Development of Mixed H_2/H_∞ Controller Design Algorithms for Singular Systems with Time Delay

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Abstract: In this paper, we consider the H_2(or guaranteed cost control) and H_∞ controller design methods for singular(or descriptor) systems with input time delay. Also, a mixed H_2/H_∞ controller design algorithm is treated by combination of the proposed H_2 and H_∞ controller design method. The sufficient conditions for the existence of controllers and controller design methods are introduced at each Lemma and Theorem. Furthermore, we present optimization problems to get the upper bound of performance measures. The proposed methods are checked by examples.

Keywords: guaranteed cost filtering, time delay systems, parameter uncertainty, LMI

I. Introduction

The special characteristics for singular(or descriptor) systems have drawn considerable attention due to extensive applications of singular systems in large scale systems, singular perturbation theory, and in particular, constraint mechanical systems. The singular form is a natural representation of linear dynamical systems, and makes it possible to analyze a larger class of systems than state space equations do[1], because state space equations cannot represent algebraic restrictions between state variables and some physical phenomena, like impulse and hysterisis which are important in circuit theory, cannot be treated properly. Many essential notions and results in control theory based on the state space form have been generalized for the descriptor form, such as LQ problem[2][3], controllability and observability [4], Lyapunov equations[3][5]-[7], and robust control [8][9], etc.

Two performance measures in optimal control theory which have been the focus of much recent research are H_2 and H_∞ norms. Recently, the descriptor H_∞ control problem has been considered by many researchers. Especially, Masubuchi *et al.*[1] considered the H_∞ control problem for descriptor systems that possibly have impulsive modes and/or jw axis zeros in order to eliminate the assumptions. Also, Takaba *et al.*[10] treated robust H_2 performance of uncertain descriptor systems. In order to get the robust performance, the control problem dealing with both H_2 and H_∞ norm measures has been formulated[11]-[14]. However, most mixed H_2/H_∞ papers did not consider the problem of singular systems, which is the first motivation of this paper. Therefore, we want to present the mixed H_2/H_∞ controller design algorithm for singular systems. In this paper, we consider just optimal H_2 performance measure instead of general H_2 method in H_2 control part. This part of H_2 control is somewhat similar to the guaranteed cost control problem or LQ(linear Quadratic) control problem.

Since the stability analysis and control of dynamic systems with time delay are problems of recurring interest as time delay often are the causes for instability and poor performance of control systems, the study of time delay systems has received considerable attention over the past years[15][16]. However, there are no papers considering H_2, H_∞, and mixed H_2/H_∞ controller design methods for singular systems with time delay, which is the second motivation of this paper. The second aim is to present not only the sufficient conditions for the existence of controllers but also the controller design algorithms for singular systems with time delay in H_2, H_∞, and mixed H_2/H_∞ control.

In this paper, we propose H_2 control method, H_∞ control technique, and mixed H_2/H_∞ control law for singular systems with input time delay by using Riccati inequality and linear matrix inequality approaches. At each section, the sufficient conditions of controller existence, the controller design algorithm, and the optimization problem to get the upper bound of performance measures are treated. Since the proposed conditions are linear matrix inequality form in terms of all finding variables, all solutions including controller gain and upper bound of performance measures can be calculated simultaneously. Also, the obtained controllers guarantee not only asymptotic stability of the closed loop system but also minimization of the upper bound of performance measures.

The following notations will be used in this paper. $(\cdot)^T$, $(\cdot)^{-1}$, $\deg(\cdot)$, $det(\cdot)$, $tr(\cdot)$, and $rank(\cdot)$ denote the transpose, inverse, degree, determinant, trace, and rank of a matrix. A positive definite matrix (negative definite matrix) X is denoted as $X>0$ ($X<0$). An identity matrix with proper dimensions is denoted as I. I_r denotes an identity matrix with $r\times r$ dimension. $x_r(t)$ means $r\times 1$ vector.

