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ON A FUNCTIONAL EQUATION ON GROUPS

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ABSTRACT. We present the general solution of the functional equation

$$\begin{aligned} & f(x_1y_1, x_2y_2) + f(x_1y_1^{-1}, x_2) + f(x_1, x_2y_2^{-1}) \\ = \quad & f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2). \end{aligned}$$

Furthermore, we also prove the Hyers-Ulam stability of the above functional equation.

1. Introduction

In a paper of Kannappan and Sahoo [8], it was shown that if the quadratic polynomial $f(x,y) = ax^2 + bx + cy^2 + dy + exy + \alpha$ with $a \neq 0$ and $c \neq 0$, is a solution of the functional differential equation

$$\begin{aligned} & f(x+h, y+k) - f(x,y) \\ = \quad & hf_x(x+\theta h, y+\theta k) + kf_y(x+\theta h, y+\theta k), \quad 0 < \theta < 1, \end{aligned}$$

for all $x, y, h, k \in \mathbb{R}$ (the set of reals) with $h^2 + k^2 \neq 0$, then $\theta = 1/2$. Conversely, if a function f satisfies the above equation with $\theta = 1/2$, then the only solution is a quadratic polynomial. Here f_x and f_y represent the first partial derivatives of f with respect to x and y, respectively.

To establish this result, Kannappan and Sahoo [8] used the following functional equation

$$\begin{aligned} & f(x_1+y_1, x_2+y_2) + f(x_1-y_1, x_2) + f(x_1, x_2-y_2) \\ = \quad & f(x_1-y_1, x_2-y_2) + f(x_1+y_1, x_2) + f(x_1, x_2+y_2) \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. In the present paper we solve the above functional equation on groups without any regularity assumption about f. We shall not assume the group to be abelian, so we write the above equation as

$$\begin{aligned} & f(x_1y_1, x_2y_2) + f(x_1y_1^{-1}, x_2) + f(x_1, x_2y_2^{-1}) \\ = \quad & f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2) \end{aligned} \tag{1}$$

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for all $x_1, x_2, y_1, y_2 \in \mathbf{G}$. Throughout this paper, $\mathbb{C}$ will denote the set of complex numbers.

This paper is organized as follows. In Section 2, we give some preliminary results that will be used in solving the functional equation (1). In Section 3, we present the general solution of (1). In Section 4, we prove the Hyers-Ulam stability of the functional equations (2) and (3), and then apply these results to prove the stability of the functional equation (1).

2. Some preliminary results

The following lemma was given by Aczel, Chung and Ng [1] and can also be obtained from Chung, Ebanks, Ng and Sahoo [2].

LEMMA 1. *Let* $\mathbf{G}$ *be a group in which* $x^2 = y$ *has a solution for all* $x, y \in \mathbf{G}$. *The general solution* $f : \mathbf{G} \to \mathbb{C}$ *of the functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x) \qquad (x, y \in \mathbf{G}) \tag{2}$$

satisfying also

$$f(xy) = f(yx) \qquad (x, y \in \mathbf{G})$$

is given by

$$f(x) = \phi(x) + a,$$

where $a \in \mathbb{C}$ *is an arbitrary constant and* ϕ *an arbitrary homomorphism of* $\mathbf{G}$ *into the additive group* $(\mathbb{C}, +)$ *of* $\mathbb{C}$.

Next, we generalize the above lemma. In fact, this lemma can also be deduced from Chung, Ebanks, Ng and Sahoo [2]. However, we give a short proof for the sake of completeness.

LEMMA 2. *Let* $\mathbf{G}$ *be a group in which* $x^2 = y$ *has a solution for all* $x, y \in \mathbf{G}$. *The general solution,* $f_1, f_2, f_3 : \mathbf{G} \to \mathbb{C}$, *of*

$$f_1(xy) + f_2(xy^{-1}) = f_3(x) \qquad (x, y \in \mathbf{G}) \tag{3}$$

satisfying also

$$f_1(xy) = f_1(yx), \quad f_2(xy) = f_2(yx) \quad (x, y \in \mathbf{G}) \tag{4}$$

are given by

$$f_1(x) = \phi(x) + a, \quad f_2(x) = \phi(x) + b, \quad f_3(x) = 2\phi(x) + a + b, \tag{5}$$

