mathbf{u}_\epsilon\in\mathbf{V}_\alpha$, $p\in S_\alpha$, and $\mathbf{t}_\epsilon\in\mathbf{W}_\alpha^*$ satisfying

$$\begin{aligned}\nu a_\alpha(\mathbf{u}_\epsilon,\mathbf{v})+b_\alpha(\mathbf{v},p_\epsilon)+c_\alpha(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v})&\\ -<\mathbf{t}_\epsilon,\gamma_g\mathbf{v}>_{-1/2,\Gamma_g}=<\mathbf{f},\mathbf{v}>_{-1}&\quad\forall\mathbf{v}\in\mathbf{V}_\alpha\,,\end{aligned} \tag{2.3}$$

$$b_\alpha(\mathbf{u}_\epsilon,q)=\epsilon(p_\epsilon,q)_0\quad\forall q\in S_\alpha \tag{2.4}$$

and

$$<\mathbf{s}^*,\mathbf{u}_\epsilon>_{-1/2,\Gamma_g}=<\mathbf{s}^*,\mathbf{g}>_{-1/2,\Gamma_g}\quad\forall\mathbf{s}^*\in\mathbf{W}_\alpha^*\,. \tag{2.5}$$

In showing that (2.3) is a weak formulation of (1.1), it is convenient to replace the viscous term in the latter with $2\nu\nabla\cdot(D(\mathbf{u}_\epsilon))$. Also, since Γ_g is smooth, the trace mapping $\gamma_g:\mathbf{V}_\alpha\to\mathbf{W}_\alpha$ is well–defined and $\mathbf{W}_\alpha=\gamma_g(\mathbf{V}_\alpha)$ for each $\alpha\in\mathcal{U}_{ad}$; hence, (2.5) is well–justified. Note that the inhomogeneous boundary condition on the velocity is enforced weakly as in [6].

By eliminating the pressure term p_ϵ, (2.3)–(2.5) can be simply written by

$$\begin{aligned}\nu a_\alpha(\mathbf{u}_\epsilon,\mathbf{v})+c_\alpha(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v})+\frac{1}{\epsilon}d_\alpha(\mathbf{u}_\epsilon,\mathbf{v})&\\ -<\mathbf{t}_\epsilon,\gamma_g\mathbf{v}>_{-1/2,\Gamma_g}=<\mathbf{f},\mathbf{v}>_{-1}&\quad\forall\mathbf{v}\in\mathbf{V}_\alpha\,,\\ <\mathbf{s}^*,\mathbf{u}_\epsilon>_{-1/2,\Gamma_g}=<\mathbf{s}^*,\mathbf{g}>_{-1/2,\Gamma_g}&\quad\forall\mathbf{s}^*\in\mathbf{W}_\alpha^*\,.\end{aligned} \tag{2.6}$$

It can be easily shown that (2.6) is a weak formulation of (1.5) and that, in the sense of distributions, $\mathbf{t}_\epsilon$ is given by

$$\mathbf{t}_\epsilon=-p_\epsilon\mathbf{n}+2\nu D(\mathbf{u}_\epsilon)\cdot\mathbf{n}=\frac{1}{\epsilon}(\nabla\cdot\mathbf{u}_\epsilon)\mathbf{n}+2\nu D(\mathbf{u}_\epsilon)\cdot\mathbf{n}\quad\text{on }\Gamma_g\,. \tag{2.7}$$

This corresponds to the stress force along the inhomogeneous boundary Γ_g due to the penalized deformation.

In [8], we have shown the existence and convergence results by employing the nonlinear functional setting. In regular branch the penalized solutions have the similar pattern with the unpenalized solutions([8]). For the completeness, we state main results of [8] modified to fit into our problem.

We first invoke the nonlinear functional formulation as in [2] and [5] and then we recast the unpenalized primal Navier–Stokes systems into the corresponding functional setting. We take $\mathcal{X} = \mathbf{V}_\alpha \times S_\alpha \times \mathbf{W}_\alpha^*$, $\mathcal{Y} = \mathbf{V}_\alpha^* \times \mathbf{W}_\alpha$ and $\mathcal{Z} = \mathbf{L}^{3/2}(\Omega_\alpha) \times \{\mathbf{0}\}$. For the parameter, we take $\lambda = \dfrac{1}{\nu} = Re \in \Lambda \subset \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the nonnegative real numbers and Λ a compact interval in $\mathbf{R}^+ - \{0\}$. We define the solution operator $\mathfrak{Q} : \mathcal{Y} \to \mathcal{X}$ for the Stokes problem with inhomogeneous boundary conditions by $\mathfrak{Q}(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}) = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{\mathbf{t}})$ if and only if

$$
(2.8)\qquad \begin{aligned}
a_\alpha(\widehat{\mathbf{u}}, \mathbf{v}) + b_\alpha(\mathbf{v}, \widehat{p}) - < \widehat{\mathbf{t}}, \gamma_g \mathbf{v} >_{-1/2,\Gamma_g} = < \widehat{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V}_\alpha\,, \\
b_\alpha(\widehat{\mathbf{u}}, q) = 0 \quad \forall q \in S_\alpha\,, \\
< \mathbf{s}^*, \widehat{\mathbf{u}} >_{-1/2,\Gamma_g} = < \mathbf{s}^*, \widehat{\mathbf{g}} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^* \in \mathbf{W}_\alpha^*\,.
\end{aligned}
$$

The nonlinearity of the Navier–Stokes systems is taken into account by the mapping $\mathcal{G} : \Lambda \times \mathcal{X} \to \mathcal{Y}$ ($(\lambda, (\mathbf{w}, q, \boldsymbol{\zeta})) \mapsto (\boldsymbol{\eta}, \boldsymbol{\phi})$) defined by

$$
(2.9)\qquad \begin{aligned}
< \boldsymbol{\eta}, \mathbf{v} >_{-1} = \lambda c_\alpha(\mathbf{w}, \mathbf{w}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{V}_\alpha\,, \\
< \mathbf{s}^*, \boldsymbol{\phi} >_{-1/2,\Gamma_g} = - < \mathbf{s}^*, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \mathbf{s}^* \in \mathbf{W}_\alpha^*\,,
\end{aligned}
$$

where $(\mathbf{f}, \mathbf{g})$ is given in $\mathbf{V}_\alpha^* \times \mathbf{W}_\alpha$.

Since the weak formulation of the Navier–Stokes equations can be written by

$$
(2.10)\qquad \begin{aligned}
a_\alpha(\mathbf{u}, \mathbf{v}) + b_\alpha(\mathbf{v}, \lambda p) - < \lambda \mathbf{t}, \mathbf{v} >_{-1/2,\Gamma_g} \\
= - \left[\lambda c_\alpha(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} \right] \quad \forall \mathbf{v} \in \mathbf{V}_\alpha\,, \\
b_\alpha(\mathbf{u}, \lambda q) = 0 \quad \forall q \in S_\alpha\,, \\
< \mathbf{s}^*, \mathbf{u} >_{-1/2,\Gamma_g} = - \left[- < \mathbf{s}^*, \mathbf{g} >_{-1/2,\Gamma_g} \right] \quad \forall \mathbf{s}^* \in \mathbf{W}_\alpha^*\,,
\end{aligned}
$$

and the mapping $\mathcal{G}$ corresponds to the weak formulation of

$$
\begin{cases}
\boldsymbol{\eta} = \lambda(\mathbf{w} \cdot \nabla)\mathbf{w} - \lambda \mathbf{f}\,, \\
\boldsymbol{\phi} = -\mathbf{g}\,.
\end{cases}
$$

Substituting $\mathbf{w} = \mathbf{u}$, we obtain from (2.10) that $q = \lambda p$, $\boldsymbol{\zeta} = \lambda \mathbf{t}$ and $(\mathbf{u}, \lambda p, \lambda \mathbf{t}) = -\mathfrak{Q}\mathcal{G}(\mathbf{u}, \lambda p, \lambda \mathbf{t})$. Hence, we have

$$(2.11) \qquad (\mathbf{u}, \lambda p, \lambda \mathbf{t}) + \mathfrak{Q}\mathcal{G}(\lambda, (\mathbf{u}, \lambda p, \lambda \mathbf{t})) = 0,$$

which is equivalent to the weak variational form (2.10) of the primal stationary incompressible Navier–Stokes system.

