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CONDITIONAL INDEPENDENCE AND TENSOR PRODUCTS OF CERTAIN HILBERT L^∞-MODULES

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ABSTRACT. For independent σ-algebras $\mathcal{A}$ and $\mathcal{B}$ on X, $L^2(X, \mathcal{A} \vee \mathcal{B})$, $L^2(X \times X, \mathcal{A} \times \mathcal{B})$, and the Hilbert space tensor product $L^2(X, \mathcal{A}) \otimes L^2(X, \mathcal{B})$ are isomorphic. In this note, we show that various Hilbert C^*-algebra tensor products provide the analogous roles when independence is weakened to conditional independence.

1. Introduction

Suppose that $(X, \mathcal{F}, \mu)$ is a probability space, and $\mathcal{A}$ and $\mathcal{B}$ are independent sub sigma algebras of $\mathcal{F}$. One can then show by standard rectangle approximation that $L^2(X \times X, \mathcal{A} \times \mathcal{B}, \mu \times \mu)$ and $L^2(X, \mathcal{A} \vee \mathcal{B}, \mu)$ are unitarily equivalent via a measure algebra isomorphism induced unitary operator. Moreover, there is a natural equivalence of these spaces with the (unique) Hilbert space tensor product $L^2(X, \mathcal{A}, \mu) \underset{Hilbert}{\otimes} L^2(X, \mathcal{B}, \mu)$. In this note, we consider analogous situations for the case that $\mathcal{A}$ and $\mathcal{B}$ are assumed only to be conditionally independent given $\mathcal{A} \cap \mathcal{B}$. The role of L^2 spaces is taken by certain Hilbert C^*-modules. As conditional independence may be easily described in terms of conditional expectations, and these conditional expectations are the basic building blocks of the Hilbert modules being investigated, this seems like the proper setting for this analysis. Moreover, there is a strong connection between the von Neumann algebra generated by the composition operator determined by a measurable transformation of X, and a pair of conditionally independent sigmal algebras. This relation is examined briefly at the end of this note.

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2. Notation and conventions

• Let $(X, \mathcal{F}, \mu)$ be a probability space. Upper case script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. will generally be used to name sub sigma algebras of $\mathcal{F}$. Then $\mathcal{A} \vee \mathcal{B}$ represents the smallest sigma algebra in $\mathcal{F}$ which contains both $\mathcal{A}$ and $\mathcal{B}$.

• For $\mathcal{S}$ a collection of sets in $\mathcal{F}$, $\sigma(\mathcal{S})$ is the smallest sigma algebra containing all the sets in $\mathcal{S}$.

• For a sigma algebra $\mathcal{A} \subset \mathcal{F}$, $E^{\mathcal{A}}$ is the conditional expectation operator. We will only be concerned with $E^{\mathcal{A}}$ acting on $L^2(\mathcal{F})$, so that for each $f \in L^2(\mathcal{F})$, $E^{\mathcal{A}} f$ is the unique function in $L^2(\mathcal{A})$ satisfying $\int_A E^{\mathcal{A}} f \, d\mu = \int_A f \, d\mu$ for every $A \in \mathcal{A}$. It is worth noting that $E^{\mathcal{A}}$ is the orthogonal projection onto $L^2(X, \mathcal{A}, \mu)$ and that $E^{\mathcal{A}} L^\infty(\mathcal{F}) = L^\infty(\mathcal{A})$.

• f^* means $\bar{f}$, the complex conjugate of f.

3. Tensor products of Hilbert C^* modules

Chapter 4 of E. C. Lance's text [4] provides a detailed treatment of the general theory of Hilbert C^* module tensor products. We will be concerned here with certain specific examples. In general, when applied to vector spaces, $\otimes$ refers only to the algebraic tensor product (i.e., tensor product over $\mathbb{C}$). Recall that if $\mathcal{E}$ and $\mathcal{F}$ are Hilbert $\mathcal{Y}$ modules, where $\mathcal{Y}$ is a C^* algebra, and ϕ is a *homomorphism from $\mathcal{Y}$ to $\mathcal{L}(\mathcal{F})$ (the space of adjointable maps on $\mathcal{F}$) then the quotient of $\mathcal{E} \otimes \mathcal{F}$ by the linear span of

$$\{e \otimes \phi(y)(f) - ye \otimes f : e \in \mathcal{E}, f \in \mathcal{F}, y \in \mathcal{Y}\}$$

is a $\mathcal{Y}$ valued inner product space, where the inner product is given on elementary tensors by

$$\langle e_1 \otimes f_1, \, e_2 \otimes f_2 \rangle = \langle f_1, \, \phi(\langle e_1, \, e_2 \rangle_{\mathcal{E}})(f_2) \rangle_{\mathcal{F}}.$$