where $a, b \in \mathbb{C}$ *are arbitrary constants and* $\phi : \mathbf{G} \to \mathbb{C}$ *is a solution of*

$$\phi(xy) = \phi(x) + \phi(y) \qquad (x, y \in \mathbf{G}). \tag{6}$$

Proof. It is obvious that (5) with (6) satisfies (3) and (4). In order to prove the converse, we first replace y by y^{-1} in (3) to obtain

$$f_1(xy^{-1}) + f_2(xy) = f_3(x). \tag{7}$$

Setting $y = e$ (the identity of $\mathbf{G}$) in (7), we get

$$f_1(x) + f_2(x) = f_3(x). \tag{8}$$

Adding (3) and (7) and then using (8), we have

$$f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) = 2(f_1(x) + f_2(x)). \tag{9}$$

Using Lemma 1, from (9) and (4) we obtain

$$f_1(x) + f_2(x) = 2\phi(x) + a + b. \tag{10}$$

Subtracting (7) from (3), we get

$$f_1(xy) - f_2(xy) = f_1(xy^{-1}) - f_2(xy^{-1})$$

for all $x, y \in \mathbf{G}$. Hence

$$f_1(x) - f_2(x) = a - b. \tag{11}$$

From (10), (11) and (8) we obtain (5). This completes the proof of the lemma. □

3. General solutions of Eq. (1)

THEOREM 1. *Let $\mathbf{G}$ be a group in which $x^2 = y$ has a solution for all $x, y \in \mathbf{G}$. The general solution $f : \mathbf{G}^2 \to \mathbb{C}$ of the functional equation (1) satisfying*

$$f(x_1y_1, x_2) = f(y_1x_1, x_2) \text{ and } f(x_1, x_2y_2) = f(x_1, y_2x_2) \tag{12}$$

for all $x_1, x_2, y_1, y_2 \in \mathbf{G}$, is given by

$$f(x_1, x_2) = A(x_1, x_2) + \alpha(x_1) + \beta(x_2), \tag{13}$$

where $\alpha, \beta : \mathbf{G} \to \mathbb{C}$ are arbitrary functions satisfying $\alpha(x_1x_2) = \alpha(x_2x_1)$ and $\beta(x_1x_2) = \beta(x_2x_1)$ for all $x_1, x_2 \in \mathbf{G}$, and $A : \mathbf{G}^2 \to \mathbb{C}$ is additive in each variable.

Proof. It is easy to check that (13) satisfies (1) and (12). In order to prove the converse, we rewrite (1) as

$$\begin{aligned} f(x_1y_1, x_2y_2) - f(x_1y_1, x_2) - \big(f(x_1y_1^{-1}, x_2y_2^{-1}) - f(x_1y_1^{-1}, x_2)\big) \\ = f(x_1, x_2y_2) - f(x_1, x_2y_2^{-1}) \end{aligned} \tag{14}$$

and define $F_i : \mathbf{G}^3 \to \mathbb{C}$ $(i = 1, 2, 3)$ by

$$F_1(x_1, x_2, y_2) := f(x_1, x_2y_2) - f(x_1, x_2), \tag{15}$$

$$F_2(x_1, x_2, y_2) := -f(x_1, x_2y_2^{-1}) + f(x_1, x_2), \tag{16}$$

$$F_3(x_1, x_2, y_2) := f(x_1, x_2y_2) - f(x_1, x_2y_2^{-1}). \tag{17}$$

The equation (14) can be written as

$$F_1(x_1y_1, x_2, y_2) + F_2(x_1y_1^{-1}, x_2, y_2) = F_3(x_1, x_2, y_2).$$

From (12), (15) and (16), we get

$$\begin{aligned} F_1(x_1y_1, x_2, y_2) &= F_1(y_1x_1, x_2, y_2), \\ F_2(x_1y_1, x_2, y_2) &= F_2(y_1x_1, x_2, y_2) \end{aligned}$$

