A note on fuzzy knowledge spaces

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요 약
최근 수학구조 및 교수습습과 관련된 연구에 지식공간 이론을 응용하고자 하는 논문들이 많이 나오고 있다. 실제로 유의미한 학습과 관련된 수학문제와 수학문제를 붙는 능력에 관련 평가를 연구하는데 지식구조가 응용되고 있지만 이를 활용하는데는 많은 제약사항이 있으며 이를 보완하기 위한 여러 가지 방법이 연구되어오고 있다. 특히, Schrepp 교수는 수학문제의 경 우에 제한하여 지식공간을 응용한 일반화된 수학구조의 연구방법을 제시하였다. 본 논문에서는 수학적 지식의 평가를 개선할 수 있는 수학구조 및 공간에 관한 연구를 하여서 효과적으로 응용될 수 있는 수학적 지식공간에 관한 체계적인 기초 이론을 정립하고 그 성질들을 연구하고자 한다.

Key words: Knowledge spaces, fuzzy knowledge spaces, fuzzy sets, fuzzy surmise relations, fuzzy surmise functions.

1. Introduction

Knowledge space theory was initiated by Doignon and Falmagne [2]. A knowledge domain is represented by a finite set \( Q \) of problems and a knowledge state is the set of problems a person is capable of solving. A family \( \mathcal{T} \) of knowledge states is called a knowledge structure if the empty set and \( Q \) are elements of \( \mathcal{T} \). When \( \mathcal{T} \) is closed under union, the structure is called a knowledge space (see [1,2,3,4,5]).

We consider two main applications of knowledge space theory in M. Schrepp [4,5]. First, knowledge spaces can be used for an efficient assessment of knowledge. Second, knowledge spaces can be used to test psychological models of problem solving processes. Psychological models of problem solving processes describe the basic cognitive abilities subjects must possess in order to solve problems from the underlying knowledge domain. Such a detailed analysis of the cognitive processes allows one often not only to predict whether or not a subject with specific abilities will solve a problem, but also to predict how far the subject will come in his/her effort to solve problem. Therefore, the assumption that every problem is solved either correctly or incorrectly by a subject is too restrictive. M. Schrepp [4] obtained that the assumption were generalized to problem domains in which solutions were evaluated on a linear scale concerning their equality.

In references [7, 8, 9], we have studied some new concepts and their applications in fuzzy set theory. Fuzzy set theory is very useful tool to the issue which concerns the effects of vagueness. So, we will use fuzzy set theory to generalize knowledge space theory. In particular, we define new concepts of fuzzy knowledge structures, fuzzy knowledge spaces, quasi-ordinal fuzzy knowledge spaces, fuzzy surmise relations, fuzzy surmise functions and investigate some properties of them.

2. New concepts and basic properties.

In this section, we introduce new concepts of fuzzy knowledge space theory. Using this concept of a fuzzy set in [10], we define a fuzzy knowledge structure and a fuzzy knowledge space. Let \( \mathcal{S}(Q) \) be the power set of a finite set \( Q \) of problems. A fuzzy set \( \mathcal{V} \) of a set \( \mathcal{S}(Q) \) is defined by

\[
\mathcal{V} = \{(A, m_\mathcal{V}(A)) | A \in \mathcal{S}(Q)\}
\]

where \( m_\mathcal{V} : \mathcal{S}(Q) \rightarrow [0,1] \) is a function and it is called the membership function of a fuzzy set \( \mathcal{S}(Q) \).

Definition 2.1 A fuzzy set of \( \mathcal{S}(Q) \) is called a fuzzy knowledge structure \( \mathcal{V} \) on \( Q \) if \( m_\mathcal{V}(\emptyset) = 1 \) and \( m_\mathcal{V}(Q) = 1 \).

Definition 2.2 A fuzzy knowledge structure \( \mathcal{V} \) is called a fuzzy knowledge space on \( Q \) if for each \( F_1, F_2 \subset \mathcal{S}(Q) \backslash \{\emptyset\} \),

\[
m_\mathcal{V}(F_1 \cup F_2) = \max\{m_\mathcal{V}(F_1), m_\mathcal{V}(F_2)\}.
\]

Definition 2.3 A fuzzy knowledge space \( \mathcal{V} \) is called a quasi-ordinal fuzzy knowledge space on \( Q \) if for each \( F_1, F_2 \subset \mathcal{S}(Q) \backslash \{\emptyset\} \),
m_\Psi(F_1 \cup F_2) = \min(m_\Psi(F_1), m_\Psi(F_2)).

Let 0 ≤ α ≤ 1. We note that

Ψ^α = \{F ∈ Ψ(\mathbb{Q}) | m_\Psi(F) ≥ α\}

is called the α-level knowledge space of a fuzzy knowledge space Ψ. Clearly, we then have the following theorem.