II. Problem formulation

Let us a linear time invariant singular(or descriptor) system with input time delay

$$\begin{aligned} E\dot{x}(t) &= Ax(t)+B_1u(t)+B_du(t-d)+B_2w(t) \\ z_1(t) &= C_1x(t) \\ z_2(t) &= C_2x(t) \\ x(t) &= \phi(t), \quad -d\le t\le 0 \end{aligned} \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the descriptor variable, $z_i(t)\in\mathbf{R}^q$, (i=1,2) is the controlled output variable, $u(t)\in\mathbf{R}^m$ is the

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control input variable, $w(t) \in \mathbf{R}^p$ is the disturbance input variable, $\phi(t)$ is an initial value function, and all matrices have proper dimensions. Here, time delay(d) is non-negative real number. We assume that E is singular matrix with $rank(E) = r \le n$ and the matrix (E, A) is regular. The property of regularity guarantees the existence and uniqueness of solution for any specified initial condition. In the following, we summarize some definitions and useful properties. If $det(sE-A)$ is not identically zero, a pencil $sE-A$(or a pair (E, A)) is regular. The singular system has no impulsive mode(or impulse free) if and only if

$$rank(E) = deg\, det(sE-A). \tag{2}$$

The assumption of impulse free ensures that singular system has no infinite poles. Associated with the system (1), we propose the following control law

$$u(t) = Kx(t) = -\frac{1}{\rho} B_1^T PEx(t). \tag{3}$$

When we apply the control (3) to the system (1), the resulting closed loop system is given by

$$\begin{aligned} E\dot{x}(t) &= A_K x(t) + B_d Kx(t-d) + B_2 w(t) \\ z_1(t) &= C_1 x(t) \\ z_2(t) &= C_2 x(t) \end{aligned} \tag{4}$$

where, $A_K = A + B_1 K = A - \frac{1}{\rho} B_1 B_1^T PE$. Also, we introduce H_2 performance(or guaranteed cost function) measure and H_∞ performance measure as follows:

$$J_1 = \int_0^\infty z_1(t)^T z_1(t) dt : \ H_2 \text{ performance measure}, \tag{5}$$

$$J_2 = \int_0^\infty [z_2(t)^T z_2(t) - \gamma^2 w(t)^T w(t)] dt \\ : \ H_\infty \text{ performance measure}. \tag{6}$$

Without loss of generality, we assume that the system matrices in (1) and some matrices have the following singular value decomposition form[1,2]:

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \ A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}, \\ B_d = \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix}, \ B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \\ C_1 = [C_{11} \ C_{12}], \ C_2 = [C_{21} \ C_{22}], \ Q_i = \begin{bmatrix} Q_{ir} & 0 \\ 0 & 0 \end{bmatrix}, (i = 1, 2), \tag{7}$$

where all decomposed matrices have appropriate dimensions and Q_{ir} is a positive definite matrix.

III. Main results

In this paper, we explain the H_2 control method in Lemma 1 and Theorem 1, H_∞ control technique in Lemma 2 and Theorem 2, and mixed H_2/H_∞ control problem in Theorem 3 for delayed singular systems, respectively.

1. H_2 control(or Guaranteed cost control)

In this section, we introduce the sufficient condition, H_2 controller design method, and the upper bound of H_2 performance measure. The objective is to minimize the H_2 performance measure satisfying the asymptotic stability of the closed loop system.

Definition 1: Consider the input delayed singular system (1) with zero disturbance input and the structure (7), if there exist a control law $u(t)^* = Kx(t) = -(1/\rho) B_1^T PEx(t)$ and a positive scalar J^* such that the closed loop system is asymptotically stable and the closed loop value of H_2 performance measure satisfies $J_1 \le J^*$, then J^* is said to be an upper bound of guaranteed cost function and $u(t)^*$ is said to be an H_2 control law for the system (1).

Lemma 1: Consider the system (1) with the structure (7) and assume that the disturbance input is zero and $C_{12} = 0$. If there exist a symmetric matrix P, $S > 0$, and a controller gain K satisfying

$$\begin{aligned} & E^T PE \ge 0 \\ & \begin{bmatrix} A_K^T PE + E^T PA_K + K^T SK + C_1^T C_1 & E^T PB_d \\ B_d^T PE & -S \end{bmatrix} \le 0 \end{aligned} \tag{8}$$

and if there exists a positive scalar satisfying $J_1 \le J_1^*$, then the control law $u(t)^* = Kx(t)$ is an H_2 control.