The completion of this space is the interior tensor product of $\mathcal{E}$ and $\mathcal{F}$ with respect to ϕ. This space is a Hilbert $\mathcal{Y}$ module. We will only be concerned with the case that ϕ is the left representation of $\mathcal{Y}$ as multiplication operators on $\mathcal{F}$.

With $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{Y}$ as above, we may also form the exterior tensor product of $\mathcal{E}$ and $\mathcal{F}$. This inner product is given on elementary tensors by

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle e_1, e_2 \rangle_{\mathcal{E}} \otimes \langle f_1, f_2 \rangle_{\mathcal{F}}.$$

The completion of this space is then a Hilbert $\mathcal{Y}\otimes\mathcal{Y}$ module (in this context $\mathcal{Y}\otimes\mathcal{Y}$ is the spatial, or minimal C^* algebra tensor product.).

The specific tensor products encountered in this note are:

$\otimes'$: The interior tensor product $\overset{\otimes}{{}_{L^{\infty}(\mathcal{C})}}$ of two Hilbert $L^{\infty}(\mathcal{C})$ modules, with respect to left multiplication.

$\otimes''$: The interior tensor product $\overset{\otimes}{{}_{L^{\infty}(\mathcal{C}\times\mathcal{C})}}$ of two Hilbert $L^{\infty}(\mathcal{C}\times\mathcal{C})$ modules, with respect to left multiplication.

$\otimes'''$: The exterior tensor product $\overset{\otimes}{{}_{L^{\infty}(\mathcal{C})\otimes L^{\infty}(\mathcal{C})}}$ of two Hilbert $L^{\infty}(\mathcal{C})$ modules.

4. Conditional independence

Chapter 2 of M. M. Rao's text [5] provides a detailed treatment of the material herein outlined. Given three sub sigma algebras $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ of $\mathcal{F}$, $\mathcal{J}$ and $\mathcal{K}$ are *conditionally independent given* $\mathcal{L}$ if $\forall J\in\mathcal{J}$, $E^{\mathcal{K}\vee\mathcal{L}}\chi_J = E^{\mathcal{L}}\chi_J$. This may be restated in operator form as $E^{\mathcal{K}\vee\mathcal{L}}E^{\mathcal{J}} = E^{\mathcal{L}}E^{\mathcal{J}}$. In the special case of $\mathcal{C}=\mathcal{A}\cap\mathcal{B}$: $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{C}$ if and only if $E^{\mathcal{A}}\,E^{\mathcal{B}} = E^{\mathcal{C}}$; equivalently $E^{\mathcal{A}}\,E^{\mathcal{B}} = E^{\mathcal{B}}\,E^{\mathcal{A}}$.

We will employ the notation " $\mathcal{A}$ and $\mathcal{B}$ are $ci\mid\mathcal{C}$ " for conditional independence. We will also make repeated use of the following:

LEMMA 0. *If a is $\mathcal{A}$ measurable and b is $\mathcal{B}$ measurable, and $\mathcal{A}$ and $\mathcal{B}$ are $ci\mid\mathcal{C}$, then*

$$E^{\mathcal{C}}(ab) = (E^{\mathcal{B}}a)\cdot(E^{\mathcal{A}}b) = (E^{\mathcal{C}}a)\cdot(E^{\mathcal{C}}b).$$

Proof.
$$\begin{aligned} E^{\mathcal{C}}(ab) &= E^{\mathcal{A}}E^{\mathcal{B}}(ab) = E^{\mathcal{A}}[(E^{\mathcal{B}}a)b] \\ &= E^{\mathcal{A}}\left[\left(E^{\mathcal{B}}E^{\mathcal{A}}a\right)b\right] = E^{\mathcal{A}}\left[\left(E^{\mathcal{A}}E^{\mathcal{B}}a\right)b\right] \\ &= \left(E^{\mathcal{A}}E^{\mathcal{B}}a\right)\cdot\left(E^{\mathcal{A}}b\right) = \left(E^{\mathcal{B}}E^{\mathcal{A}}a\right)\cdot\left(E^{\mathcal{A}}b\right) \\ &= \left(E^{\mathcal{B}}a\right)\cdot\left(E^{\mathcal{A}}b\right). \end{aligned}$$