for $x_1, x_2, y_1, y_2 \in \mathbf{G}$. So by Lemma 2, (15) and (16) we get

$$f(x_1, x_2y_2) - f(x_1, x_2) = \phi_1(x_1, x_2, y_2) + b(x_2, y_2) \tag{18}$$

and

$$\begin{aligned} & f(x_1, x_2y_2^{-1}) - f(x_1, x_2) \\ & \quad = \; -\phi_1(x_1, x_2, y_2) + a(x_2, y_2) - b(x_2, y_2), \end{aligned} \tag{19}$$

where $\phi_1(x_1, x_2, y_2)$ satisfies

$$\phi_1(x_1y_1, x_2, y_2) = \phi_1(x_1, x_2, y_2) + \phi_1(y_1, x_2, y_2).$$

Adding (18) and (20) we get

$$f(x_1, x_2y_2) + f(x_1, x_2y_2^{-1}) - 2f(x_1, x_2) = a(x_2, y_2). \tag{20}$$

Setting $x_1 = e$ in (20) we have

$$a(x_2, y_2) = \beta_1(x_2y_2) + \beta_1(x_2y_2^{-1}) - 2\beta_1(x_2), \tag{21}$$

where $\beta_1(x_2) := f(e, x_2)$. Further, by (12), we also have $\beta_1(x_2y_2) = \beta_1(y_2x_2)$. Substituting (21) into (20), we get

$$F(x_1, x_2y_2) + F(x_1, x_2y_2^{-1}) = 2F(x_1, x_2),$$

where $F(x_1, x_2) := f(x_1, x_2) - \beta_1(x_2)$ satisfies $F(x_1, x_2y_2) = F(x_1, y_2x_2)$. By Lemma 1, we obtain

$$F(x_1, x_2) = \phi_2(x_1, x_2) + \alpha(x_1),$$

that is

$$f(x_1, x_2) = \phi_2(x_1, x_2) + \alpha(x_1) + \beta_1(x_2), \tag{22}$$

where $\phi_2(x_1, x_2)$ satisfies

$$\phi_2(x_1, x_2y_2) = \phi_2(x_1, x_2) + \phi_2(x_1, y_2). \tag{23}$$

Since f satisfies (12), using (22) we obtain

$$\phi_2(x_1y_1, x_2) + \alpha(x_1y_1) = \phi_2(y_1x_1, x_2) + \alpha(y_1x_1) \tag{24}$$

for all $x_1, y_1, x_2 \in \mathbf{G}$. Replacing x_2 by x_2y_2 in (24), we get

$$\phi_2(x_1y_1, x_2y_2) + \alpha(x_1y_1) = \phi_2(y_1x_1, x_2y_2) + \alpha(y_1x_1). \tag{25}$$

Hence from (23) and (25), we see that

$$\begin{aligned} &\phi_2(x_1y_1, x_2) + \phi_2(x_1y_1, y_2) + \alpha(x_1y_1) \\ = \ &\phi_2(y_1x_1, x_2) + \phi_2(y_1x_1, y_2) + \alpha(y_1x_1). \end{aligned}$$

Thus the last equation together with (24) yields

$$\phi_2(x_1y_1, y_2) = \phi_2(y_1x_1, y_2) \tag{26}$$

for all $x_1, y_1, y_2 \in \mathbf{G}$. From (24) and (26), we see that $\alpha(x_1x_2) = \alpha(x_2x_1)$ for all $x_1, x_2 \in \mathbf{G}$. Substituting (22) into (1) and using (23), we get

$$\phi_2(x_1y_1, y_2) + \phi_2(x_1y_1^{-1}, y_2) = 2\phi_2(x_1, y_2). \tag{27}$$

From Lemma 1, (27) and (26), we get

$$\phi_2(x_1, x_2) = A(x_1, x_2) + \beta_2(x_2), \tag{28}$$

where $A : \mathbf{G}^2 \to \mathbb{C}$ satisfies

$$A(x_1y_1, x_2) = A(x_1, x_2) + A(y_1, x_2). \tag{29}$$

Rewriting (28), we have

$$A(x_1, x_2) = \phi_2(x_1, x_2) - \beta_2(x_2). \tag{30}$$

Using (30) in (29), we obtain

$$\phi_2(x_1y_1, x_2) = \phi_2(x_1, x_2) + \phi_2(y_1, x_2) - \beta_2(x_2). \tag{31}$$