Existence and convergence results when ϵ tends to $0+$ for solutions of the system (2.3)–(2.5) are contained in the following theorem; for a proof, one may consult [8].

THEOREM 2.1. *Let* $\alpha \in \mathcal{U}_{ad}$ *and let the data satisfy* $\mathbf{f} \in \mathbf{V}_\alpha^*$, $\mathbf{g} \in \mathbf{W}_\alpha$ *and the compatibility condition* (1.4). *Let* $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \lambda \mathbf{t}(\lambda))) \mid \lambda = \frac{1}{\nu} \in \Lambda\}$ *be a branch of regular solutions of* (2.10). *Then, there exists a neighborhood* $\mathcal{O}$ *of the origin in* $\mathbf{V}_\alpha \times S_\alpha \times \mathbf{W}_\alpha^*$ *and for* $\epsilon \le \epsilon_0$ *small enough, a unique* C^2 *branch* $\{(\lambda, (\mathbf{u}_\epsilon(\lambda), \lambda p_\epsilon(\lambda), \lambda \mathbf{t}_\epsilon(\lambda))) \mid \lambda \in \Lambda\}$ *of the penalized system* (2.3)–(2.5) *such that* $\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda) \in \mathcal{O}$ *for all* $\lambda \in \Lambda$. *Moreover, there exists a positive constant* C, *independent of* ϵ *and* λ, *such that*

$$(2.12) \qquad \begin{aligned} \|\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda)\|_{1,\Omega_\alpha} &+ \|p_\epsilon(\lambda) - p(\lambda)\|_{0,\Omega_\alpha} \\ &+ \|\mathbf{t}_\epsilon(\lambda) - \mathbf{t}(\lambda)\|_{-1/2,\Gamma_g} \le C\epsilon \qquad \forall \lambda \in \Lambda . \end{aligned}$$

2.2. The extremal problem

In the notation introduced in §1.2 and §2.1, the cost functional $\mathcal{J}$ defined in (1.6) can be expressed in the form

$$(2.13) \quad \mathcal{J}(\alpha) = \mathcal{J}\Big(\alpha, \mathbf{u}_\epsilon(\alpha)\Big) = 2\nu \int_{\Omega_\alpha} D(\mathbf{u}_\epsilon) : D(\mathbf{u}_\epsilon)\, d\Omega = \nu a_\alpha(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon) .$$

We introduce the admissibility set of controls and velocities

$$\mathcal{V}_{ad} = \Big\{ (\alpha, \mathbf{u}_\epsilon(\alpha)) \in \mathcal{U}_{ad} \times \mathbf{V}_\alpha \;\Big|\; \mathcal{J}(\alpha, \mathbf{u}_\epsilon(\alpha)) < \infty, \text{ and there exists } \mathbf{t}_\epsilon(\alpha) \in \mathbf{W}_\alpha^* \text{ such that } (\mathbf{u}_\epsilon(\alpha), \mathbf{t}_\epsilon(\alpha)) \text{ is a solution of (2.6)} \Big\} .$$

Then, the extremal problem (1.7) can be restated in the following precise form:

$$(2.14) \qquad \min_{(\alpha, \mathbf{u}_\epsilon(\alpha)) \in \mathcal{V}_{ad}} \mathcal{J}(\alpha, \mathbf{u}_\epsilon(\alpha)) .$$

The existence of optimal solutions for the problem (2.14) can be shown in the similar manner as in [7].

THEOREM 2.2. *There exists at least one optimal solution* $(\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \in \mathcal{V}_{ad}$ *for the problem* (2.14).

Proof. The nonemptiness of $\mathcal{V}_{ad}$ follows from Theorem 2.1 for the existence of regular branches of the penalized solutions. All the other concerns are similar to [7] except that $\mathbf{t}_\epsilon(\alpha^*)$ may be evaluated directly from the weak lower limit of minimizers in $\mathcal{V}_{ad}$. □

3. Shape sensitivity by using Lagrange multipliers

In this section, we are mainly concerned with the shape sensitivity analysis for the problem (2.13). Sensitivity analysis in a shape control problem is the study of the effects on the shape functional and potential constraints due to variations of the shape parameters. Given any shape, a sensitivity analysis is used to determine if it is a stationary point in design space for the relevant shape control problem. Otherwise, one may try to improve the given design of shape locally. Improvement of performance can be achieved iteratively by following the gradient of the shape functional. This is based on the existence of Gateaux derivative of the shape functional and constraints in the direction of perturbation of shape parameters.

In this section, we wish to derive the information for the shape gradient. For this purpose, we employ the Lagrange multipliers technique to get the adjoint systems, and with the help of the material derivative method, we obtain the shape gradient for the functional $\mathcal{J}$. For the concrete structure for the material derivative method, one may refer to [11] and [13]. Strict mathematical justification for the existence of Lagrange multipliers will be verified in ensuing papers.

3.1. Domain perturbations

To begin with, we parameterize the shape perturbations. We consider the following homotopy to describe the domain perturbations to fit into our purpose.

$$\mathcal{F}_t(\mathbf{p}) = \mathbf{p} + t\,\mathbf{V}(0,\mathbf{p}) = (\mathcal{I} + t\,\mathbf{V})(0,\mathbf{p}) \quad \text{for} \quad 0 \le t < \sigma, \tag{3.1}$$

where $\sigma > 0$ is small enough to ensure the diffeomorphism of the homotopy $\mathcal{F}_t$. From the second expression, we can regard $\mathcal{F}_t(\mathbf{p})$ as a first order perturbation of the identity operator over the reference domain. The choice of $\mathbf{V}$ is crucial in the shape sensitivity analysis. In our problem, we want to keep the variation of $\Gamma_b(\alpha)$ within the rectangular region Ω_0 depicted by the shaded region in Figure 2, i.e., we want that $\Gamma_b(\alpha) \subset \overline{\Omega}_0$ for every $\alpha \in \mathcal{U}_{ad}$. Note that then $\Omega_\alpha \subset \widehat{\Omega}$, where the latter is the rectangular

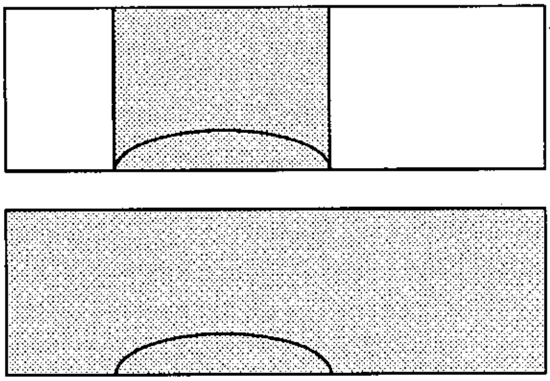

FIG. 2. *The domains* Ω_0 *(top) and* $\widehat{\Omega}$ *(bottom).*

domain also depicted in Figure 2. An appropriate choice for the velocity is then given by $\mathbf{V} = (0, V_2)^T$.