Proof: Firstly, we define a Lyapunov functional candidate as

$$V(Ex(t)) = x(t)^T E^T PEx(t) + \int_{t-d}^{t} x(\tau)^T K^T SKx(\tau) d\tau. \tag{9}$$

Here, P is a symmetric matrix satisfying $E^T PE \ge 0$, S is a positive definite matrix. Taking the derivative of (9) along the solution of the closed loop system (4) yields

$$\begin{aligned} \dot{V}(Ex(t)) &= \dot{x}(t)^T E^T PEx(t) + x(t)^T E^T PE\dot{x}(t) \\ &+ x(t)^T K^T SKx(t) - x(t-d)^T K^T SKx(t-d). \end{aligned} \tag{10}$$

The matrix inequality (8) implies

$$\dot{V}(Ex(t)) \le -z_1(t)^T z_1(t) < 0. \tag{11}$$

Therefore, we have

$$\begin{bmatrix} x(t) \\ Kx_d \end{bmatrix}^T \begin{bmatrix} A_K^T PE + E^T PA_K + K^T SK + C_1^T C_1 & E^T PB_d \\ B_d^T PE & -S \end{bmatrix} \times \begin{bmatrix} x(t) \\ Kx_d \end{bmatrix} \le 0 \tag{12}$$

which ensures the asymptotic stability of the closed loop system (4). Here, $x_d = x(t-d)$. Furthermore, by integrating both sides of the inequality (11) from 0 to T_f and using initial condition, we obtain

$$-\int_0^{T_f} z_1(t)^T z_1(t) dt > x(T_f)^T E^T PEx(T_f) - x(0)^T E^T PEx(0) \\ + \int_{T_f-d}^{T_f} x(\tau)^T K^T SKx(\tau) d\tau - \int_{-d}^{0} x(\tau)^T K^T SKx(\tau) d\tau. \tag{13}$$

As the closed loop system (4) is asymptotically stable, when $T_f \to \infty$,

$$x(T_f)^T E^T P E x(T_f) \to 0, \quad \int_{T_f-d}^{T_f} x(\tau)^T K^T S K x(\tau) d\tau \to 0. \tag{14}$$

Hence, we get

$$\int_0^\infty z_1(t)^T z_1(t) dt \le \phi(0)^T E^T P E \phi(0) + \int_{-d}^{0} \phi(\tau)^T K^T S K \phi(\tau) d\tau := J_1^* \tag{15}$$

This (15) is an upper bound of H_2 performance measure. ■

In the following Theorem 1, we present the optimization problem to get the upper bound of H_2 performance measure and the H_2 controller design method.

Theorem 1: If the following optimization problem

$$\text{minimize} \quad \{\alpha + tr(N^T G N)\} \quad \text{subject to} \tag{16}$$

i) $\begin{bmatrix} XA_1^T + A_1X - 2\varepsilon B_{11}B_{11}^T + B_{d1}YB_{d1}^T & XC_{11}^T & \varepsilon B_{11} \\ C_{11}X & -I & 0 \\ \varepsilon B_{11}^T & 0 & -Y \end{bmatrix} < 0,$

ii) $\begin{bmatrix} -\alpha & \phi_r(0)^T \\ \phi_r(0) & -X \end{bmatrix} < 0,$

iii) $\begin{bmatrix} -G & \varepsilon B_{11} \\ \varepsilon B_{11}^T & -Y \end{bmatrix} < 0,$

iv) $X > I$

has a positive definite solution, X, Y, G, α, ε, then (3) is an optimal H_2 controller(or optimal guaranteed cost controller) and $J^* = \alpha + tr(N^T G N)$ is an optimal guaranteed cost of H_2 performance measure. Here, some notations are defined as

$$X = P_1^{-1}, \quad Y = S^{-1}, \quad \varepsilon = \frac{1}{\rho}, \quad \phi(t) = \begin{bmatrix} \phi_r(t) \\ \phi_{n-r}(t) \end{bmatrix}, \quad \int_{-d}^{0} \phi_r(\tau)\phi_r(\tau)^T d\tau = NN^T. \tag{17}$$