Replacing x_1 by x_1y_1 in (23) and using (31), we see that

$$\beta_2(x_2y_2) = \beta_2(x_2) + \beta_2(y_2). \tag{32}$$

From (23), (30) and (32), we obtain

$$A(x_1, x_2y_2) = A(x_1, x_2) + A(x_1, y_2).$$

Putting (28) into (22), we get

$$f(x_1, x_2) = A(x_1, x_2) + \alpha(x_1) + \beta(x_2),$$

where $\beta(x_2) = \beta_1(x_2) + \beta_2(x_2)$ which proves (13). It is easy to check that β satisfies $\beta(x_1x_2) = \beta(x_2x_1)$ for all $x_1, x_2 \in \mathbf{G}$. This concludes the proof of the theorem. □

REMARK 1. Theorem 1 also remains true if $\mathbb{C}$ is replaced by a 2-cancellative abelian group $\mathbf{H}$.

4. Stability results

In this section, we first prove the Hyers-Ulam stability of the functional equations (2) and (3) and apply these results to the proof of the Hyers-Ulam stability of the equation (1). Throughout this section, we assume $\mathbf{G}$ to be an abelian group.

LEMMA 3. *Let $\mathbf{G}$ be an abelian group. If a function $f : \mathbf{G} \to \mathbb{C}$ satisfies the functional inequality*

$$|f(xy) + f(xy^{-1}) - 2f(x)| \leq \delta \qquad (x, y \in \mathbf{G}) \tag{33}$$

for some given $\delta \geq 0$, then there exist an arbitrary constant $a \in \mathbb{C}$ and a group homomorphism ϕ of $\mathbf{G}$ into the additive group $(\mathbb{C}, +)$ of $\mathbb{C}$ such that

$$|f(x) - \phi(x) - a| \leq 2\delta$$

for all x in $\mathbf{G}$.

Proof. By e we will denote the unit element of G. If we define $\varphi(x) = f(x) - f(e)$, then $\varphi(e) = 0$. From (33) we have

$$|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)| \leq \delta \qquad (x, y \in \mathbf{G}). \tag{34}$$

We put $y = x$ in (34) to get

$$|\varphi(x^2) - 2\varphi(x)| \leq \delta \qquad (x \in \mathbf{G}). \tag{35}$$

Substituting $x^{-1}y^{-1}$ for y in (34), we have

$$|\varphi(y^{-1}) + \varphi(x^2y) - 2\varphi(x)| \leq \delta \qquad (x, y \in \mathbf{G}). \tag{36}$$

By (35) and (36), we obtain

$$\begin{aligned} & |\varphi(y^{-1}) + \varphi(x^2y) - \varphi(x^2)| \\ \leq\ & |\varphi(y^{-1}) + \varphi(x^2y) - 2\varphi(x)| + |2\varphi(x) - \varphi(x^2)| \\ \leq\ & 2\delta \qquad (x, y \in \mathbf{G}). \end{aligned} \tag{37}$$

If we put $u = y^{-1}$ and $v = x^2y$ in (37), then we have

$$|\varphi(u) + \varphi(v) - \varphi(uv)| \leq 2\delta \qquad (u, v \in \mathbf{G}).$$

By a theorem of Rätz [9] (or see also Hyers and Rassias [6], Forti [4], Jung [7] or Hyers, Isac and Rassias [5]), there exists a group homomorphism ϕ of $\mathbf{G}$ into the additive group $(\mathbb{C}, +)$ of $\mathbb{C}$ such that

$$|\varphi(x) - \phi(x)| \leq 2\delta \qquad (x \in \mathbf{G}),$$

and hence

$$|f(x) - \phi(x) - a| \leq 2\delta \qquad (x \in G),$$

where $a = f(e)$. □

LEMMA 4. *Let* $\mathbf{G}$ *be an abelian group in which* $x^2 = y$ *has a solution for all* $x, y \in \mathbf{G}$. *If a function* $f : \mathbf{G} \to \mathbb{C}$ *satisfies the functional inequality*

$$|f_1(xy) + f_2(xy^{-1}) - f_3(x)| \leq \delta \qquad (x, y \in \mathbf{G}) \tag{38}$$

for some $\delta \geq 0$, *then there exist constants* $a, b \in \mathbb{C}$ *and a group homomorphism* $\phi : \mathbf{G} \to (\mathbb{C}, +)$ *such that*

$$\begin{aligned} |f_1(x) - \phi(x) - a| &\leq 5\delta, \\ |f_2(x) - \phi(x) - b| &\leq 5\delta, \\ |f_3(x) - 2\phi(x) - a - b| &\leq 11\delta \end{aligned}$$

for all $x \in \mathbf{G}$.