Utilizing the mapping technique, V_2 can be characterized as follows. For a fixed $\alpha \in \mathcal{U}_{ad}$, we associate a bijection

$$F_\alpha : \widehat{\Omega} \longrightarrow \Omega_\alpha \ ((\widehat{x}_1, \widehat{x}_2) \mapsto (p_1, p_2))$$

via

$$p_1 = \widehat{x}_1 \quad \text{and} \quad p_2 = \begin{cases} L + \dfrac{(\widehat{x}_2 - L)(L - \alpha(\widehat{x}_1))}{L} & \text{if } M_1 \le \widehat{x}_1 \le M_2 \\ \widehat{x}_2 & \text{otherwise}. \end{cases}$$

Let $\vartheta \in C^{0,1}([M_1, M_2])$ such that $\vartheta(M_1) = \vartheta(M_2) = 0$ and there exists $\sigma > 0$ such that the graph of $\alpha + t\vartheta$ lies in $\overline{\widehat{\Omega}}$ for $0 \le t < \sigma$. We may extend ϑ to $[0, M]$ by defining $\vartheta = 0$ over $[0, M_1] \cup [M_2, M]$. If we consider a bijection

$$F_{\alpha+t\vartheta} : \Omega_\alpha \longrightarrow \Omega(\alpha + t\vartheta) \ \ ((\widehat{x}_1, \widehat{x}_2) \mapsto (x_1, x_2)),$$

the composite $F_{\alpha+t\vartheta} \circ F_\alpha^{-1} : \Omega_\alpha \longrightarrow \Omega(\alpha+t\vartheta)$ $((p_1, p_2) \mapsto (x_1, x_2))$ is given by

$$x_1 = p_1 \text{ and } x_2 = \begin{cases} p_2 + t\dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)} & \text{if } M_1 \le p_1 \le M_2 \\ p_2 & \text{otherwise}. \end{cases} \tag{3.2}$$

Since $0 \le \alpha(p_1) < L$ for all $p_1 \in [M_1, M_2]$, the mapping (3.2) is well–defined and $(x_1, x_2) = (p_1, p_2) + t\,(0, V_2(p_1, p_2))$, where

$$V_2(p_1, p_2) = \begin{cases} \dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)} & \text{if } M_1 \le p_1 \le M_2 \\ 0 & \text{otherwise}. \end{cases} \tag{3.3}$$

Hence, for the perturbation of the domain, it is reasonable to consider the transformation

$$\mathcal{F}_t(p_1,p_2) = (p_1,p_2) + t\,\mathbf{V}(p_1,p_2) = F_{\alpha+t\vartheta} \circ F_\alpha^{-1}(p_1,p_2)\,,$$

where $\mathbf{V} = (0, V_2)^T$ is an autonomous vector field. Clearly, $\mathcal{F}_t$ is a one–to–one transformation from Ω_α onto $\Omega(\alpha + t\vartheta)$ whose inverse is given by $\mathcal{F}_t^{-1}(x_1,x_2) = (p_1,p_2)$, where

$$p_1 = x_1 \text{ and } p_2 = \begin{cases} x_2 + t\dfrac{(L-x_2)\vartheta(x_1)}{(\alpha(x_1) - L + t\vartheta(x_1))} & \text{if } M_1 \le x_1 \le M_2 \\ x_2 & \text{otherwise}\,. \end{cases}$$

Note that $\mathbf{V}(p_1,\alpha(p_1)) = (0,\vartheta(p_1))^T$ for all $p_1 \in [M_1,M_2]$. Thus, $\mathbf{V} = (0,\vartheta)^T$ along $\Gamma_b(\alpha)$ and $\mathbf{V} = \mathbf{0}$ along $\partial\Omega_\alpha - \Gamma_b(\alpha)$.

3.2. Lagrange multipliers technique and adjoint equations

We now want to define *adjoint variables* which will enable one to compute the shape gradient without having to directly consider the equations dealing with the shape sensitivities. Formally, one may derive equations for the adjoint variables by introducing the Lagrangian $\mathcal{L} : \mathcal{U}_{ad}\times\mathbf{V}_\alpha\times\mathbf{V}_\alpha\times\mathbf{W}_\alpha^* \to \mathbb{R}$ defined by

$$\begin{aligned} &\mathcal{L}(\alpha, \mathbf{u}_\epsilon, \boldsymbol{\mu}_\epsilon, \boldsymbol{\tau}_\epsilon) \\ =&\mathcal{J}(\alpha,\mathbf{u}_\epsilon) - \{\,\nu a_\alpha(\mathbf{u}_\epsilon,\boldsymbol{\mu}_\epsilon) + c_\alpha(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\boldsymbol{\mu}_\epsilon) - \frac{1}{\epsilon}d_\alpha(\mathbf{u}_\epsilon,\boldsymbol{\mu}_\epsilon) \\ &- <\mathbf{t}_\epsilon,\boldsymbol{\mu}_\epsilon>_{-1/2,\Gamma_g} - <\mathbf{f},\boldsymbol{\mu}_\epsilon>_{-1} - <\boldsymbol{\tau}_\epsilon,\mathbf{u}_\epsilon - \mathbf{g}>_{-1/2,\Gamma_g}\}\,, \end{aligned}$$

where $\mathbf{t}_\epsilon \in \mathbf{W}_\alpha^*$ is given by (2.7) and $(\boldsymbol{\mu}_\epsilon,\boldsymbol{\tau}_\epsilon) \in \mathbf{V}_\alpha \times S_\alpha \times \mathbf{W}_\alpha^*$ are the adjoint variables. Formally, the adjoint equations are defined from the Euler–Lagrange equations for the Lagrangian.

Clearly, variations in the Lagrange multipliers $\boldsymbol{\mu}_\epsilon$ and $\boldsymbol{\tau}_\epsilon$ recover the constraints (2.6). From the variation in the state variable $\mathbf{u}_\epsilon$, one can derive the adjoint state equations by assigning a suitable boundary condition $\boldsymbol{\mu}_\epsilon = \mathbf{0}$ on the whole boundary $\partial\Omega_\alpha$;

(3.4)

$$\begin{aligned} &\nu\int_{\Omega_\alpha} \nabla\boldsymbol{\mu}_\epsilon : \nabla\mathbf{w}\,d\Omega + \int_{\Omega_\alpha} (\mathbf{w}\cdot\nabla)\,\mathbf{u}_\epsilon\cdot\boldsymbol{\mu}_\epsilon\,d\Omega + \int_{\Omega_\alpha} (\mathbf{u}_\epsilon\cdot\nabla)\,\mathbf{w}\cdot\boldsymbol{\mu}_\epsilon\,d\Omega \\ &+ \frac{1}{\epsilon}\int_{\Omega_\alpha} (\nabla\cdot\boldsymbol{\mu}_\epsilon)(\nabla\cdot\mathbf{w})\,d\Omega + <\boldsymbol{\tau}_\epsilon,\mathbf{w}>_{-1/2,\Gamma_g} = 2\nu\int_{\Omega_\alpha} \nabla\mathbf{u}_\epsilon : \mathbf{w}\,d\Omega \end{aligned}$$

for every $\mathbf{w} \in \mathbf{V}_\alpha$ and

$$< \mathbf{s}^*, \boldsymbol{\mu}_\epsilon >_{-1/2,\partial\Omega_\alpha} = 0 \qquad \forall \mathbf{s}^* \in \mathbf{W}_\alpha^* . \tag{3.5}$$