Proof: By Schur complements, (8) is transformed into

$$A_K^T PE + E^T P A_K + K^T S K + C_1^T C_1 + E^T P B_d S^{-1} B_d^T P E \le 0. \tag{18}$$

In order to solve the above Riccati inequality, (18) is changed to

$$A_K^T PE + E^T P A_K + K^T S K + C_1^T C_1 + E^T P B_d S^{-1} B_d^T P E + Q_1 = 0. \tag{19}$$

If we apply (7) to (19), (19) can be expressed as follows:

$$\begin{aligned} &A_1^T P_1 + P_1 A_1 - (2/\rho)(P_1 B_{11} + P_2 B_{12})(P_1 B_{11} + P_2 B_{12})^T \\ &+ (1/\rho^2)(P_1 B_{11} + P_2 B_{12}) S (P_1 B_{11} + P_2 B_{12})^T + C_{11}^T C_{11} \\ &+ (P_1 B_{d1} + P_2 B_{d2}) S^{-1} (P_1 B_{d1} + P_2 B_{d2})^T + Q_{1_r} = 0 \end{aligned} \tag{20}$$

$$P_2 A_4^T + C_{11}^T C_{12} = 0 \tag{21}$$

$$A_4 P_2^T + C_{12}^T C_{11} = 0 \tag{22}$$

$$C_{12}^T C_{12} = 0 \tag{23}$$

Therefore, $C_{12} = 0$ from (23). And then, $P_2 = 0$ from (21) and (22) by letting $C_{12} = 0$ in the input delayed singular systems (1). (20) is expressed as

$$\begin{aligned} &A_1^T P_1 + P_1 A_1 - (2/\rho) P_1 B_{11} B_{11}^T P_1 + (1/\rho^2) P_1 B_{11} S B_{11}^T P_1 \\ &+ C_{11}^T C_{11} + P_1 B_{d1} S^{-1} B_{d1}^T P_1 = -Q_{1_r} < 0. \end{aligned} \tag{24}$$

Since the positive definite matrix Q_{1_r} can be selected, (24) can be transformed into i) of (16) using the Schur complements and changes of variables, $X = P_1^{-1}$, $Y = S^{-1}$, $\varepsilon = 1/\rho$. In the first term of (15), $\phi(0)^T E^T P E \phi(0) < \alpha$ is equivalent to ii) of (16). The second term of right hand side in (15) has the following relations.

$$\begin{aligned} &\int_{-d}^{0} \phi(\tau)^T K^T S K \phi(\tau) d\tau \\ &= (1/\rho^2) \int_{-d}^{0} tr(\phi_r(\tau)^T P_1 B_{11} S B_{11}^T P_1 \phi_r(\tau)) d\tau \\ &= (1/\rho^2) tr(NN^T P_1 B_{11} S B_{11}^T P_1) \\ &= (1/\rho^2) tr(N^T P_1 B_{11} S B_{11}^T P_1 N) \\ &< tr(N^T P_1 G P_1 N). \end{aligned} \tag{25}$$

Therefore, $-G + (1/\rho^2) B_{11} S B_{11}^T < 0$ is equivalent to iii) of (16) by Schur complements. It follows from (15) that

$$J_1^* < \alpha + tr(N^T P_1 G P_1 N) < \alpha + tr(N^T G N) := J^* \tag{26}$$

because of the condition (iv) in (16). In other words,

$$\begin{aligned} &P_1 G P_1 < G \\ &\Leftrightarrow XGX > G \;(\text{by } P_1^{-1} = X) \\ &\Leftrightarrow (X - I) G (X + I) > 0 \;(\text{by } X > I). \end{aligned} \tag{27}$$

■

Remark 1: It is well known that the given H_2 performance measure (5) can be changed to the LQ performance measure by simple modifications as follows:

$$\int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt. \tag{28}$$

Therefore, the considering problem includes the guaranteed cost control and LQ control methods.

2. H_∞ control

In this section, we present the sufficient condition, H_∞ controller design method, and the upper bound of H_∞ norm bound of the closed loop system. The objective is to minimize the H_∞ norm bound and guarantee the asymptotic stability of the closed loop system.