**Theorem 2.4** (1) If Ψ is a fuzzy knowledge space on Q and 0 ≤ α ≤ 1, then the α-level knowledge space Ψ^α on Q is a knowledge space.

(2) If Ψ is a quasi-ordinal fuzzy knowledge space on Q and 0 ≤ α ≤ 1, then the α-level quasi-ordinal knowledge space Ψ^α on Q is a quasi-ordinal knowledge space.

**Proof.** (1) If F_1, F_2 ∈ Ψ^α, then we have that m_\Psi(F_1) ≥ α and m_\Psi(F_2) ≥ α. Thus

m_\Psi(F_1 \cup F_2) = \max(m_\Psi(F_1), m_\Psi(F_2)) ≤ α.

That is, F_1 \cup F_2 ∈ Ψ^α.

(2) By (1), Ψ^α is a knowledge space. Let F_1, F_2 ∈ Ψ^α. We then have that m_\Psi(F_1) ≥ α and m_\Psi(F_2) ≥ α. Thus

m_\Psi(F_1 \cap F_2) = \min(m_\Psi(F_1), m_\Psi(F_2)) ≥ α.

That is, F_1 \cap F_2 ∈ Ψ^α.

3. Fuzzy surmise relations and fuzzy surmise functions.

Let (Q, 7) be a knowledge structure and ∈ is a surmise relation on Q defined by

p ∈ q ⇔ p ∈ 7 q

where p, q ∈ Q and 7 q = \{F ∈ 7 | q ∈ F\}. In this section, we will define a fuzzy surmise relation and discuss some their properties.

**Definition 3.1** Let (Q, Ψ) be a fuzzy knowledge structure and 0 ≤ α ≤ 1.

(1) A fuzzy surmise relation ⊆^α on Q is defined by

p ⊆^α q ⇔ p ∈ 7^α q

where 7^α = \{F ∈ Ψ(Q) | q ∈ F and m_Ψ(F) > 0\}.

(2) The α-level surmise relation ⊆^α on Q of ∈^α is defined by

p ⊆^α q ⇔ p ∈ 7^α q.

Since Ψ^α is a knowledge space, it is easily to show that ⊆^α is a surmise relation on Q for each a = (0, 1].

**Theorem 3.2** If Ψ is a fuzzy knowledge space on Q and q ∈ Q, then we have that

(1) Ψ_q is a finite set.

(2) inf{m_Ψ(F) | F ∈ Ψ_q} > 0, and

(3) there exists a_0 ∈ (0, 1) such that Ψ_q = Ψ^a_0.

**Proof.** (1) Since Q is a finite set, the power set 8(Q) of Q is finite.

(2) By (1), Ψ_q is a finite set and hence

\{m_Ψ(F) ∈ (0, 1) | F ∈ Ψ_q\}

is a finite set. That is, there exists F_0 ∈ Ψ_q such that

inf{m_Ψ(F) ∈ (0, 1) | F ∈ Ψ_q} = m_Ψ(F_0) > 0.

(3) Clearly, Ψ^a_0 ⊆ Ψ_q. Let F ∈ Ψ_q. By (2), there exists

a_0 = inf{m_Ψ(F) | F ∈ Ψ_q} > 0

and hence m_Ψ(F) ≥ a_0. So F ∈ Ψ^a_0. That is, Ψ_q ⊆ Ψ^a_0.

We remark that since Ψ_q is a finite set, there exists a set \{a_1, a_2, ..., a_n\} such that 0 ≤ a_1 < a_2 < ... < a_n ≤ 1 and

Q ∩ Ψ^a_1 ⊆ Ψ^a_2 ⊆ ... ⊆ Ψ^a_n ⊆ Ψ_q.

**Theorem 3.3** If Ψ is a fuzzy knowledge space on Q and if any p, q ∈ Q and p ⊆^a q and a_0 is as in Theorem 3.2 (3), then there exists a_1 = [a_0, 1] such that p ⊆^a_1 q.

**Proof.** By Theorem 3.2 (3), there exists a_2 ∈ (0, 1] such that Ψ_q = Ψ^a_2. We can choose a_1 with

a_1 = \inf{\{a ∈ [a_0, 1] | p ⊆^a q\}}.

Since \{a ∈ [a_0, 1] | p ⊆^a q\} is finite,

a_1 = \inf{\{a ∈ [a_0, 1] | p ⊆^a q\}}.

We then have p ⊆^a q and so p ⊆^a_1 q.

Now, we define a fuzzy relation ∈_\mathcal{A} by

p ⊆^a_1 q ⇔ m_Ψ(p, q) > 0

where m_\mathcal{A} : Q × Q → [0, 1] is a function.