To derive the equations for $\boldsymbol{\mu}_\epsilon$ and $\boldsymbol{\tau}_\epsilon$, we first note that

$$\begin{aligned}
&\int_{\Omega_\alpha} (\mathbf{w}\cdot\nabla)\mathbf{u}_\epsilon \cdot \boldsymbol{\mu}_\epsilon \, d\Omega + \int_{\Omega_\alpha} (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{w} \cdot \boldsymbol{\mu}_\epsilon \, d\Omega \\
&\quad = \sum_{i,j=1,2}^{2} \int_{\Omega_\alpha} \left(w_j \frac{\partial u_{\epsilon i}}{\partial x_j} \mu_{\epsilon i} + u_{\epsilon j} \frac{\partial w_i}{\partial x_j} \mu_{\epsilon i} \right) d\Omega \\
&\quad = \sum_{i,j=1,2}^{2} \int_{\Omega_\alpha} \left(\mu_{\epsilon j} \frac{\partial u_{\epsilon j}}{\partial x_i} w_i - u_{\epsilon j} \frac{\partial \mu_{\epsilon i}}{\partial x_j} w_i - \frac{\partial u_{\epsilon j}}{\partial x_j} \mu_{\epsilon i} w_i \right) d\Omega \\
&\quad = \int_{\Omega_\alpha} \left(\boldsymbol{\mu}_\epsilon \cdot (\nabla \mathbf{u}_\epsilon)^T - \mathbf{u}_\epsilon \cdot (\nabla \boldsymbol{\mu}_\epsilon) - (\nabla \cdot \mathbf{u}_\epsilon)\, \boldsymbol{\mu}_\epsilon \right) \cdot \mathbf{w} \, d\Omega ,
\end{aligned}$$

using integration by parts and $\boldsymbol{\mu}_\epsilon = \mathbf{0}$ on $\partial\Omega_\alpha$. Applying Green's formula, the (3.4)–(3.5) yields

$$\begin{aligned}
&-\nu\Delta\boldsymbol{\mu}_\epsilon + \boldsymbol{\mu}_\epsilon \cdot (\nabla \mathbf{u}_\epsilon)^T - \mathbf{u}_\epsilon \cdot (\nabla \boldsymbol{\mu}_\epsilon) - (\nabla \cdot \mathbf{u}_\epsilon)\, \boldsymbol{\mu}_\epsilon \\
&\qquad -\frac{1}{\epsilon}\nabla(\nabla \cdot \boldsymbol{\mu}_\epsilon) = -2\nu\Delta\mathbf{u}_\epsilon \quad \text{in } \Omega_\alpha ,
\end{aligned} \tag{3.6}$$

$$\boldsymbol{\mu}_\epsilon = \mathbf{0} \qquad \text{on } \partial\Omega_\alpha \tag{3.7}$$

and

$$\boldsymbol{\tau}_\epsilon = 2\nu \frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} - \nu \frac{\partial \boldsymbol{\mu}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}(\nabla \cdot \boldsymbol{\mu}_\epsilon)\mathbf{n} \quad \text{on } \partial\Omega_\alpha . \tag{3.8}$$

The equations (3.6)–(3.8) can be interpreted as a penalized version of linearized adjoint incompressible Navier–Stokes equations given by

$$\begin{aligned}
-\nu\Delta\mathbf{q} + \mathbf{q} \cdot (\nabla \mathbf{u})^T - \mathbf{u} \cdot (\nabla \mathbf{q}) + \nabla\xi &= -2\nu\Delta\mathbf{u} \quad \text{in } \Omega_\alpha , \\
\nabla \cdot \mathbf{q} &= 0 \qquad \text{in } \Omega_\alpha , \\
\mathbf{q} &= \mathbf{0} \quad \text{on } \partial\Omega_\alpha ,
\end{aligned} \tag{3.9}$$

where $\mathbf{u}$ is a solution of the incompressible Navier–Stokes equations (2.10) and ξ corresponds to the adjoint to the pressure p. The penalty term for

(3.9) is introduced by $\xi_\epsilon = -\frac{1}{\epsilon}(\nabla \cdot \boldsymbol{\mu}_\epsilon)$. In this case, $\boldsymbol{\tau}_\epsilon$ of (3.8) corresponds to the penalized adjoint stress vector of

$$\boldsymbol{\tau} = 2\nu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \nu\frac{\partial \mathbf{q}}{\partial \mathbf{n}} + \xi\mathbf{n} \qquad \text{on } \partial\Omega_\alpha . \tag{3.10}$$

Hence, (3.4)–(3.5) can be interpreted as an adjoint of the weak penalized formulation of the linearized equations (3.9) together with

$$< \boldsymbol{\tau}_\epsilon, \mathbf{s} >_{-1/2,\Gamma_g} = < 2\nu\frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} - \nu\frac{\partial \boldsymbol{\mu}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}(\nabla \cdot \mathbf{u}_\epsilon)\mathbf{n}, \mathbf{s} >_{-1/2,\Gamma_g} \tag{3.11}$$

for every $\mathbf{s} \in \mathbf{W}_\alpha$.

3.3. Derivation of shape gradient

For the specific problem considered in this section, the following two canonical types of shape functionals are useful:

$$\mathfrak{J}_1(\mathcal{D}_t) = \int_{\mathcal{D}_t} y_t \, d\mathcal{D}_t \quad \text{and} \quad \mathfrak{J}_2(\mathcal{D}_t) = \int_{\partial\mathcal{D}_t} y_t \, d\partial\mathcal{D}_t ,$$

where $y_t(\mathbf{x}) = y(t, \mathbf{x})$ is a function defined on $\mathcal{D}_t \subset \widehat{\mathcal{D}}$ or $\partial\mathcal{D}_t$, respectively. Let $\widehat{y}$ be a uniform extension of y_t in $\widehat{\mathcal{D}}$. Then, under some reasonable assumptions on the regularity for the feasible domains and the class of functions, one can obtain

$$d\mathfrak{J}_1(\mathcal{D}; \mathbf{V}) = \int_{\mathcal{D}} \frac{\partial \widehat{y}}{\partial t} \, d\mathcal{D} + \int_{\partial\mathcal{D}} (\mathbf{V}(0, \cdot) \cdot \mathbf{n}) y_0 d\partial\mathcal{D} \tag{3.12}$$

and

$$d\mathfrak{J}_2(\mathcal{D}; \mathbf{V}) = \int_{\partial\mathcal{D}} \Big[\frac{\partial \widehat{y}}{\partial t} + (\mathbf{V}(0, \cdot) \cdot \mathbf{n}) \Big(\frac{\partial y_0}{\partial \mathbf{n}} + \kappa y_0 \Big) \Big] \, d\partial\mathcal{D} . \tag{3.13}$$

Here, κ denotes the curvature of the boundary curve $\partial\mathcal{D}$ when the spatial dimension of the domain is 2 and the mean curvature of the boundary surface $\partial\mathcal{D}$ when the spatial dimension is 3. These formulations were introduced by many authors. For derivations, one may refer to [11] and [13].

In these two standard examples of functionals, $d\mathfrak{J}_i(\mathcal{D}; \mathbf{V})$ consists of two main components: a linear term $\mathbf{V} \cdot \mathbf{n}$ on the boundary and a *shape derivative* term $\widehat{y}' = \widehat{y}'(\mathcal{D}; \mathbf{V}) = \frac{\partial \widehat{y}}{\partial t}$. In order to obtain the shape gradients, it should be justified that $\mathbf{V} \mapsto \dfrac{\partial \widehat{y}}{\partial t}(\mathcal{D};\mathbf{V})$ is linear and continuous over appropriate admissible vector fields. This implies that $\widehat{y}'(\mathcal{D};\mathbf{V})$ should be represented as a linear function of $\mathbf{V}$.