Definition 2: Consider the input delayed singular system (1) with the structure (7), if there exist a control law $u(t)^* = Kx(t) = -(1/\rho) B_1^T P E x(t)$ and a positive scalar γ^* such that the closed loop system is asymptotically stable and the closed loop value of H_∞ performance measure satisfies $\gamma \le \gamma^*$, then γ^* is said to be an upper bound of H_∞ norm and $u(t)^*$ is said to be an H_∞ control law for system (1).

Lemma 2: Consider the system (1) with the structure (7) and assume that $C_{22} = 0$. If there exist a symmetric matrix P, $S > 0$, $\gamma > 0$, and controller gain K satisfying

$$E^TPE \geq 0$$

$$\begin{bmatrix} A_K^TPE+E^TPA_K+K^TSK+C_1^TC_1 & E^TPB_d & E^TPB_2 \\ B_d^TPE & -S & 0 \\ B_2^TPE & 0 & -\gamma^2 I \end{bmatrix} \leq 0 \tag{29}$$

and if there exists a positive scalar satisfying $\gamma \leq \gamma^*$, then the control law $u(t)^* = Kx(t)$ is an H_∞ control law.

Proof: Similarly to the proof of Lemma 1, we take same Lyapunov functional (8). The matrix inequality (29) implies

$$\dot{V}(Ex(t)) \leq -z_2(t)^Tz_2(t)+\gamma^2 w(t)^Tw(t) < 0. \tag{30}$$

Therefore, we have

$$\begin{bmatrix} x(t) \\ Kx_d \\ w(t) \end{bmatrix}^T \times \begin{bmatrix} A_K^TPE+E^TPA_K+K^TSK+C_2^TC_2 & E^TPB_d & E^TPB_2 \\ B_d^TPE & -S & 0 \\ B_2^TPE & 0 & -\gamma^2 I \end{bmatrix} \times \begin{bmatrix} x(t) \\ Kx_d \\ w(t) \end{bmatrix} \leq 0. \tag{31}$$

■

In the following Theorem 2, we consider the optimization problem to get the upper bound of H_∞ norm of the closed loop system and the H_∞ controller design method.

Theorem 2: If the following optimization problem

$$\text{minimize } \beta \text{ subject to} \tag{32}$$

$$\begin{bmatrix} XA_1^T+A_1X-2\varepsilon B_{11}B_{11}^T+B_{d1}YB_{d1}^T & XC_{21}^T & \varepsilon B_{11} & B_{21} \\ C_{21}X & -I & 0 & 0 \\ \varepsilon B_{11}^T & 0 & -Y & 0 \\ B_{21}^T & 0 & 0 & -\beta I \end{bmatrix} < 0,$$

has a positive definite solution X, Y, ε, β, then (3) is an H_∞ controller and $\gamma^* = \sqrt{\beta}$ is an upper bound of H_∞ norm bound. Here, some notations are defined as

$$X = P_1^{-1},\ Y = S^{-1},\ \varepsilon = 1/\rho,\ \beta = \gamma^2. \tag{33}$$

Proof: Similarly to the proof of Theorem 1, (29) is changed to

$$A_K^TPE+E^TPA_K+K^TSK+C_2^TC_2 + E^TPB_dS^{-1}B_d^TPE+(1/\gamma^2)E^TPB_2B_2^TPE \leq 0. \tag{34}$$

In order to solve the above Riccati inequality, (34) is changed to

$$A_K^TPE+E^TPA_K+K^TSK+C_2^TC_2 + E^TPB_dS^{-1}B_d^TPE+(1/\gamma^2)E^TPB_2B_2^TPE+Q_2 = 0. \tag{35}$$

If we apply (7) to (35), (35) can be expressed as follows:

$$\begin{aligned} &A_1^TP_1+P_1A_1-(2/\rho)(P_1B_{11}+P_2B_{12})(P_1B_{11}+P_2B_{12})^T \\ &+(1/\rho^2)(P_1B_{11}+P_2B_{12})S(P_1B_{11}+P_2B_{12})^T+C_{21}^TC_{21} \\ &+(P_1B_{d1}+P_2B_{d2})S^{-1}(P_1B_{d1}+P_2B_{d2})^T \\ &+(1/\gamma^2)(P_1B_{21}+P_2B_{22})(P_1B_{21}+P_2B_{22})^T+Q_{2_r}=0 \end{aligned} \tag{36}$$

