An Eeffective Mesh Generation Algorithm Using Singular Shape Functions

Hyeong Seon Yoo, Jun-Hwan Jang and Soo Bum Pyun

Abstract: In this paper, we propose a simplified pollution adaptive mesh generation algorithm using singular elements. The algorithm based on the element pollution error indicator concentrate on boundary nodes. The automatic mesh generation method is followed by either a node-relocation or a node-insertion method. The boundary node relocation phase is introduced to reduce pollution error estimates without increasing the boundary nodes. The node insertion phase greatly improves the error and the factor with the cost of increasing the node numbers. It is shown that the suggested r-h version algorithm combined with singular elements converges more quickly than the conventional one.

Keywords: pollution error, local error, singularity, modifred shape-function, r-h method, mesh generation

I. Introduction

Most engineering problems are described in polygonal domains with geometric singularities. These singularities make the solution diverge to infinity and cause the conventional error estimators to severely underestimate the error in any patch outside the neighborhood of the singular point. Since Babuska's works about error estimators and pollution errors it is known that the pollution error estimates are much more than the local error ones [1]-[4]. It was demonstrated that the conventional Zienkiewicz-Zhu error estimator [5]-[8] was insufficient and should include a pollution error indicator [1][4]. The pollution-adaptive feedback algorithm employs both local error indicators and pollution error indicators to refine the mesh outside a larger patch, which includes a patch and one to two surrounding mesh layers [2][3]. The conventional pollution adaptive algorithm bisects the element for every iteration and needs a lot of iterations to converge.

Special elements with singular shape functions were developed to overcome singularites in finite element alalysis [9]. It seems that the elements can be combined to accelerate in the pollution adaptive algorithms. We concentrate only on a problem boundary since the singularities exist on the boundary and mesh sizes change gradually regardless of the mesh generation algorithm. A mesh generation algorithm, which uses a node relocation method (r-method) as well as h-method of the finite element method for boundary elements, is proposed. The algorithm employs a boundary-node relocation at first and then does a node insertion based on the pollution error indicator.

II. The model problem

Consider a typical L-shaped polygon in two dimension, $\Omega \subseteq R^2$, with mixed boundaries $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \{\}$ where Γ_D is the *Dirichlet* and Γ_N is the *Newmann* boundary (Fig.1).

We will consider Laplacian with mixed boundary conditions. Let us consider the Hilbert space satisfying boundary condition $H^1_{\Gamma_D} \equiv \{u \in H^1(\Omega) \mid u = 0 \ \ on \ \ \Gamma_D\}$. Then the variational formulation of this model problem satisfies (1). And the solution space will be S^p_{h,Γ_D} combining the Hilbert and a trial function space S^p_h , [1].

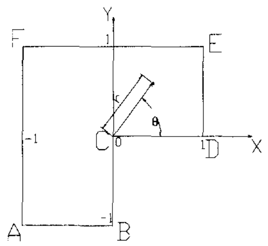

Fig. 1. The L-shaped domain for the model problem.

Find $u_h \in S^p_{h,\Gamma_D}(\Omega) := H^1_{\Gamma_D} \cap S^p_h$ such that

$$B_\Omega(u_h, v_h) = \int_{\Gamma_N} g v_h \quad \forall v_h \in S^p_{h,\Gamma_D} \tag{1}$$

where, $S^p_{h,\Gamma_D} \equiv \left\{ v \in C^0(\Omega) \middle| \ v|_\tau \in p_p(\tau) \ \forall v \in T_h, v = 0 \ \ on \ \Gamma_D \right\}$

A patch error was expressed only by a local error, but it was demonstrated that the pollution error should include the patch error. The local error was improved by considering a mesh patch ω_h with a few surrounding mesh layers. The equilibrated residual functional is the same for the local error and the pollution error. But the pollution error was calculated by considering the outside of the larger patch, ω_h .