We now consider the variation in the shape parameter $\alpha \in \mathcal{U}_{ad}$. We notice that

$$\min_{\alpha\in\mathcal{U}_{ad}} \mathcal{J}(\alpha) = \min_{\alpha\in\mathcal{U}_{ad}} \mathcal{L}(\alpha, \mathbf{u}_\epsilon, \boldsymbol{\mu}_\epsilon, \boldsymbol{\tau}_\epsilon),$$

whenever $(\boldsymbol{\mu}_\epsilon, \boldsymbol{\tau}_\epsilon)$ is a solution of (3.4)–(3.5). Thus, computations of the design sensitivity may involve the sensitivity of the state variables and adjoint state variables. Recall that $\mathbf{V} = (0, \vartheta)$ along $\Gamma_b(\alpha)$ for any $\vartheta \in \mathcal{C}^{0,1}([M_1, M_2])$ such that $\vartheta(M_1) = \vartheta(M_2) = 0$. Since the perturbation of a domain is determined by the variation of the boundary part Γ_α, for the computation of $\inf_{\alpha\in\mathcal{U}_{ad}} \mathcal{J}(\alpha)$, we try to find a semi–derivative

$$d\mathcal{J}(\alpha;\vartheta) \equiv \frac{d}{dt}\,\mathcal{J}(\alpha + t\,\vartheta)\Big|_{t=0+} = \lim_{t\to 0+} \frac{\mathcal{J}(\alpha_t) - \mathcal{J}(\alpha)}{t},$$

where $\alpha_t = \alpha + t\,\vartheta$ for $\vartheta \in \mathcal{C}^{0,1}([M_1, M_2])$. From this, we wish to derive the information for the gradient of the design functional.

Throughput, we set $\Omega_t{=}\Omega_{\alpha_t}$ and $\mathbf{V}(0,\cdot) = \mathbf{V}(0)$ for the sake of brevity. Let $\mathbf{u}_\epsilon(\alpha_t) \in \mathbf{H}^1(\Omega_t)$ be a solution of the penalized incompressible Navier–Stokes equations over Ω_t, which is represented by the following integral formulations:

$$\begin{aligned}
\nu \int_{\Omega_t} \nabla \mathbf{u}_\epsilon(\alpha_t) : \nabla \mathbf{w}\, d\Omega_t &+ \int_{\Omega_t} (\mathbf{u}_\epsilon(\alpha_t)\cdot\nabla)\mathbf{u}_\epsilon(\alpha_t)\cdot \mathbf{w}\, d\Omega_t \\
&+ \frac{1}{\epsilon}\int_{\Omega_t} (\nabla\cdot\mathbf{u}_\epsilon(\alpha_t))(\nabla\cdot\mathbf{w})\, d\Omega_t - \nu \int_{\Gamma_t} \frac{\partial \mathbf{u}_\epsilon(\alpha_t)}{\partial \mathbf{n}_t}\cdot \mathbf{w}\, d\Gamma_t \\
&- \frac{1}{\epsilon}\int_{\Gamma_t} (\nabla\cdot\mathbf{u}_\epsilon(\alpha_t))(\mathbf{w}\cdot\mathbf{n}_t)\, d\Gamma_t = \int_{\Omega_t} \mathbf{f}\cdot\mathbf{w}\, d\Omega_t,
\end{aligned} \tag{3.14}$$

for $\mathbf{w} \in \mathbf{H}^1(\Omega_t)$ and

$$\mathbf{u}_\epsilon(\alpha_t) = \mathbf{g} \qquad \text{on } \Gamma_t. \tag{3.15}$$

Here, $\Gamma_t = \partial\Omega_t$, and $\mathbf{n}_t$ denotes the outward unit normal vector along Γ_t. The function space $\mathbf{H}^1(\Omega_t)$ is dependent on time t. To remove of this dependence, we apply the uniform extension property(c.f. [1] or [4]).

Let $\widehat{\mathbf{u}}_\epsilon(t,\mathbf{x}) = P_{\widehat{\Omega}}(\mathbf{u}_\epsilon(\alpha_t)\circ\mathcal{F}_t)\circ\mathcal{F}_t^{-1}(\mathbf{x})$. Then, $\widehat{\mathbf{u}}_\epsilon(t,\cdot)$ is a uniform extension of $\mathbf{u}_\epsilon(\alpha_t)$ to $\widehat{\Omega}$ such that $\mathbf{u}_\epsilon(\alpha_t) = \widehat{\mathbf{u}}\big|_{\{t\}\times\Omega_t}$. From (3.12), we have that

$$
\begin{aligned}
d\mathcal{J}(\alpha;\vartheta) &= \left\{ \frac{d}{dt} 2\nu \int_{\Omega_t} D(\mathbf{u}_\epsilon(\alpha_t)) : D(\mathbf{u}_\epsilon(\alpha_t))\, d\Omega_t \right\}\Bigg|_{t=0^+} \\
&= 4\nu \int_{\Omega_\alpha} D(\mathbf{u}(\alpha)) : D(\widehat{\mathbf{u}}'_\epsilon)\, d\Omega + 2\nu \int_{\partial\Omega_\alpha} D(\mathbf{u}_\epsilon(\alpha)) : D(\mathbf{u}_\epsilon(\alpha))\, \mathbf{V}(0)\cdot \mathbf{n}\, d\Gamma\,, \\
&= 2\nu \int_{\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \widehat{\mathbf{u}}'_\epsilon\, d\Omega + \nu \int_{\partial\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{u}_\epsilon(\alpha) \mathbf{V}(0)\cdot \mathbf{n}\, d\Gamma\,,
\end{aligned}
$$

where $\widehat{\mathbf{u}}'_\epsilon$ denotes the shape derivative for the uniform extension $\widehat{\mathbf{u}}_\epsilon$ in $\widehat{\Omega}$ of $\mathbf{u}_\epsilon$ in Ω_α.

Note that $\mathbf{n} = \left(\frac{\alpha'}{\sqrt{1+\alpha'^2}}, -\frac{1}{\sqrt{1+\alpha'^2}} \right)$ over Γ_α and $d\Gamma = \sqrt{1+\alpha'^2}\, dx_1$, where $\alpha'(x_1) = \frac{d\alpha(x_1)}{dx_1}$.

Since $\mathbf{V} = (0, \vartheta)$ on $\Gamma_b(\alpha)$ and $\mathbf{V}(0) = \mathbf{0}$ on $\partial\Omega_\alpha - \Gamma_b(\alpha)$,

$$
\begin{aligned}
&\int_{\partial\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{u}_\epsilon(\alpha)\, \mathbf{V}(0)\cdot \mathbf{n}\, d\Gamma \\
&= -\int_{M_1}^{M_2} \nabla \mathbf{u}_\epsilon(x_1, \alpha(x_1)) : \nabla \mathbf{u}_\epsilon(x_1, \alpha(x_1))\, \vartheta(x_1)\, dx_1\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
d\mathcal{J}(\alpha;\vartheta) =& 2\nu \int_{\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \widehat{\mathbf{u}}_\epsilon{}'\, d\Omega \\
&- \int_{M_1}^{M_2} \nabla \mathbf{u}_\epsilon(x_1, \alpha(x_1)) : \nabla \mathbf{u}_\epsilon(x_1, \alpha(x_1)) \vartheta(x_1)\, dx_1\,.
\end{aligned}
\tag{3.16}
$$

Since $d\mathcal{J}(\alpha;\vartheta)$ contains a shape derivative term $\widehat{\mathbf{u}}'_\epsilon$, we may use the state equations and its adjoint equations to eliminate it.