$$P_2A_4^T+C_{21}^TC_{22}=0 \tag{37}$$

$$A_4P_2^T+C_{22}^TC_{21}=0 \tag{38}$$

$$C_{22}^TC_{22}=0 \tag{39}$$

Therefore, $C_{22}=0$ from (39), and $P_2=0$ from (37) and (38) by letting $C_{22}=0$ in the input delayed singular systems (1). (36) is expressed as

$$\begin{aligned} &A_1^TP_1+P_1A_1-(2/\rho)P_1B_{11}B_{11}^TP_1+(1/\rho^2)P_1B_{11}SB_{11}^TP_1 \\ &+C_{21}^TC_{21}+P_1B_{d1}S^{-1}B_{d1}^TP_1+(1/\gamma^2)P_1B_{21}B_{21}^TP_1=-Q_{2_r}<0. \end{aligned} \tag{40}$$

Since the positive definite matrix Q_{2_r} can be chosen, (40) can be transformed into (32) using the Schur complements and changes of variables, $X=P_1^{-1}$, $Y=S^{-1}$, $\varepsilon=1/\rho$, $\beta=\gamma^2$. ■

3. Mixed H_2/H_∞ control

In this section, we propose the mixed H_2/H_∞ controller design method by combination of 3.1 and 3.2. The aim is to minimize the H_2 performance measure under satisfying the prescribed H_∞ norm bound of the closed loop system.

Definition 3: Consider the input delayed singular system (1) with structure (7), if there exist a control law $u(t)^*=Kx(t)=-(1/\rho)B_1^TPEx(t)$ and a positive scalar J^* satisfying the H_∞ norm bound within a prescribed γ, then J^* is said to be an upper bound of H_2 performance measure and $u(t)^*$ is said to be a mixed H_2/H_∞ control law for system (1).

Theorem 3: For a given positive scalar γ, if the following optimization problem

$$\text{minimize } \{\alpha+tr(N^TGN)\} \text{ subject to} \tag{41}$$

$$\text{(i)}\ \begin{bmatrix} XA_1^T+A_1X-2\varepsilon B_{11}B_{11}^T+B_{d1}YB_{d1}^T & XC_{11}^T & \varepsilon B_{11} \\ C_{11}X & -I & 0 \\ \varepsilon B_{11}^T & 0 & -Y \end{bmatrix} < 0,$$

$$\text{(ii)}\ \begin{bmatrix} XA_1^T+A_1X-2\varepsilon B_{11}B_{11}^T+B_{d1}YB_{d1}^T & XC_{21}^T & \varepsilon B_{11} & B_{21} \\ C_{21}X & -I & 0 & 0 \\ \varepsilon B_{11}^T & 0 & -Y & 0 \\ B_{21}^T & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0$$

$$\text{(iii)}\ \begin{bmatrix} -\alpha & \phi_r(0)^T \\ \phi_r(0) & -X \end{bmatrix} < 0,$$

$$\text{(iv)}\ \begin{bmatrix} -G & \varepsilon B_{11} \\ \varepsilon B_{11}^T & -Y \end{bmatrix} < 0,$$

$$\text{(v)}\ X > I$$

has a positive definite solution, X, Y, G, α, ε, then (3) is a mixed H_2/H_∞ controller and $J^*=\alpha+tr(N^TGN)$ is an optimal guaranteed cost of H_2 performance measure.

Proof: The proof follows in a straightforward way from the proofs of Lemma 1, Lemma 2, Theorem 1, and Theorem 2. ■

Remark 2: The optimization problem in Theorem 1, Theorem 2, and Theorem 3 can be easily solvable using the command of 'mincx' in LMI Toolbox[17].

Remark 3: The proposed results can be extended the time-varying delay systems by simple modifications[15]. Also, the results can be applicable singular systems without time delay directly.