$$e_h|_{\omega_h} = V_1^{\omega_h} + V_2^{\overset{\vee}{\omega}_h} \tag{2}$$

where, $V_1^{\omega_h}$; local error on ω_h

$V_2^{\omega_h}$; pollution error on ω_h

ω_h ; ω_h + a few mesh layers

Let us denote $\|v\|_S = \sqrt{B_S(v,v)}$ energy norm over any domain $S \subseteq \Omega$, then the equation (3) can be a pollution estimator with $\bar{x} \in \omega_h$, [1,2,3].

$$\left\| V_2^{\omega_h} \right\|_{\omega_h} \cong \sqrt{|\omega_h|} \sqrt{\left(\frac{\partial V_2^{\omega_h}}{\partial x_1}(\bar{x}) \right)^2 + \left(\frac{\partial V_2^{\overset{x}{\omega}_h}}{\partial x_2}(\bar{x}) \right)^2} \tag{3}$$

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III. The proposed algorithm

1. The basic idea

For adaptive control of the pollution error in a patch of interest, the conventional algorithm fixes meshes in the patch and refines meshes outside the patch especially near singularities. The algorithm calculates an element pollution indicator and regularly divides $\gamma\%$ of elements whose pollution indicators are high [2]. This algorithm is as following Fig.2.

```
Initialize mesh T_h = T_h^0
  Compute the finite element solution on T_h , ε_{ω_h} and M_{ω_h} ;
  While ( M_{ω_h} > t% of ε_{ω_h} ) do
     For(each element )do
         Compute μ_τ , τ ∈ T_h, τ ∉ ω_h ;
         If( μ_τ ≥ γ max μ_τ )
            Subdivided τ regularly;
         Endif
     Endfor
     Compute the finite element solution on T_h and
  ε_{ω_h} , M_{ω_h} ;
  Endwhile
```

Fig. 2. Structure of the conventional algorithm.

In Fig.2 we denote the element pollution error M_{ω_h}, the local error ε_{ω_h} and the element pollution indicator μ_τ [2]. Since the conventional algorithm bisects the element length, it could be accelerated if we have smaller boundary elements near the singular points. Therefore it is natural to think about combining r and h method.

In our proposed algorithm, special elements with singular shape functions were adopted to overcome singularites. The special elements can be combined to accelerate in the pollution adaptive algorithms. Our algorithm employs two ideas. The first is to adopt special elements with singular shape functions. The other is to use r-h algorithm in which the node relocation phase and the node insertion phase are employed alternatively. In the relocation phase, the new boundary element length is calculated by using the following relationship between the pollution error estimator and the element size h, [1].

$$\text{Let } \left\|V_2^{\omega_h}\right\|_{\omega_h} \approx h^{2\lambda+1} \tag{4}$$

where λ ; the exponent for singular point.

From this expression, we can deduce old and new element length as following,

$$\left\|V_2^{\omega_h}\right\|_{\omega_h,old} = Ch_{old}^{2\lambda+1} \tag{4'}$$

$$\left\|V_2^{\omega_h}\right\|_{\omega_h,new} = Ch_{new}^{2\lambda+1} \tag{4''}$$

Combining two equations, we obtain the new element size h_{new},

$$h_{new} = h_{old} \times \left(\left\|V_2^{\omega_h}\right\|_{\omega_h,old} / \left\|V_2^{\omega_h}\right\|_{\omega_h,new} \right)^{-\frac{1}{2\lambda+1}} \tag{5}$$

In order to get the pollution error smaller than the local error we use $t\varepsilon_{\omega_h} \approx t\left\|V_1^{\omega_h}\right\|_{\omega_h}$ instead of $\left\|V_2^{\omega_h}\right\|_{\omega_h,new}$. t is a user-specified constant between 0 and 1. And $\left\|V_2^{\omega_h}\right\|_{\omega_h,old}$ will be $\mu_\tau \approx \left\|V_2^{\omega_h}\right\|_{\omega_h} / \sqrt{|\omega_h|}$ since the pollution error consists of the element pollution error indicators outside $\tilde{\omega}_h$. Finally the new element size becomes,