Let us consider the state equations (3.14). One may take $\mathbf{w} \in \mathbf{H}^1(\widehat{\Omega}) \cap \mathbf{V}_\alpha$. Take the derivative of both sides of (3.14) with respect to time t. Since $\mathbf{f}$ and $\mathbf{w}$ are independent of t, by the similar computation to (3.12), we obtain the following equation at $t = 0^+$:

$$
\begin{aligned}
&\nu \int_{\Omega_\alpha} \nabla \widehat{\mathbf{u}}'_\epsilon : \nabla \mathbf{w}\, d\Omega + \int_{\Omega_\alpha} (\widehat{\mathbf{u}}'_\epsilon \cdot \nabla)\mathbf{u}_\epsilon(\alpha) \cdot \mathbf{w}\, d\Omega \\
&\quad + \int_{\Omega_\alpha} (\mathbf{u}_\epsilon(\alpha) \cdot \nabla)\, \widehat{\mathbf{u}}'_\epsilon \cdot \mathbf{w}\, d\Omega + \frac{1}{\epsilon} \int_{\Omega_\alpha} (\nabla \cdot \widehat{\mathbf{u}}'_\epsilon)(\nabla \cdot \mathbf{w})\, d\Omega \\
&\quad - \int_{M_1}^{M_2} \Big(\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{w} + \frac{1}{\epsilon} (\nabla \cdot \mathbf{u}_\epsilon(\alpha))(\nabla \cdot \mathbf{w}) \Big) \vartheta(x_1)\, dx_1 \\
&\qquad - \frac{d}{dt} \Big(\nu \int_{\Gamma_t} \frac{\partial \mathbf{u}_\epsilon(\alpha_t)}{\partial \mathbf{n}_t} \cdot \mathbf{w}\, d\Gamma \Big) \Big|_{t=0^+} \\
&\qquad - \frac{d}{dt} \Big(\frac{1}{\epsilon} \int_{\Gamma_t} (\nabla \cdot \mathbf{u}_\epsilon(\alpha_t))(\mathbf{w} \cdot \mathbf{n}_t)\, d\Gamma_t \Big) \Big|_{t=0^+} \\
&\qquad + \int_{\partial \Omega_\alpha} ((\mathbf{u}_\epsilon(\alpha) \cdot \nabla)\, \mathbf{u}_\epsilon(\alpha) \cdot \mathbf{w})\, (\mathbf{V}(0) \cdot \mathbf{n})\, d\Gamma \\
&\qquad\qquad = \int_{\partial \Omega_\alpha} (\mathbf{f} \cdot \mathbf{w})\ , \mathbf{V}(0) \cdot \mathbf{n}\, d\Gamma\,.
\end{aligned}
\tag{3.17}
$$

Since $\mathbf{w} = \mathbf{0}$ on Γ_α and $\mathbf{V}(0) = \mathbf{0}$ on Γ_g, we have

$$\int_{\partial \Omega_\alpha} (\mathbf{f} \cdot \mathbf{w}) \mathbf{V}(0) \cdot \mathbf{n}\, d\Gamma = \int_{\Gamma_g} (\mathbf{f} \cdot \mathbf{w})\, \mathbf{V}(0) \cdot \mathbf{n}\, d\Gamma = 0\,.$$

Similarly, $\displaystyle\int_{\partial \Omega_\alpha} ((\mathbf{u}_\epsilon(\alpha) \cdot \nabla)\mathbf{u}_\epsilon(\alpha) \cdot \mathbf{w})\, (\mathbf{V}(0) \cdot \mathbf{n})\, d\Gamma = 0.$

Next, we consider $\displaystyle\frac{d}{dt} \int_{\Gamma_t} \frac{\partial \mathbf{u}_\epsilon(\alpha_t)}{\partial \mathbf{n}_t} \cdot \mathbf{w}\, d\Gamma_t$ and $\displaystyle\frac{d}{dt} \int_{\Gamma_t} (\nabla \cdot \mathbf{u}_\epsilon(\alpha_t))(\mathbf{w} \cdot \mathbf{n}_t)\, d\Gamma_t$. For these computations, we need the surface measure of the transformation.

LEMMA 3.1. *Let Ω_t be a domain in $\mathbf{R}^n$ which is transported by a one–to–one transformation $\mathcal{F}_t$ and let Γ_t be the boundary of Ω_t. If h is an integrable function defined on Γ_t, we have the following formula for the transformation of boundary integrals:*

$$\int_{\Gamma_t} h\, d\Gamma_t = \int_{\Gamma} (h \circ \mathcal{F}_t)\ \det(D\mathcal{F}_t) |(D\mathcal{F}_t^{-1})^T \mathbf{n}|_{\mathbf{R}^n}\, d\Gamma\,. \tag{3.18}$$

For the proof, one may refer to [11] or [13].

Here, $\varpi(t) = \det(D\mathcal{F}_t)|(D\mathcal{F}_t^{-1})^T \mathbf{n}|_{\mathbf{R}^n}$ is the cofactor of the Jacobian matrix $D\mathcal{F}_t$, and $\varpi(t)\, d\Gamma$ denotes the surface measure due to the transformation $\mathcal{F}_t$. It is easy to check that $\varpi(0) = 1$. For our purpose, we need the following facts.

LEMMA 3.2. *It follows that*

$$\frac{d}{dt}\det(D\mathcal{F}_t)\Big|_{t=0^+} = \nabla\cdot\mathbf{V}(0)\,. \tag{3.19}$$

Proof. Let

$$\mathcal{F}_t : \mathbf{R}^n \ni (p_1,\cdots,p_n) \longmapsto (x_1,\cdots,x_n)\in\mathbf{R}^n, \quad \text{where}$$
$$x_i = x_i(t,p_1,\cdots,p_n)\,,\quad i=1,\cdots,n\,.$$

We can write $\det(D\mathcal{F}_t) = \sum_{\sigma\in S_n}(sgn\sigma)\prod_{i=1}^{n}\frac{\partial x_i}{\partial p_{\sigma(i)}}$, where S_n denotes the permutations over $\{1,\cdots,n\}$ and $sgn\sigma$ the sign of a permutation σ.

$$\begin{aligned}\frac{d}{dt}\det(D\mathcal{F}_t)\Big|_{t=0^+} &= \sum_{j=1}^{n}\sum_{\sigma\in S_n}(sgn\sigma)\Big(\frac{\partial}{\partial p_{\sigma(j)}}\frac{\partial x_j}{\partial t}\prod_{i=1(,\neq j)}^{n}\frac{\partial x_i}{\partial p_{\sigma(i)}}\Big)\Big|_{t=0^+}\\ &= \sum_{j=1}^{n}\sum_{\sigma\in S_n}(sgn\sigma)\Big(\frac{\partial}{\partial p_{\sigma(j)}}\mathbf{V}_j(0)\prod_{i=1(,\neq j)}^{n}\delta_{i,\sigma(i)}\Big)\\ &= \sum_{j=1}^{n}\frac{\partial}{\partial p_j}\mathbf{V}_j(0) = \nabla\cdot\mathbf{V}(0)\,,\end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker delta. □

LEMMA 3.3. *If* $[0,\sigma)\ni t\longmapsto\varpi(t)$ *is differentiable,*

$$\varpi'(0)\equiv\frac{d}{dt}\varpi(t)\Big|_{t=0^+} = \nabla\cdot\mathbf{V}(0) - (D\mathbf{V}(0)\mathbf{n})\cdot\mathbf{n}\,. \tag{3.20}$$

Proof. We first note that

$$D\mathcal{F}_t\Big|_{t=0^+} = \mathcal{I},\quad \frac{d}{dt}(D\mathcal{F}_t)\Big|_{t=0^+} = D\mathbf{V}(0)$$

and

$$\frac{d}{dt}(D\mathcal{F}_t^{-1})\Big|_{t=0^+} = -(D\mathbf{V}(0))^T\,.$$

It is clear that $\varpi(t)^2 = (\det D\mathcal{F}_t)^2((D\mathcal{F}_t^{-1})^T D\mathcal{F}_t^{-1}\mathbf{n})\cdot\mathbf{n}$. Taking derivatives with respect to t and evaluating at $t = 0^+$, it follows from (3.19) that

$$\begin{aligned} 2\varpi(0)\varpi'(0) &= 2(\nabla\cdot\mathbf{V}(0) - \Big((D\mathbf{V}(0))\mathbf{n} + (D\mathbf{V}(0))^T\mathbf{n}\Big)\cdot\mathbf{n} \\ &= 2(\nabla\cdot\mathbf{V}(0)) - 2\Big((D\mathbf{V}(0))\mathbf{n}\Big)\cdot\mathbf{n}\,. \end{aligned}$$