Proof. If we replace y by y^{-1} in (38), then we have

$$|f_1(xy^{-1}) + f_2(xy) - f_3(x)| \leq \delta. \tag{39}$$

Put $y = e$ in (38) to get

$$|f_1(x) + f_2(x) - f_3(x)| \leq \delta. \tag{40}$$

This inequality, together with (38) and (39), yields

$$\begin{aligned} &|f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) - 2f_1(x) - 2f_2(x)| \\ &\leq |f_1(xy) + f_2(xy^{-1}) - f_3(x)| + |f_1(xy^{-1}) + f_2(xy) - f_3(x)| \\ &\quad + 2\,|f_3(x) - f_1(x) - f_2(x)| \\ &\leq 4\delta. \end{aligned}$$

By Lemma 3, we can conclude that there exists a group homomorphism $\phi : \mathbf{G} \to (\mathbb{C}, +)$ such that

$$|f_1(x) + f_2(x) - 2\phi(x) - f_1(e) - f_2(e)| \leq 8\delta \tag{41}$$

for any x in $\mathbf{G}$. By (38) and (39), we get

$$\begin{aligned} &|f_1(xy) - f_2(xy) - f_1(xy^{-1}) + f_2(xy^{-1})| \\ &\leq |f_1(xy) + f_2(xy^{-1}) - f_3(x)| + |f_3(x) - f_1(xy^{-1}) - f_2(xy)| \\ &\leq 2\delta. \end{aligned}$$

Putting $y = x$ in the above inequality, we obtain

$$|f_1(x^2) - f_2(x^2) - f_1(e) + f_2(e)| \leq 2\delta.$$

Since the equation $x^2 = y$ is solvable for all x and y in $\mathbf{G}$, the last inequality can be rewritten as

$$|f_1(x) - f_2(x) - f_1(e) + f_2(e)| \leq 2\delta \tag{42}$$

for each $x \in \mathbf{G}$. It then follows from (41) and (42) that

$$\begin{aligned} & 2\,|f_1(x) - \phi(x) - f_1(e)| \\ & \quad \leq \ |f_1(x) + f_2(x) - 2\phi(x) - f_1(e) - f_2(e)| \\ & \qquad + |f_1(x) - f_2(x) - f_1(e) + f_2(e)| \\ & \quad \leq \ 10\delta, \end{aligned}$$

that is,

$$|f_1(x) - \phi(x) - f_1(e)| \ \leq \ 5\delta \tag{43}$$

for all $x \in \mathbf{G}$. Analogously, we have

$$|f_2(x) - \phi(x) - f_2(e)| \ \leq \ 5\delta. \tag{44}$$

By (40), (43) and (44), we see

$$\begin{aligned} & |f_3(x) - 2\phi(x) - f_1(e) - f_2(e)| \\ & \quad \leq \ |f_3(x) - f_1(x) - f_2(x)| + |f_1(x) - \phi(x) - f_1(e)| \\ & \qquad + |f_2(x) - \phi(x) - f_2(e)| \\ & \quad \leq \ 11\delta \end{aligned}$$

which completes the proof. □

We are now ready to prove the main theorem of this section concerning the Hyers-Ulam stability of the functional equation (1).

THEOREM 2. *Let $\mathbf{G}$ be an abelian group in which $x^2 = y$ has a solution for all $x, y \in \mathbf{G}$. If a function $f : \mathbf{G}^2 \to \mathbb{C}$ satisfies the functional inequality*

$$\begin{aligned} & |f(x_1y_1, x_2y_2) + f(x_1y_1^{-1}, x_2) + f(x_1, x_2y_2^{-1}) \\ & \qquad - f(x_1y_1^{-1}, x_2y_2^{-1}) - f(x_1y_1, x_2) - f(x_1, x_2y_2)| \ \leq \ \delta \end{aligned} \tag{45}$$

for some $\delta \geq 0$ and for all $x_1, y_1, x_2, y_2 \in \mathbf{G}$, then there exist arbitrary functions $\alpha, \beta : \mathbf{G} \to \mathbb{C}$ and $A : \mathbf{G}^2 \to \mathbb{C}$ additive in each variable such that

$$|f(x_1, x_2) - A(x_1, x_2) - \alpha(x_1) - \beta(x_2)| \ \leq \ 14982\delta \tag{46}$$

for all $x_1, x_2 \in \mathbf{G}$.