$$h_{new} = h_{old} \times (\zeta_\tau)^{-\frac{1}{2\lambda+1}} \tag{6}$$

$$\text{where } \zeta_\tau \equiv \frac{\mu_\tau}{t\varepsilon_{\omega_h}}$$

This new element size has an effect on the location of the boundary node, especially the nodes on BC and CD in Fig 1. If the ratio of the element length $(\zeta_\tau)^{-\frac{1}{2\lambda+1}}$ is less than 1, the algorithm moves the node to the singular point. But if it is greater than 1, the new length is discarded and the location of the node remains fixed to have stable solution. This relocation method is for reducing the number of iteration to get the final mesh. The boundary node insertion phase takes part in a high quality of the error estimator, this phase is the same as others [1]-[3].

2. The proposed algorithm

For singular shape functions, we follow a generation scheme developed by Huges and Akin [9]. The r-directional shape function of node i, $N_i(r)$ can be expressed as the following algorithm.

$$\text{Step 1. } N_{m+1}(r) \leftarrow \frac{N_{m+1}(r) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r)}{N_{m+1}(r_{m+1}) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r_{m+1})} \tag{7}$$

Step 2. $N_a(r) \leftarrow N_a(r) - N_a(r_{m+1}) N_{m+1}(r)$ $a = 1,2,n,m$

Step 3. If $m+1 < n$ (number of nodes), replace m by $m+1$ and repeat steps 1-3

If $m+1 = n$, stop.

A binary number Flag is employed to alternate the boundary relocation and the node insertion process. If the flag is 0, the relocation phase is performed. Figure 3 shows the entire procedure.

The algorithm starts with the initial mesh and set Flag 0. The boundary node relocation is controlled by $(\zeta_\tau)^{-\frac{1}{2\lambda+1}}$. If the value is below 1, the element shrinks to singular point. In the node insertion phase, a new node is added on the middle of the boundary element. This r-h method makes fewer nodes on the boundary than the h-version.

The interior mesh generation phase is following the control of nodes on boundaries. This step is performed by the constrained Delaunay method. And the finite element analysis and error estimations are following.

```
Initialize mesh  T_h = T_h^0  and set Flag = 0
 Compute the finite element solution on  T_h , ε_{ω_h} and  M_{ω_h} ;
 While ( M_{ω_h} > t% ε_{ω_h} ) do
   Switch ( Flag )
     Case  0 :  /* relocation phase*/
       For ( each element on boundary) do
          Calculate μ_τ ;
          Calculate ζ_τ and h^(k+1) ;
          If( (ζ_τ)^(-2λ+1) < 1.0 )  do
          Relocate the node of the element on boundary;
          Endif
       Endfor
       Set Flag = 1;
       Break ;
     Case  1:    /* node-insertion phase */
      /* The same as Fig.2 */
      Set Flag = 0;
  Endswitch
  Generate mesh using nodes on boundary ;
    Compute the finite element solution on  T_h  and
 ε_{ω_h} , M_{ω_h} ;
 Endwhile
```

Fig. 3. Structure of the proposed algorithm.

IV. Numerical results and discussions

We considered the mixed boundary-valued problem for the Laplacian over a L-shaped domain and applied boundary conditions consistent with the exact solution $u(r,\theta) = r^{\frac{1}{3}}\sin(\frac{1}{3}\theta)$ [1]. An interior patch element ω_h far from the singular point as in Fig.4 is chosen. The L-shaped domain is meshed by uniform quadratic triangles (p=2) with h=0.125. In table.1, we show the numerical results for the model problem. Though the local error estimate (ε_{ω_h}) is almost constant, the pollution error decreases dramatically with iteration in the r-h version.

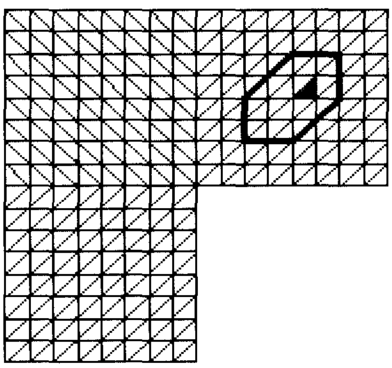

Fig. 4. The initial mesh for numerical example.