IV. Example

In order to check the validities of the proposed methods, we consider an input delayed singular system

$$
\begin{aligned}
\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\dot{x}(t) &= \begin{bmatrix}-3&1&0\\1&1&0\\0&0&1\end{bmatrix}x(t)+\begin{bmatrix}1\\2\\1\end{bmatrix}u(t) \\
&\quad+\begin{bmatrix}0.1\\0.3\\0.2\end{bmatrix}u(t-d)+\begin{bmatrix}1\\1\\0\end{bmatrix}w(t) \\
z_1(t) &= [1\ 2\ 0]x(t) \\
z_2(t) &= [3\ 1\ 0]x(t) \\
d &= 2,\ \phi(t) = [e^{t+1}\ 0\ 0]^T.
\end{aligned} \tag{42}
$$

All solutions can be calculated at the same time from the LMI Toolbox because the proposed optimization problems are LMI forms regarding finding all variables.

(H_2 control)

The solutions satisfying Theorem 1 are as follows:

$$
\begin{aligned}
X &= \begin{bmatrix}2.8406 & -0.0003\\ -0.0003 & 0.0011\end{bmatrix},\ Y = 0.0018, \\
G &= \begin{bmatrix}0.0014 & -0.0006\\ -0.0006 & 0.0015\end{bmatrix},\ \alpha = 2.6016, \\
\varepsilon &= 3.4713\times 10^{-4}.
\end{aligned} \tag{43}
$$

Therefore, the H_2 control law and the upper bound of H_2 performance measure are

$$
\begin{aligned}
u(t)^* &= [-0.0002\ \ -0.6185\ \ 0]x(t), \\
J^* &= 2.6067.
\end{aligned} \tag{44}
$$

The obtained H_2 controller(or guaranteed cost controller) guarantees asymptotic stability of the closed loop system in spite of time delay. The trajectories of states and controlled output signal are shown in Fig. 1.

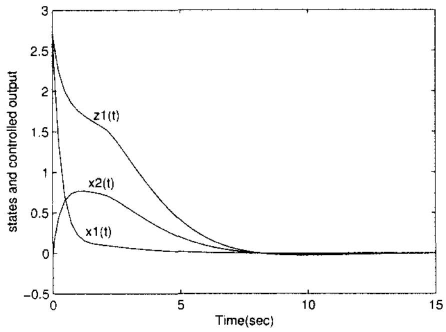


Fig. 1. The trajectories of states and controlled output with H_2 controller.

(H_∞ control)

The solutions satisfying Theorem 2 are as follows:

$$
\begin{aligned}
X &= 10^8\times\begin{bmatrix}2.9548 & -1.4775\\ -1.4775 & 0.7388\end{bmatrix},\ Y = 7.2394\times 10^8, \\
\varepsilon &= 5.6853\times 10^8,\ \beta = 5.6601\times 10^{-10}.
\end{aligned} \tag{45}
$$

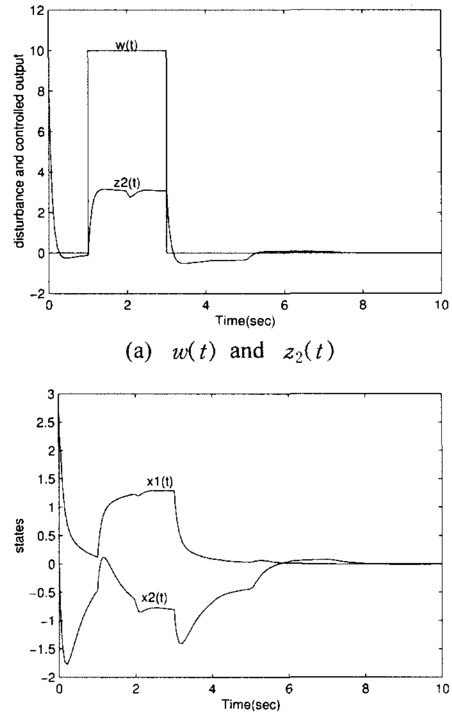
Therefore, the H_∞ control law and the upper bound of γ performance measure are

$$
\begin{aligned}
u(t)^* &= 10^5\times[-2.0937\ \ -4.1872\ \ 0]x(t), \\
\gamma^* &= 2.3791\times 10^{-5}.
\end{aligned} \tag{46}
$$