Proof. Define $F_i : \mathbf{G}^3 \to \mathbb{C}$ $(i = 1, 2, 3)$ by (15), (16) and (17). Then, the inequality (45) can be written as

$$|F_1(x_1y_1, x_2, y_2) + F_2(x_1y_1^{-1}, x_2, y_2) - F_3(x_1, x_2, y_2)| \ \leq \ \delta$$

for any $x_1, y_1, x_2, y_2 \in \mathbf{G}$. According to Lemma 4, by considering (15), (16) and (17), the last inequality implies that there exists a function $\phi_1 : \mathbf{G}^3 \to$

$\mathbb{C}$ such that

$$\phi_1(x_1y_1, x_2, y_2) = \phi_1(x_1, x_2, y_2) + \phi_1(y_1, x_2, y_2),$$

$$|f(x_1, x_2y_2) - f(x_1, x_2) - \phi_1(x_1, x_2, y_2) - b(x_2, y_2)| \le 5\delta \tag{47}$$

and

$$|f(x_1, x_2y_2^{-1}) - f(x_1, x_2) + \phi_1(x_1, x_2, y_2) - a(x_2, y_2) + b(x_2, y_2)| \le 5\delta \tag{48}$$

for all $x_1, y_1, x_2, y_2 \in \mathbf{G}$. Combining (47) and (48), we get

$$|f(x_1, x_2y_2) + f(x_1, x_2y_2^{-1}) - 2f(x_1, x_2) - a(x_2, y_2)| \le 10\delta \tag{49}$$

for any $x_1, x_2, y_2 \in \mathbf{G}$. Putting $x_1 = e$ and setting $\beta_1(x_2) = f(e, x_2)$ in (49) yield

$$|a(x_2, y_2) - \beta_1(x_2y_2) - \beta_1(x_2y_2^{-1}) + 2\beta_1(x_2)| \le 10\delta. \tag{50}$$

By defining $F(x_1, x_2) = f(x_1, x_2) - \beta_1(x_2)$, it follows from (49) and (50) that

$$|F(x_1, x_2y_2) + F(x_1, x_2y_2^{-1}) - 2F(x_1, x_2)| \le 20\delta$$

for any $x_1, x_2, y_2 \in \mathbf{G}$. Hence, by Lemma 3, there exist an arbitrary function $\alpha : \mathbf{G} \to \mathbb{C}$ and a function $\phi_2 : \mathbf{G}^2 \to \mathbb{C}$ such that

$$\phi_2(x_1, x_2y_2) = \phi_2(x_1, x_2) + \phi_2(x_1, y_2) \tag{51}$$

and

$$|f(x_1, x_2) - \phi_2(x_1, x_2) - \alpha(x_1) - \beta_1(x_2)| \le 40\delta \tag{52}$$