Since $\varpi(0) = 1$, the result follows immediately. □

The expression (3.20) defines a differential operator on the boundary surface which is called the *tangential divergence*. This introduces an operator $\nabla_\Gamma\cdot$ on the boundary:

$$\nabla_\Gamma\cdot\mathbf{V} = \nabla\cdot\mathbf{V} - (D\mathbf{V}\mathbf{n})\cdot\mathbf{n}\,.$$

REMARK. The corresponding (pseudo-)adjoint to ∇_Γ is the *tangential gradient* ∇_Γ which is defined by $\nabla_\Gamma\varphi = \nabla\varphi - \dfrac{\partial\varphi}{\partial\mathbf{n}}\cdot\mathbf{n}$, i.e., ∇_Γ assigns φ the tangential component of its gradient. Combined with the following formula for the boundary integral

$$\begin{aligned} &\int_\Gamma \nabla_\Gamma\cdot(\varphi\mathbf{V})\,d\Gamma \\ &= \int_\Gamma(\nabla_\Gamma\mathbf{V})\varphi\,d\Gamma + \int_\Gamma\mathbf{V}\cdot\nabla_\Gamma\varphi\,d\Gamma = \int_\Gamma\kappa\varphi\mathbf{V}\cdot\mathbf{n}\,d\Gamma\,, \end{aligned} \tag{3.21}$$

they are fundamental tools to deal with variational problems defined on the boundary surface of a domain (c.f. [10] and [13]).

We return to the computation for the boundary integrals. From (3.18), we have

$$\begin{aligned} \int_{\Gamma_t}\frac{\partial\mathbf{u}_\epsilon(\alpha_t)}{\partial\mathbf{n}_t}\cdot\mathbf{w}\,d\Gamma_t &= \sum_{i,j=1}^{2}\int_{\Gamma_t} n_{jt}\frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j}w_i\,d\Gamma_t \\ &= \sum_{i,j=1}^{2}\int_{\Gamma_g} n_j\frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j}w_i\,\varpi(t)\,d\Gamma\,, \end{aligned}$$

for $\mathbf{w} = \mathbf{0}$ along Γ_α. Since $\mathbf{n}_t = \mathbf{n} =$ constant along Γ_g for all $0 \le t < \sigma$,

$$\frac{d}{dt}\Big(\sum_{i,j=1}^{2}\int_{\Gamma_g} n_j \frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j} w_i\, \varpi(t)\, d\Gamma\Big)\Big|_{t=0^+}$$
$$= \sum_{i,j=1}^{2}\int_{\Gamma_g}\Big[n_j\Big(\Big(\frac{\partial \widetilde{u}_{\epsilon i}}{\partial x_j}\Big)' + \nabla\Big(\frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j}\Big)\cdot \mathbf{V}(0)\Big)\, w_j$$
$$+ n_j \frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j}(\nabla w_i \cdot \mathbf{V}(0)) + n_j \frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j} w_i\, \nabla_\Gamma \mathbf{V}(0)\Big]\, d\Gamma\,.$$

Since $\mathbf{V}(0) = \mathbf{0}$ along Γ_g, this computation is reduced to

$$\frac{d}{dt}\int_{\Gamma_t} \frac{\partial \mathbf{u}_\epsilon(\alpha_t)}{\partial \mathbf{n}_t}\cdot \mathbf{w}\, d\Gamma_t\Big|_{t=0^+} = \int_{\Gamma_g} \frac{\partial \widehat{\mathbf{u}}'_\epsilon}{\partial \mathbf{n}}\cdot \mathbf{w}\, d\Gamma\,. \tag{3.22}$$

In a similar manner, we can show that

$$\frac{d}{dt}\int_{\Gamma_t} \nabla(\mathbf{u}_\epsilon(\alpha_t))\,\mathbf{w}\cdot \mathbf{n}_t\, d\Gamma_t\Big|_{t=0^+} = \int_{\Gamma_g} \nabla(\widehat{\mathbf{u}}'_\epsilon)\,\mathbf{w}\cdot \mathbf{n}\, d\Gamma\,. \tag{3.23}$$

Therefore, from (3.22)–(3.23), (3.17) is simplified to

$$\begin{aligned}
&\nu\int_{\Omega_\alpha} \nabla\widehat{\mathbf{u}}'_\epsilon : \nabla\mathbf{w}\, d\Omega + \int_{\Omega_\alpha} (\widehat{\mathbf{u}}'_\epsilon\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{w}\, d\Omega \\
&+ \int_{\Omega_\alpha} (\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widehat{\mathbf{u}}'_\epsilon\cdot\mathbf{w}\, d\Omega + \frac{1}{\epsilon}\int_{\Omega_\alpha} (\nabla\cdot\widehat{\mathbf{u}}'_\epsilon)(\nabla\cdot\mathbf{w})\, d\Omega \\
&- \nu\int_{\Gamma_g} \frac{\partial\widehat{\mathbf{u}}'_\epsilon}{\partial\mathbf{n}}\cdot\mathbf{w}\, d\Gamma - \frac{1}{\epsilon}\int_{\Gamma_g} (\nabla\cdot\widehat{\mathbf{u}}'_\epsilon)\mathbf{w}\cdot\mathbf{n}\, d\Gamma \\
&- \int_{M_1}^{M_2} \Big(\nu\nabla\mathbf{u}_\epsilon(\alpha):\nabla\mathbf{w} + \frac{1}{\epsilon}(\nabla\cdot\mathbf{u}_\epsilon(\alpha))(\nabla\cdot\mathbf{w})\Big)\vartheta(x_1)\, dx_1 = 0\,.
\end{aligned} \tag{3.24}$$

Next, we consider the adjoint equations (3.4)–(3.5). If we substitute $\mathbf{w} = \widehat{\mathbf{u}}'_\epsilon$, then (3.4) may be written in the integral form :

$$\begin{aligned}
&\nu\int_{\Omega_\alpha} \nabla\boldsymbol{\mu}_\epsilon : \nabla\widehat{\mathbf{u}}'_\epsilon\, d\Omega + \int_{\Omega_\alpha} (\widehat{\mathbf{u}}'_\epsilon\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\boldsymbol{\mu}_\epsilon\, d\Omega \\
&+ \int_{\Omega_\alpha} (\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widehat{\mathbf{u}}'_\epsilon\cdot\boldsymbol{\mu}_\epsilon\, d\Omega + \frac{1}{\epsilon}\int_{\Omega_\alpha} (\nabla\cdot\boldsymbol{\mu}_\epsilon)(\nabla\cdot\widehat{\mathbf{u}}'_\epsilon) d\Omega \\
&+ \int_{\Gamma_g} \boldsymbol{\tau}_\epsilon\cdot\widehat{\mathbf{u}}'_\epsilon\, d\Gamma = 2\nu\int_{\Omega_\alpha} \nabla\mathbf{u}_\epsilon(\alpha):\nabla\widehat{\mathbf{u}}'_\epsilon\, d\Omega\,.
\end{aligned} \tag{3.25}$$

By substituting $\mathbf{w} = \boldsymbol{\mu}_\epsilon$ into (3.24) and using the fact that $\boldsymbol{\mu}_\epsilon = \mathbf{0}$ along $\partial\Omega_\alpha$, we get

$$\begin{aligned}
&\nu \int_{\Omega_\alpha} \nabla \widehat{\mathbf{u}}'_\epsilon : \nabla \boldsymbol{\mu}_\epsilon \, d\Omega + \int_{\Omega_\alpha} (\widehat{\mathbf{u}}'_\epsilon \cdot \nabla)\, \mathbf{u}_\epsilon(\alpha) \cdot \boldsymbol{\mu}_\epsilon \, d\Omega \\
&\quad + \int_{\Omega_\alpha} (\mathbf{u}_\epsilon(\alpha) \cdot \nabla)\, \widehat{\mathbf{u}}'_\epsilon \cdot \boldsymbol{\mu}_\epsilon \, d\Omega + \frac{1}{\epsilon} \int_{\Omega_\alpha} (\nabla \cdot \widehat{\mathbf{u}}'_\epsilon)(\nabla \cdot \boldsymbol{\mu}_\epsilon)\, d\Omega \\
&\quad - \int_{M_1}^{M_2} \Big(\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \boldsymbol{\mu}_\epsilon + \frac{1}{\epsilon} (\nabla \cdot \mathbf{u}_\epsilon(\alpha))(\nabla \cdot \boldsymbol{\mu}_\epsilon) \Big)\, \vartheta(x_1)\, dx_1 = 0 .
\end{aligned} \tag{3.26}$$