On the other hand, $E^{\mathcal{B}}a = E^{\mathcal{B}}E^{\mathcal{A}}a = E^{\mathcal{C}}a$, and similarly $E^{\mathcal{A}}b = E^{\mathcal{C}}b$. □

5. The Hilbert L^∞-modules associated with nested sigma algebras

For $\mathcal{C} \subset \mathcal{A} \subset \mathcal{F}$, let

$$\mathbb{L}(\mathcal{C},\mathcal{A}) = \{f : f \text{ is } \mathcal{A} \text{ measurable and } E^{\mathcal{C}}|f|^2 \in L^\infty\};$$
$$\|f\|_{\mathbb{L}} = \|f\|_{\mathbb{L}(\mathcal{C},\mathcal{A})} = \|E^{\mathcal{C}}|\, f\,|^2\|_\infty^{1/2}.$$

It was shown in [3] that $(\mathbb{L}(\mathcal{C},\mathcal{A}), \|\,\|_{\mathbb{L}})$ is a Banach space satisfying $L^\infty(\mathcal{A}) \subset \mathbb{L}(\mathcal{C},\mathcal{A}) \subset L^2(\mathcal{F})$. Moreover, $\mathbb{L}(\mathcal{C},\mathcal{A})$ is a Hilbert $L^\infty(\mathcal{A})$-module which is a realization of the localization via $E^{\mathcal{A}}$ of $L^\infty(\mathcal{F})$. (see [4] for a discussion of Hilbert C* modules in general and localization in particular.)

Let $\mathcal{A}$ and $\mathcal{B}$ be sub sigma algebras of $\mathcal{F}$, and let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$. Assume $\mathcal{A}$ and $\mathcal{B}$ are $ci|\mathcal{C}$. We note that in the case that $\mathcal{C}$ is the trivial algebra consisting of all $\mathcal{F}$-sets of measure 0 or 1, then conditional independence given $\mathcal{C}$ reduces to the probabilistic concept of independence. Also, Since $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{C}$, for a$\in L^2(\mathcal{A})$ and b$\in L^2(\mathcal{B})$,

(Lemma 0) $$E^{\mathcal{C}}\left(|ab|^2\right) = \left(E^{\mathcal{C}}|a|^2\right) \cdot \left(E^{\mathcal{C}}|b|^2\right)$$

so

$$\mathbb{L}(\mathcal{C},\mathcal{A}) \cdot \mathbb{L}(\mathcal{C},\mathcal{B}) \subset \mathbb{L}(\mathcal{C},\mathcal{F}).$$

Moreover, $\mathbb{L}(\mathcal{C},\mathcal{A})$ is a complemented Hilbert $L^\infty(\mathcal{C})$ submodule of $(\mathcal{C}, \mathcal{F})$ because the projection $E^{\mathcal{A}}$ is an adjointable operator on $\mathbb{L}(\mathcal{C},\mathcal{F})$. Consider the interior tensor product $\otimes'$ of $\mathbb{L}(\mathcal{C},\mathcal{A})$ and $\mathbb{L}(\mathcal{C},\mathcal{B})$ (with respect to the representation of $L^\infty(\mathcal{C})$ as left multiplication on $\mathbb{L}(\mathcal{C},\mathcal{B})$) where $\mathcal{A}$ and $\mathcal{B}$ are $ci|\mathcal{C}$:

$$\begin{aligned} & \langle a_1 \otimes b_1, a_2 \otimes b_2\rangle \stackrel{\text{def.}}{=} \langle b_1, \langle a_1, a_2\rangle_{\mathrm{L}} \cdot b_2\rangle_{\mathrm{L}} \\ = \; & E^{\mathcal{C}}\left(b_1^* \cdot \langle a_1, a_2\rangle_{\mathrm{L}} \cdot b_2\right) = E^{\mathcal{C}}\left(b_1^* \cdot E^{\mathcal{C}}\left(a_1^* \cdot a_2\right) \cdot b_2\right) \\ = \; & E^{\mathcal{C}}\left(a_1^* \cdot a_2\right) \cdot E^{\mathcal{C}}\left(b_1^* \cdot b_2\right). \end{aligned}$$