for all $x_1, x_2, y_2 \in \mathbf{G}$. Using (45), (51) and (52), we obtain

$$\begin{aligned}
&| - \phi_2(x_1y_1, y_2) - \phi_2(x_1y_1^{-1}, y_2) + 2\phi_2(x_1, y_2)| \\
&\le \ |f(x_1y_1, x_2y_2) - \phi_2(x_1y_1, x_2y_2) - \alpha(x_1y_1) - \beta_1(x_2y_2)| \\
&\quad + |f(x_1y_1^{-1}, x_2) - \phi_2(x_1y_1^{-1}, x_2) - \alpha(x_1y_1^{-1}) - \beta_1(x_2)| \\
&\quad + |f(x_1, x_2y_2^{-1}) - \phi_2(x_1, x_2y_2^{-1}) - \alpha(x_1) - \beta_1(x_2y_2^{-1})| \\
&\quad + | - f(x_1y_1^{-1}, x_2y_2^{-1}) + \phi_2(x_1y_1^{-1}, x_2y_2^{-1}) + \alpha(x_1y_1^{-1}) + \beta_1(x_2y_2^{-1})| \\
&\quad + | - f(x_1y_1, x_2) + \phi_2(x_1y_1, x_2) + \alpha(x_1y_1) + \beta_1(x_2)| \\
&\quad + | - f(x_1, x_2y_2) + \phi_2(x_1, x_2y_2) + \alpha(x_1) + \beta_1(x_2y_2)| \\
&\quad + | - f(x_1y_1, x_2y_2) - f(x_1y_1^{-1}, x_2) - f(x_1, x_2y_2^{-1}) \\
&\qquad + f(x_1y_1^{-1}, x_2y_2^{-1}) + f(x_1y_1, x_2) + f(x_1, x_2y_2)| \\
&\le \ 241\delta.
\end{aligned}$$

By Lemma 3 again, there exist an arbitrary function $\beta_2 : \mathbf{G} \to \mathbb{C}$ and a function $A_1 : \mathbf{G}^2 \to \mathbb{C}$ such that

$$A_1(x_1y_1, x_2) = A_1(x_1, x_2) + A_1(y_1, x_2) \tag{53}$$

and

$$|\phi_2(x_1,x_2) - A_1(x_1,x_2) - \beta_2(x_2)| \le 482\delta \tag{54}$$

for all $x_1, y_1, x_2 \in \mathbf{G}$. By considering (53), it follows from (54) that

$$\begin{aligned} &|\phi_2(x_1y_1,x_2) - \phi_2(x_1,x_2) - \phi_2(y_1,x_2) + \beta_2(x_2)| \\ &\quad\le |\phi_2(x_1y_1,x_2) - A_1(x_1y_1,x_2) - \beta_2(x_2)| \\ &\qquad + |-\phi_2(x_1,x_2) + A_1(x_1,x_2) + \beta_2(x_2)| \\ &\qquad + |-\phi_2(y_1,x_2) + A_1(y_1,x_2) + \beta_2(x_2)| \\ &\quad\le 1446\delta. \end{aligned} \tag{55}$$

By replacing x_2 by y_2 respectively by x_2y_2 in (55) and combining (55) and the resulting inequalities and then considering (51), we have

$$|\beta_2(x_2) + \beta_2(y_2) - \beta_2(x_2y_2)| \le 4338\delta$$

for any $x_2, y_2 \in \mathbf{G}$. By a theorem of Forti [3] or Rätz [9] (see also Forti [4], Hyers and Isac and Rassias [5], Hyers and Rassias [6] and Jung [7]), there exists an additive function $A_2 : \mathbf{G} \to \mathbb{C}$ such that

$$|\beta_2(x_2) - A_2(x_2)| \le 4338\delta \tag{56}$$

for all $x_2 \in \mathbf{G}$. By (54) and (56), we have

$$|\phi_2(x_1,x_2) - A_1(x_1,x_2) - A_2(x_2)| \le 4820\delta \tag{57}$$

for any $x_1, x_2 \in \mathbf{G}$. By replacing x_2 by y_2 and by x_2y_2 separately in (57), and combining (57) and the resulting inequalities, and then considering (51) and the fact that A_2 is additive, we get

$$|A_1(x_1,x_2y_2) - A_1(x_1,x_2) - A_1(x_1,y_2)| \le 14460\delta$$

for any $x_1, x_2, y_2 \in \mathbf{G}$. According to a theorem of Forti [3] or Rätz [9] again, if we define a function $A : \mathbf{G}^2 \to \mathbb{C}$ by

$$A(x_1,x_2) = \lim_{n\to\infty} 2^{-n} A_1\left(x_1, x_2^{2^n}\right)$$

for any $x_1, x_2 \in \mathbf{G}$, then A is additive in the second variable and also in the first variable, since A_1 has this property (see (53)). Furthermore,

$$|A_1(x_1,x_2) - A(x_1,x_2)| \le 14460\delta \tag{58}$$

for all $x_1, x_2 \in \mathbf{G}$. By (52), (54), (58) and putting $\beta(x_2) = \beta_1(x_2) + \beta_2(x_2)$, we see that the inequality (46) holds true. □

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