In the case of applications, the designer can select a value of γ according to the desired performance. If we set $\gamma=1$, all solutions and H_∞ control law are obtained as follows:

$$
\begin{aligned}
X &= \begin{bmatrix}0.2043 & -0.2275\\ -0.2275 & 0.8799\end{bmatrix},\ Y = 1.9086, \\
\varepsilon &= 0.5679, \\
u(t)^* &= [-5.9214\ \ -2.8216\ \ 0]x(t).
\end{aligned} \tag{47}
$$

If we define the disturbance input like (a) of Fig. 2 for computer simulation, then the states and controlled output are shown in Fig. 2. Therefore, the obtained H_∞ controller guarantees not only asymptotic stability of the closed loop system but also H_∞ norm within a prescribed bound against input time delay and disturbance input.



(a) $w(t)$ and $z_2(t)$

(b) $x_1(t)$ and $x_2(t)$

Fig. 2. The trajectories of disturbance input, controlled output, and states with H_∞ controller.

(Mixed H_2/H_∞ control)

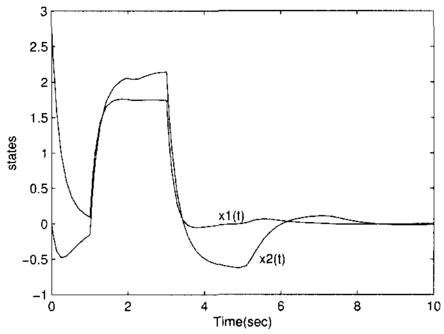
For a given $\gamma=1$, the solutions satisfying Theorem 3 are as follows:

$$
\begin{aligned}
X &= \begin{bmatrix}0.8274 & -0.2913\\ -0.2913 & 0.6963\end{bmatrix},\ Y = 1.7396, \\
G &= \begin{bmatrix}0.2138 & 0.2963\\ 0.2963 & 0.6980\end{bmatrix},\ \alpha = 10.4748,\ \varepsilon = 0.5343.
\end{aligned} \tag{48}
$$

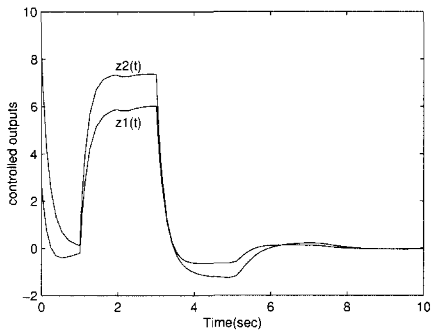
Therefore, the mixed H_2/H_∞ control law and the upper bound of H_2 performance measure are

$$
\begin{aligned}
u(t)^* &= [-1.3909\ \ -2.1165\ \ 0]x(t), \\
J^* &= 11.2501.
\end{aligned} \tag{49}
$$

Similarly to the previous two examples, the obtained mixed H_2/H_∞ controller guarantees the desired two performances and asymptotic stability of the closed loop system. The trajectories of states and controlled outputs are displayed in Fig. 3 for the same disturbance input, i.e., (a) of Fig. 2. Therefore, if we can minimize H_2 performance measure(or guaranteed cost function) and H_∞ norm of the closed loop system from $w(t)$ to $z_2(t)$, then the H_2 norm in the closed loop system from $w(t)$ to $z_1(t)$ can be reduced at second hand. However, this is not a pure mixed H_2/H_∞ control method. A future research would be to develop the pure mixed H_2/H_∞ controller design method which minimizes the H_2 norm of a closed loop system from $w(t)$ to $z_1(t)$ satisfying a prescribed H_∞ norm bound on another closed loop system from $w(t)$ to $z_2(t)$.



(a) $x_1(t)$ and $x_2(t)$



(b) $z_1(t)$ and $z_2(t)$

Fig. 3. The trajectories of states and controlled outputs with mixed H_2/H_∞ controller.

V. Conclusion

This paper considered the design problems of H_2, H_∞, and mixed H_2/H_∞ controller for singular systems with time delay by Riccati inequality and linear matrix inequality approach. The presented controllers guaranteed the asymptotic stability and the minimization of upper bound in performance measures. The sufficient conditions expressed by linear matrix inequality form, the controller design methods, and the optimization problems to get the upper bound of performance measures were proposed. Finally, the validities of the controller design methods were checked by numerical examples.

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