As noted in Lemma 0, since $\mathcal{A}$ and $\mathcal{B}$ are ci$|\mathcal{C}$, we have

(1) $$\begin{aligned} \langle a_1 \otimes b_1, a_2 \otimes b_2\rangle &= E^{\mathcal{C}}\left(a_1^* \cdot a_2 \cdot b_1^* \cdot b_2\right) \\ &= E^{\mathcal{C}}\left((a_1 b_1)^* \cdot (a_2 b_2)\right). \end{aligned}$$

In order to form the interior tensor product, we must consider the subspace $\mathbb{N}$ of the algebraic tensor product generated by all elements of the form

$(ca) \otimes b - a \otimes (cb)$ and take the completion of $(\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes \mathbb{L}(\mathcal{C},\mathcal{B}))/\mathbb{N}$ with respect to the norm determined by Eq. 1. This space is denoted $\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes' \mathbb{L}(\mathcal{C},\mathcal{B})$.

THEOREM 1. *$\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes' \mathbb{L}(\mathcal{C},\mathcal{B})$ and $\mathbb{L}(\mathcal{C},\mathcal{A} \vee \mathcal{B})$ are isometrically isomorphic as Hilbert L^∞-modules.*

Proof. For finite sequences $\{a_i\}$ and $\{b_i\}$ in $\mathbb{L}(\mathcal{C},\mathcal{A})$ and $\mathbb{L}(\mathcal{C},\mathcal{B})$, respectively, we have (via Eq. 1)

$$\begin{aligned}
\langle \Sigma a_i \cdot b_i, \Sigma a_i \cdot b_i \rangle_{\mathbb{L}(\mathcal{C},\mathcal{A}\vee\mathcal{B})} &= \sum_{i,j} \langle a_i \cdot b_i, a_j \cdot b_j \rangle_{\mathbb{L}(\mathcal{C},\mathcal{A}\vee\mathcal{B})} \\
&= \sum_{i,j} E^{\mathcal{C}}\left((a_i \cdot b_i)^* \cdot (a_j \cdot b_j)\right) \\
&= \sum_{i,j} \langle a_i \otimes' b_i, a_j \otimes' b_j \rangle \\
&= \langle \Sigma a_i \otimes' b_i, \Sigma a_i \otimes' b_i \rangle_{\mathbb{L}(\mathcal{C},\mathcal{A})\otimes'\mathbb{L}(\mathcal{C},\mathcal{B})}.
\end{aligned}$$

This insures that the map S:$\Sigma a_i \otimes' b_i \mapsto \Sigma a_i \cdot b_i$ extends to an $L^\infty(\mathcal{C})$-module isometry from $\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes' \mathbb{L}(\mathcal{C},\mathcal{B})$ onto $\mathbb{L}(\mathcal{C},\mathcal{A} \vee \mathcal{B})$. To see that S is in fact an adjointable map, note that for $a_i \in \mathbb{L}(\mathcal{C},\mathcal{A})$ and $b_i \in \mathbb{L}(\mathcal{C},\mathcal{B})$, $i = 1, 2$ we have

$$\begin{aligned}
\left\langle a_1 \cdot b_1, S\left(a_2 \otimes' b_2\right) \right\rangle_{\mathbb{L}(\mathcal{C},\mathcal{A}\vee\mathcal{B})} &= \langle a_1 \cdot b_1, a_2 \cdot b_2 \rangle_{\mathbb{L}(\mathcal{C},\mathcal{A})} \\
&= E^{\mathcal{C}}\left(a_1^* a_2 \cdot b_1^* b_2\right) = E^{\mathcal{C}}\left(a_1^* a_2\right) E^{\mathcal{C}}\left(b_1^* b_2\right) \\
&= \langle a_1 \otimes' b_1 \,,\, a_2 \otimes' b_2) \rangle_{\mathbb{L}(\mathcal{C},\mathcal{A}\vee\mathcal{B})} \ ;
\end{aligned}$$

so indeed, S is adjointable, and S* (= S^{-1}) is given (on a fundamental set) by S*(a · b) = a⊗′b. □