**Definition** Defined by

m_\mathcal{A}(p, q) = \inf\{m_Ψ(F) | F ∈ Ψ_q\} for all p, q ∈ Q. Then, we have the following theorem.

**Theorem 3.4** (1) If Ψ is a fuzzy knowledge space on Q and if any p, q ∈ Q and p ⊆^a_1 q, then there exists a_2 ∈ (a_0, 1] such that m_\mathcal{A}(p, q) = a_2 and p ⊆^a_2 q.

(2) If Ψ is a fuzzy knowledge space on Q and if any p, q ∈ Q with p ⊆^a_1 q and m_\mathcal{A}(p, q) = a_2, then p ⊆^a_2 q.
Proof. (1) By the definition of $\rho \subseteq \gamma q$, we can choose $a_2 \in [a_1, 1]$ such that $a_2 = \rho \subseteq \gamma q$. Since $\Psi_\alpha$ is finite, $m_\alpha(F) \geq a_2$ for all $F \subseteq \Psi_\alpha$ and $p \subseteq F$.

We then have $p \subseteq \bigcap \Psi_\alpha^p$.

(2) If $p \subseteq \gamma q$, then by (1) we can take $a_2$ with $p \subseteq \bigcap \Psi_\alpha^p$. We then have $p \subseteq \gamma q$.

Using Theorem 3.3(3) and Theorem 3.4(2), we note that if $p, q \subseteq \Psi$ and $a = a_1 = a_2$, then $\alpha$-level surmise relations of these two fuzzy surmise relations are equal, that is, $\gamma q_1 = \gamma q_2$. We also will have more properties of fuzzy surmise relation.

**Theorem 3.5** If $\Psi$ is a fuzzy knowledge space on $Q$ and any $p \subseteq \Psi$, then $p \subseteq \gamma p$ and $p \subseteq \gamma p$.

**Proof.** Let $p \subseteq \Psi$. We clearly have $p \subseteq \bigcap \Psi_\alpha^p$ and so $p \subseteq \gamma p$. By Theorem 3.2(3), there exists $a_0 \in (0, 1]$ such that $\Psi_\alpha = \Psi_\alpha^{a_0}$. We note that for all $F \subseteq \Psi_\alpha$, we have $m_\alpha(F) \geq a_0$. Since $m_{\times}(p, q) = \inf \{m_\alpha(F) | p \subseteq F \text{ for all } F \subseteq \Psi_\alpha\}$, $m_{\times}(p, q) \geq a_0 > 0$ and so $p \subseteq \gamma q$.

From Theorem 3.5, we have the following corollary.

**Corollary 3.6** If $\Psi$ is a fuzzy knowledge space on $Q$ and fuzzy surmise relations $\subseteq \gamma$ and $\subseteq \gamma$ on $Q \times Q$ are reflexive.

**Theorem 3.7** If $\Psi$ is a quasi-ordinal fuzzy knowledge space on $Q$ and if $p \subseteq \gamma q$ and $q \subseteq \gamma r$ for all $p, q, r \subseteq Q$, then we have $p \subseteq \gamma r$.

**Proof.** Let $p \subseteq \gamma q$ and $q \subseteq \gamma r$ for all $p, q, r \subseteq Q$. We then have that

$m_{\times}(p, q)
= \inf \{m_\alpha(F) | p \subseteq F \text{ for all } F \subseteq \Psi_\alpha\} > 0$

and

$m_{\times}(q, r)
= \inf \{m_\alpha(F) | q \subseteq F \text{ for all } F \subseteq \Psi_\alpha\} > 0$

Since $\Psi_\alpha$ and $\Psi_\beta$ are finite, there exist $F_1 \subseteq \Psi_\alpha$ and $F_2 \subseteq \Psi_\beta$ such that $p \subseteq F_1$, $q \subseteq F_2$ and

$m_{\times}(p, q) = m_\alpha(F_1) > 0$, $m_{\times}(q, r) = m_\alpha(F_2) > 0$.

Since $\Psi$ is a quasi-ordinal fuzzy knowledge space, $p \subseteq F_1 \cap F_2 \subseteq \Psi_\alpha \cap \Psi_\beta \subseteq \Psi$, and

$m_\alpha(F_1 \cap F_2) = \min \{m_\alpha(F_1), m_\alpha(F_2)\} > 0$.