ω_h : patch, a shaded triangular element

ω_h : large patch, elements enclosed by thick hexagonal line

Table 1 shows the results of the model problem. The local error estimates change little and are nearly the same for both cases. The pollution error converges after 4 iterations for the conventional shape function case, But the modified shape function case is acceptable even for the start and changes a little.

The pollution factor is defined by the ratio of a pollution error estimate, $\beta_\tau = M_{\omega_h} / \varepsilon_{\omega_h}$. In Fig.5 we can see that the pollution factor decrease quickly for the conventional shape function case, but one iteration is enough for the modified shape function case. From this result, we note that the proposed algorithm with modified shape function case controls the pollution error and is effective.

Fig.6 shows that the effectivity index has nearly the same tendency as the pollution factor. It is almost 1 for the first iteration, but in the conventional shape function case it needs 4 iterations to converge. In Fig. 7 we show the final mesh, which is obtained by the proposed algorithm with modified shape functions. The final number of element is 1,696 and the node 3,493.

Table 1. Results of the model problem.

Iter.	ε_{ω_h} x E-05		M_{ω_h} x E-05	
	A	B	A	B
1	8.29	8.29	21.90	4.32
2	8.33	8.33	7.00	4.03
3	8.33	8.33	6.09	3.85
4	8.32	8.32	2.88	2.75

A: Conventional shape functions

B: Modified shape functions

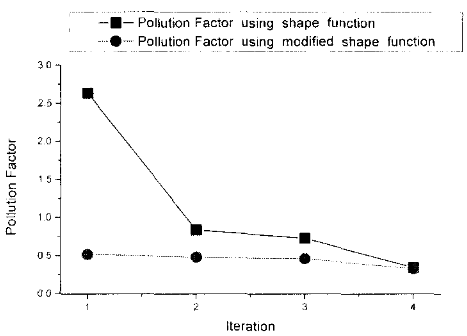

Fig. 5. The pollution factor, β_τ versus iteration.

VI. Conclusions

The pollution factor shows that the algorithm converges after 4 iterations for both conventional and singular shape function cases. But the singular shape function case shows less value than the conventional counterpart from the start. The local effectivity index shows the nearly the same tendency.

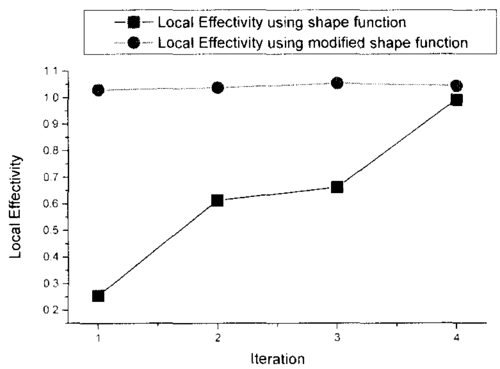

Fig. 6. The effectivity index versus iteration.

The modified shape function case shows fast convergence and is nearly 1 during iterations. The proposed r-h algorithm with modified shape functions is easy to handle since it considers only the boundary elements. The boundary node-relocation phase is very effective for this fast convergence. It is proved that the well known Delaunay method in this pollution adaptive algorithm is effective.

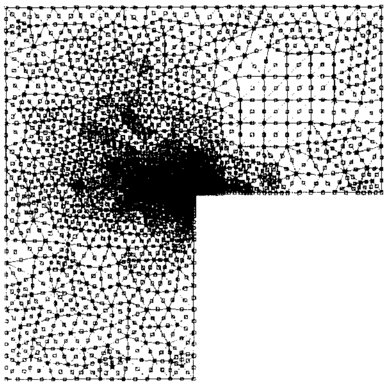

Fig. 7. The final mesh after 4 iterations by the proposed algorithm(N=3493,E=1696).

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