It was shown in [2] that $\mathbb{L}(\mathcal{C},\mathcal{A}) = L^2(X,\mathcal{A},\mu)$ if and only if $\mathcal{C}$ is generated by a finite partition. Indeed, if $\mathcal{C} = \sigma\ (C_i : 1 \leq i \leq$n), where $\{C_1, \cdots, C_n\}$ is a measurable partition of X, then

$$E^{\mathcal{C}}|f|^2 = \Sigma\Big(\frac{1}{\mu(C_i)}\Big(\int_{C_i} |f|^2 d\mu\Big)\chi_{C_i}\Big);$$

so

$$\|f\|_{\mathbb{L}}^2 = \max_{1\leq i\leq n} \frac{1}{\mu(C_i)} \int_{C_i} |f|^2 d\mu,$$

Thus the $L^2(\mathcal{A})$ and $\mathbb{L}(\mathcal{C},\mathcal{A})$ norms are equivalent. If $\mathcal{C}$ is the trivial sigma algebra $\sigma(X)$, then $E^{\mathcal{C}}$ is the unconditional expectation and $\mathcal{A}$ and $\mathcal{B}$ are actually independent. In this case, of course the L^2 and $\mathbb{L}$ norms are identical. The previous few lines may be formalized as follows.

COROLLARY 2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $ci|\mathcal{C}$ where $\mathcal{C}$ is finitely generated. Then $L^2(X, \mathcal{A}, \mu) \otimes' L^2(X, \mathcal{B}, \mu) = L^2(X, \mathcal{A} \vee \mathcal{B}, \mu)$. If $\mathcal{C}$ is trivial then $L^2(X, \mathcal{A}, \mu) \underset{Hilbert}{\otimes} L^2(X, \mathcal{B}, \mu) = L^2(X, \mathcal{A} \vee \mathcal{B}, \mu)$, where $\underset{Hilbert}{\otimes}$ denotes the (unique) Hilbert space tensor product.*

REMARK. If in fact $\mathcal{C} = \sigma(C_1, C_2, \cdots, C_n)$, where the C_i's are pairwise disjoint then we may identify $L^\infty(\mathcal{C})$ with $\mathbb{C}^n$ via the isomorphism $(\lambda_1, \cdots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i \, \chi_{C_i}$. We can then view $\mathbb{L}\ (\mathcal{C}, \mathcal{A})$ and $\mathbb{L}(\mathcal{C}, \mathcal{B})$ as Hilbert $\mathbb{C}^n$-modules.

Suppose for the moment that $\mathcal{A}$ and $\mathcal{B}$ have intersection $\mathcal{C}$ but are not necessarily $ci|\mathcal{C}$. It is easy to see that if a is $\mathcal{A}$-measurable and b is $\mathcal{B}$ -measurable and we define $a'(x,y) = a(x)$ and $b''(x,y) = b(y)$, and let $F = a' \cdot b'' : X \times X \mapsto \mathbb{C}$, then for any sigma algebras $\mathcal{P}$ and $\mathcal{Q}$ contained in $\mathcal{A}$ and $\mathcal{B}$, respectively,

$$(E^{\mathcal{P}\times\mathcal{Q}}F)(x,y) = (E^{\mathcal{P}}a)(x) \cdot (E^{\mathcal{Q}}b)(y).$$

LEMMA 3. *$\mathcal{A} \times \mathcal{C}$ and $\mathcal{C} \times \mathcal{B}$ are $ci|\mathcal{C} \times \mathcal{C}$ with respect to the measure space $(X \times X, \mathcal{A} \times \mathcal{B}, \mu \times \mu)$.*

Proof. Let a, b, and F be as in the remark immediately preceding the statement of the lemma. Then

$$E^{\mathcal{C}\times\mathcal{B}}F(x,y) = (E^{\mathcal{C}}a)(x) \cdot (E^{\mathcal{B}}b)(y) = (E^{\mathcal{C}}a)(x) \cdot b(y).$$

We may then apply the same reasoning to deduce that

$$\begin{aligned} E^{\mathcal{A}\times\mathcal{C}}\left(E^{\mathcal{C}\times\mathcal{B}}F\right)(x,y) &= \left(E^{\mathcal{A}}\left(E^{\mathcal{C}}a\right)\right)(x) \cdot \left(E^{\mathcal{C}}b\right)(y) \\ &= \left(E^{\mathcal{C}}a\right)(x) \cdot \left(E^{\mathcal{C}}b\right)(y) \qquad (\text{since} E^{\mathcal{A}}E^{\mathcal{C}} = E^{\mathcal{C}}) \\ &= \left(E^{\mathcal{C}\times\mathcal{C}}F\right)(x,y). \end{aligned}$$