Hence, it follows from (3.25) and (3.26) that

$$\begin{aligned}
&2\nu \int_{\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \widehat{\mathbf{u}}'_\epsilon \, d\Omega \\
&\quad = \int_{M_1}^{M_2} \Big(\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \boldsymbol{\mu}_\epsilon + \frac{1}{\epsilon} (\nabla \cdot \mathbf{u}_\epsilon(\alpha))(\nabla \cdot \boldsymbol{\mu}_\epsilon) \Big)\, \vartheta(x_1)\, dx_1 \\
&\qquad + \int_{\Gamma_g} \boldsymbol{\tau}_\epsilon \cdot \widehat{\mathbf{u}}'_\epsilon \, d\Gamma .
\end{aligned} \tag{3.27}$$

For the computation of $\int_{\Gamma_g} \boldsymbol{\tau}_\epsilon \cdot \widehat{\mathbf{u}}'_\epsilon \, d\Gamma$, we note that $\widehat{\mathbf{u}}_\epsilon = \mathbf{u}_\epsilon(\alpha) = \mathbf{g}$ on $\partial\Omega_\alpha$, where $\mathbf{g}$ is given. So, the material derivative of $\widehat{\mathbf{u}}$ is given by $\dot{\widehat{\mathbf{u}}}_\epsilon = \nabla \mathbf{g} \cdot \mathbf{V}(0)$ on $\partial\Omega_\alpha$. Using $\widehat{\mathbf{u}}'_\epsilon = \dot{\widehat{\mathbf{u}}}_\epsilon - \nabla \widehat{\mathbf{u}}_\epsilon \cdot \mathbf{V}(0)$, we get

$$\widehat{\mathbf{u}}'_\epsilon = \nabla(\mathbf{g} - \mathbf{u}_\epsilon(\alpha)) \cdot \mathbf{V}(0) \qquad \text{on } \partial\Omega_\alpha .$$

Since $\mathbf{g} - \mathbf{u}_\epsilon(\alpha) = \mathbf{0}$ on $\partial\Omega_\alpha$, the gradient of $\mathbf{g} - \mathbf{u}_\epsilon(\alpha)$ is parallel to the normal direction. Hence,

$$\widehat{\mathbf{u}}'_\epsilon = \frac{\partial(\mathbf{g} - \mathbf{u}_\epsilon(\alpha))}{\partial \mathbf{n}}\, \mathbf{n} \cdot \mathbf{V}(0) \qquad \text{on } \partial\Omega_\alpha . \tag{3.28}$$

However, since $\mathbf{V}(0) = \mathbf{0}$ along $\partial\Omega_\alpha - \Gamma_b(\alpha)$ and $\mathbf{g} = \mathbf{0}$ on Γ_α, we obtain

$$\int_{\Gamma_g} \boldsymbol{\tau}_\epsilon \cdot \widehat{\mathbf{u}}'_\epsilon \, d\Gamma = \int_{M_1}^{M_2} \frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} \cdot \boldsymbol{\tau}_\epsilon(\alpha)\, \vartheta(x_1)\, dx_1 . \tag{3.29}$$

Therefore, it follows from (3.16) and (3.27)–(3.29) that

$$\begin{aligned}
&d\mathcal{J}(\alpha; \vartheta) \\
&= 2\nu \int_{\Omega_\alpha} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \widehat{\mathbf{u}}'_\epsilon \, \mathbf{V}(0) \cdot \mathbf{n}\, d\Omega + \nu \int_{\Gamma} \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{u}_\epsilon(\alpha)\, \mathbf{V}(0) \cdot \mathbf{n}\, d\Gamma \\
&= \int_{M_1}^{M_2} \Bigg[-\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{u}_\epsilon(\alpha) + \Big(\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \boldsymbol{\mu}_\epsilon + \frac{1}{\epsilon} (\nabla \cdot \mathbf{u}_\epsilon)(\nabla \cdot \boldsymbol{\mu}_\epsilon) \Big) \\
&\qquad + \Big(\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} \cdot \Big(2\nu \frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu \frac{\partial \boldsymbol{\mu}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon} (\nabla \cdot \boldsymbol{\mu}_\epsilon)\, \mathbf{n} \Big) \Big) \Bigg]\ \vartheta(x_1)\, dx_1 .
\end{aligned}$$

Recall that $\mathbf{V}(0) \cdot \mathbf{n}\, d\Gamma$ corresponds to $-\vartheta(x_1)\, dx_1$. Hence in the sense of Hadamard's structure([3] or [11]), we may say that the shape gradient of the design functional $\mathcal{J}$ is given by

$$
\begin{aligned}
g_\epsilon(\Gamma_\alpha) &\equiv \nabla \mathcal{J} \\
&= \Big[\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \mathbf{u}_\epsilon(\alpha) - \Big(\nu \nabla \mathbf{u}_\epsilon(\alpha) : \nabla \boldsymbol{\mu}_\epsilon + \frac{1}{\epsilon} (\nabla \cdot \mathbf{u}_\epsilon)(\nabla \cdot \boldsymbol{\mu}_\epsilon) \Big) \\
&\qquad - \Big(\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} \cdot \Big(2\nu \frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu \frac{\partial \boldsymbol{\mu}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon} (\nabla \cdot \boldsymbol{\mu}_\epsilon) \mathbf{n} \Big) \Big) \Big]
\end{aligned} \tag{3.30}
$$

along the perturbed boundary, and $\mathbf{0}$ along the boundary of unperturbed region.

Let us summarize the above discussion in the following theorem.

THEOREM 3.4. *Let $\mathcal{J}(\alpha) = 2\nu \int_{\Omega_\alpha} D(\mathbf{u}_\epsilon(\alpha)) : D(\mathbf{u}_\epsilon(\alpha))\, d\Omega$ be the design functional which represents the energy dissipation due to the flow for a given $(\alpha, \mathbf{u}_\epsilon(\alpha)) \in \mathcal{V}_{ad}$. Then, the shape gradient of $\mathcal{J}$ is given in the form of* (3.30), *where $\mathbf{u}_\epsilon$ is a solution of the state equations* (2.6) *and $\boldsymbol{\mu}_\epsilon$ is a solution of the adjoint equations* (3.6)–(3.8) *which represent the weak linearized incompressible Navier–Stokes equations, respectively, with respect to the fluid domain Ω_α.*

REMARK. In the computation of (3.22) and (3.23), the curvature κ as for (3.13) does not appear. This is due to the choice of a trial function $\mathbf{w}$ and $\mathbf{n}_t = \mathbf{n}$ along Γ_g. For any unitary extension $\mathcal{N}$ of the normal vector field $\mathbf{n}$ on Γ, the curvature (or the mean curvature to the surface) is given by

$$\kappa = \nabla_\Gamma \mathbf{n} = \nabla \cdot \mathcal{N}.$$

In the 3-dimensional case, the computation of the mean curvature to the surface is nontrivial.

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Department of Mathematics
Kangnung National University
Kangnung 210-702, Korea
E-mail: hongchul@knusun.kangnung.ac.kr