Since the set of such F's generates all $\mathcal{A} \times \mathcal{B}$ measurable functions, we see that $E^{\mathcal{A}\times\mathcal{C}}E^{\mathcal{C}\times\mathcal{B}} = E^{\mathcal{C}\times\mathcal{C}}$. But $(\mathcal{A} \times \mathcal{C}) \cap (\mathcal{C} \times \mathcal{B}) = \mathcal{C} \times \mathcal{C}$, which guarantees that $(\mathcal{A} \times \mathcal{C})$ and $(\mathcal{C} \times \mathcal{B})$ are $ci|\ \mathcal{C} \times \mathcal{C}$. □

COROLLARY 4. *Let $\mathcal{A} \cap \mathcal{B} = \mathcal{C}$ (no assumption of conditional independence). Then $\mathbb{L}(\mathcal{C} \times \mathcal{C}, \mathcal{A} \times \mathcal{C}) \otimes'' \mathbb{L}(\mathcal{C} \times \mathcal{C}, \mathcal{C} \times \mathcal{B})$ and $\mathbb{L}\ (\mathcal{C} \times \mathcal{C}, \mathcal{A} \times \mathcal{B})$ are isometrically isomorphic as Hilbert $L^\infty(\mathcal{C} \times \mathcal{C})$-modules.*

Proof. By Lemma 3 $(\mathcal{A} \times \mathcal{C})$ and $(\mathcal{C} \times \mathcal{B})$ are $ci|\ \mathcal{C} \times \mathcal{C}$, so that Theorem 1 applies. □

When the hypothesis for Corollary 4 is strengthened so that $\mathcal{A}$ and $\mathcal{B}$ are $ci|\mathcal{C}$, then we have another representation of $\mathbb{L}(\mathcal{C} \times \mathcal{C}, \mathcal{A} \times \mathcal{B})$, as presented later in Theorem 5.

We now examine the exterior tensor product $\mathbb{L}\ (\mathcal{C},\ \mathcal{A}) \otimes''' \mathbb{L}\ (\mathcal{C},\ \mathcal{B})$. We continue to use the notational conventions $f'(\mathrm{x,y}) = f(x)$, $f''(x,y) = f(y)$, with the obvious interpretations of $\mathcal{S}'$ and $\mathcal{S}''$ for a set $\mathcal{S}$ of functions of one variable. As we are using the minimal C* algebra norm on $L^\infty(\mathcal{C}) \otimes L^\infty(\mathcal{C})$, we identify this space as $L^\infty(\mathcal{C} \times \mathcal{C})$.

THEOREM 5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sigma sub algebras of $\mathcal{F}$ with intersection $\mathcal{C}$. Then $\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes''' \mathbb{L}(\mathcal{C},\mathcal{B})$ and $\mathbb{L}(\mathcal{C}\times\mathcal{C},\mathcal{A}\times\mathcal{B})$ are isometrically isomorphic as Hilbert $L^\infty(\mathcal{C}) \otimes L^\infty(\mathcal{C})$-modules.*

Proof. The inner product for $\mathbb{L}(\mathcal{C},\mathcal{A}) \otimes''' \mathbb{L}(\mathcal{C},\mathcal{B})$ is given on elementary tensors by

$$\begin{aligned} \langle a_1 \otimes b_1 , a_2 \otimes b_2 \rangle &= \langle a_1, a_2 \rangle_{\mathbb{L}} \otimes \langle b_1 , b_2 \rangle_{\mathbb{L}} \\ &= E^{\mathcal{C}}(a_1^* a_2) \otimes E^{\mathcal{C}}(b_1^* b_2), \end{aligned}$$

from which it follows that for finite sequences $\{a_i\}$ and $\{b_i\}$,

$$\langle \Sigma a_i \otimes b_i , \Sigma a_i \otimes b_i \rangle = \sum_{i,j} E^{\mathcal{C}}(a_i^* a_j) \otimes E^{\mathcal{C}}(b_i^* b_j).$$

Thus

$$\|\Sigma\, a_i \otimes b_i\|^2_{\mathbb{L}(\mathcal{C},\mathcal{A})\otimes'''\mathbb{L}(\mathcal{C},\mathcal{B})} = \left\| \sum_{i,j} E^{\mathcal{C}}(a_i^* a_j) \otimes E^{\mathcal{C}}(b_i^* b_j) \right\|;$$

this last norm being computed with respect to the minimal C* algebra tensor product completion of $L^\infty(\mathcal{C}) \otimes L^\infty(\mathcal{C})$.

Now,

$$\begin{aligned}
& \langle \Sigma a_i' \cdot b_i'', \Sigma a_i' \cdot b_i'' \rangle_{\mathrm{L}(\mathcal{C}\times\mathcal{C},\, \mathcal{C}\times\mathcal{B})} \\
= & \sum_{i,j} E^{\mathcal{C}\times\mathcal{C}} \left(\left(a_i' \cdot b_i'' \right)^* a_j' \cdot b_j'' \right) \\
= & \sum_{i,j} E^{\mathcal{C}\times\mathcal{C}} \left(a_i'^* \cdot a_j' \cdot b_i''^* \cdot b_j'' \right) \\
= & \sum_{i,j} E^{\mathcal{C}} \left(a_i^* \cdot a_j \right) \cdot E^{\mathcal{C}} \left(b_i^* \cdot b_j \right)
\end{aligned}$$

(the last equality following from the remark preceding Lemma 3)

$$= \sum_{i,j} E^{\mathcal{C}\times\mathcal{C}} \left(a_i^* \cdot a_j \right)' \cdot E^{\mathcal{C}\times\mathcal{C}} \left(b_i^* \cdot b_j \right)''.$$

This is precisely the image of $\sum_{i,j} E^{\mathcal{C}} \left(a_i^* \, a_j \right) \otimes E^{\mathcal{C}} \left(b_i^* \, b_j \right)$ in the representation of (the minimal C* algebra tensor product completion of) $L^\infty(\mathcal{C}) \otimes L^\infty(\mathcal{C})$ as $L^\infty(\mathcal{C} \times \mathcal{C})$. □

6. An application to composition operators

Starting with the probability space $(X, \mathcal{F}, \mu)$, suppose that T is a mapping from X to X such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. If $\frac{d\mu \circ T^{-1}}{d\mu} \in L^\infty$, then the composition operator C given by the formula $Cf = f \circ T$ is a bounded linear operator on $L^2(X, \mathcal{F}, \mu)$. (In [6], R. K. Singh and J. S. Manhas give many of the basic properties of such operators). In [1], the von Neumann algebra, $\mathcal{W}$, generated by C was studied. In particular, the following results were established:

PROPOSITION. *Let $\mathcal{W}$ be the von Neumann algebra generated by C. Then*

i) *there is a sigma algebra $\mathcal{M}$ in $\mathcal{F}$ such that $[\mathcal{W}1] = L^2(\mathcal{M})$;*
ii) *for $\mathcal{A}' = \{F \in \mathcal{F} : T^{-1}F = F\}$, $[\mathcal{W}'1] = L^2(\mathcal{A}')$;*
iii) *$E^{\mathcal{M}} E^{\mathcal{A}'} = E^{\mathcal{A}'} E^{\mathcal{M}}$; and*
iv) *$E^{\mathcal{M} \vee \mathcal{A}'}$ is a central projection of $\mathcal{W}$.*
v) *$\mathcal{W}$ is a factor if and only if $\mathcal{M} \vee \mathcal{A}' = \mathcal{F}$ and $\mathcal{M} \cap \mathcal{A}'$ is the trivial sigma algebra.*

(*The square brackets above indicate closed linear span in* $L^2(X,\mathcal{F},\mu)$).

Let $\mathcal{C}=\mathcal{M}\cap\mathcal{A}'$. Then (iii) above shows that $\mathcal{M}$ and $\mathcal{A}'$ are $ci|\mathcal{C}$. Thus the tensor product results apply (where $=$ is used for Hilbert $C*$-module equivalence):

i) $\mathbb{L}(\mathcal{C},\mathcal{M})\otimes'\mathbb{L}(\mathcal{C},\mathcal{A}')=\mathbb{L}(\mathcal{C},\ \mathcal{M}\vee\mathcal{A}')$

ii) $\mathbb{L}(\mathcal{C}\times\mathcal{C},\mathcal{M}\times\mathcal{C})\otimes''\mathbb{L}(\mathcal{C}\times\mathcal{C},\mathcal{C}\times\mathcal{A}')=\mathbb{L}(\mathcal{C}\times\mathcal{C},\ \mathcal{M}\times\mathcal{A}')$

iii) $\mathbb{L}(\mathcal{C},\mathcal{M})\otimes'''\mathbb{L}(\mathcal{C},\mathcal{A}')=\mathbb{L}(\mathcal{C}\times\mathcal{C},\mathcal{M}\times\mathcal{A}')$.

Now the composition operator C actually leaves $\mathbb{L}(\mathcal{C},\mathcal{M})$ and $\mathbb{L}(\mathcal{C},\mathcal{A}')$ invariant; indeed, it is the identity operator on the latter. Consider the tensor product "equation" (i) above, and let R be the Hilbert $L^\infty(\mathcal{C})$-module isomorphism implementing the "equality." Define S on $\mathbb{L}(\mathcal{C},\mathcal{M})\otimes'\mathbb{L}(\mathcal{C},\mathcal{A}')$ by (extending the elementary tensor map) $S(m\otimes a')=m\circ T\otimes a'$. Then it is clear that $RS=CR$, and similar statements may be made for (ii) and (iii). We now show that in the case that $\mathcal{W}$ is a factor, these three module identifications coalesce and provide a special representation of T. In this case, we have $\mathcal{C}$ being the trivial sigma algebra, so that $L^\infty(\mathcal{C})$, $L^\infty(\mathcal{C}\times\mathcal{C})$, and $L^\infty(\mathcal{C})\otimes L^\infty(\mathcal{C})$ "are" $\mathbb{C}$. Then (i) reduces to the identification of $L^2(\mathcal{M})\underset{Hilbert}{\otimes}L^2(\mathcal{A}')$ with $L^2(\mathcal{M}\vee\mathcal{A}')$; while (iii) identifies this same tensor product with $L^2(\mathcal{M}\times\mathcal{A}')$. The identification of $L^2(\mathcal{M}\times\mathcal{A}')$ and $L^2(\mathcal{M}\vee\mathcal{A}')$ may be realized at the set level by the mapping $\chi_{M\times A'}\mapsto\chi_{M\cap A'}$, and verifying (via independence) that this extends in the proper way to a unitary equivalence. This unitary equivalence then establishes a model for all composition operators generating factors. We note that when $\mathcal{W}$ is a factor, $C|_{L^2}(\mathcal{M})$ is irreducible, so that part b of the following result is truly a converse to part a.

THEOREM. a) *Suppose that* $\mathcal{W}$ *is a factor* (as above). *Define* S *on* $X\times X$ *by* $S(x,y)=(Tx,y)$. *Then the composition operator* C_s *on* $L^2(X\times X,\mathcal{M}\times\mathcal{A}',\mu\times\mu)$ *is unitarily equivalent to* C_T *on* $L^2(X,\mathcal{F},\mu)$.

b) *Suppose that* T_o *is a transformation such that its corresponding composition operator* C_{T_o} *on* $L^2(X_o,\mathcal{F}_o,\mu_o)$ *is irreducible* (*so that* $\mathcal{W}_{T_o}$ *is the ring of all operators on* $L^2(X_o,\mathcal{F}_o,\mu_o)$). *Let* $(X_1,\mathcal{F}_1,\mu_1)$ *be a probability space. Define* T *on* $X_o\times X_1$ *by* $T(x,y)=(T_ox,y)$. *Then* $\mathcal{W}_T$ *is a factor.*

Proof. Part a follows directly from the comments preceding the statement under consideration. As for part b, it is easy to see that the corresponding $\mathcal{M}_{\mathrm{T}}$ and $\mathcal{A}'_{\mathrm{T}}$ sigma subalgebras of $\mathcal{F}_o \times \mathcal{F}_1$ are given by

$$\mathcal{M}_{\mathrm{T}} = \mathcal{T}_o \times \mathcal{F}_1 \quad \text{and} \quad \mathcal{A}'_{\mathrm{T}} = \mathcal{F}_o \times \mathcal{T}_1,$$

where the $\mathcal{T}_i$'s are the appropriate trivial sub sigma algebras. It then follows that

$$\mathcal{M}_{\mathrm{T}} \vee \mathcal{A}'_{\mathrm{T}} = \mathcal{F}_o \times \mathcal{F}_1 \text{ and } \mathcal{M}_{\mathrm{T}} \cap \mathcal{A}'_{\mathrm{T}} = \mathcal{T}_o \times \mathcal{T}_1,$$

this last algebra being the trivial algebra in $\mathcal{F}_o \times \mathcal{F}_1$. Thus $\mathcal{W}_{\mathrm{T}}$ is a factor. □

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