We note that since $F_1$ and $F_2$ are two elements which have smallest degrees $m_\alpha(F_1)$ and $m_\alpha(F_2)$ of $m_\alpha(F_1) = \inf \{m_\alpha(F) | p \subseteq F \text{ for all } F \subseteq \Psi_\alpha\}$

and

$m_\alpha(F_2) = \inf \{m_\alpha(F) | q \subseteq F \text{ for all } F \subseteq \Psi_\beta\}$

respectively. So, we can clearly know that $F_1 \cap F_2$ is the element which is

$m_\alpha(F_1 \cap F_2) = \inf \{m_\alpha(F) | p \subseteq F \text{ for all } F \subseteq \Psi_\alpha\}$.$E$

We then have $m_{\times}(p, r) = m_\alpha(F_1 \cap F_2) > 0$ and so $p \subseteq \gamma r$.

Finally, we will define a fuzzy surmise function and discuss some properties of them.

**Definition 3.8** Let $F(S(Q))$ be the family of all fuzzy sets of $S(Q)$. A fuzzy mapping $\delta : Q \rightarrow F(S(Q))$ is called a fuzzy surmise function if it has the following properties:

(i) $m_{\times}(E) \geq \chi_\delta(q)$,
(ii) for each $E \subseteq S(Q)$, $\exists E \subseteq S(Q)$ with $E \subseteq E$ and $\min \{m_{\times}(E), \chi_\delta(p)\} \leq m_{\times}(E)$,
(iii) for each $E, E' \subseteq S(Q)$ with $E \subseteq E'$

$1 - m_{\times}(E, m_{\times}(E')) \geq \chi_{\delta_{E(E')}}$

where $\rho, q \subseteq Q$ and $\chi_\delta$ is the characteristic function of $E$ on $Q$.

We note that a mapping $\delta : Q \rightarrow S(S(Q))$ is called a surmise function if it has the following properties:

(i) $\forall q \subseteq Q \rightarrow q \subseteq W$,
(ii) $\forall q \subseteq Q \wedge \rho \subseteq W \rightarrow E \subseteq \delta(q) \wedge W \subseteq W$,
(iii) $W \wedge \delta(q) \wedge W \subseteq W \rightarrow W = W$.

Let $a \subseteq (0, 1]$ and we consider the $\alpha$-level surmise function $\delta_{\alpha}$ of a fuzzy surmise function $\delta$ defined by

$\delta_{\alpha}(q) = \{E \subseteq S(Q) \mid m_{\times}(E) \geq a\}$.

We then easily know that Then we will prove that $\delta_{\alpha}$ is a function from $Q$ to $S(S(Q))$, and will prove that these $\alpha$-level surmise functions are surmise functions.

**Theorem 3.9** If $\delta : Q \rightarrow F(S(Q))$ is a fuzzy surmise function and $0 \alpha \leq 1$, then the $\alpha$-level surmise function $\delta_{\alpha} : Q \rightarrow S(S(Q))$ is a surmise function.

**Proof.** We will prove the conditions of the definition of surmise functions.

(i) If $F \subseteq \delta_{\alpha}(q)$, then we have

$\chi_{\delta}(q) \geq m_{\times}(F) \geq a > 0$. 


Since $\chi_F$ is the characteristic function of $F$, $\chi_F(q) = 1$ and hence $q \subseteq F$. Thus, the condition (i) of the definition of surmise functions is proved.

(ii) Let $F \subseteq \delta'_q(q)$ and $p \not\subseteq F$. By the condition (ii) of a fuzzy surmise function, there exists $F' \subseteq \delta'_q(q) (F' \subseteq F)$ and

$$\min(m_{\delta'_q(F), \chi_F(p)}) \leq \delta'_q(p) (F').$$

Since $m_{\delta'_q(F)} \geq \alpha$ and $\chi_F(p) = 1$, we have

$$\alpha \leq \min(m_{\delta'_q(F), \chi_F(p)}) \leq \delta'_q(p) (F').$$

and so $F \subseteq \delta'_q(p)$. Thus, the condition (ii) of the definition of surmise functions is proved.

(iii) Let $F \subseteq F'$ and $F, F' \subseteq \delta'_q(q)$. We then have $m_{\delta'_q(F)} \geq \alpha$ and $m_{\delta'_q(F')} \geq \alpha$, and so

$$\alpha \leq \min(m_{\delta'_q(F), m_{\delta'_q(F')}}).$$

Thus, we have

$$\chi_{F \cap F'} \leq 1 - \min(m_{\delta'_q(F), m_{\delta'_q(F')}}) = 1 - \alpha < 1,$$

where $F'$ is the complement of a set $F$. This means that $\chi_{F \cap F'} = 0$ and hence $F \subseteq F' \subseteq \delta'_q(q)$. Since $F \subseteq F'$, $F \neq F'$. Thus, the condition (iii) of the definition of surmise functions is proved.

4. Remarks

Using new concepts of fuzzy knowledge space theory, we can apply to the connection of quasi-order fuzzy knowledge spaces with fuzzy surmise relations and to a similar connection between fuzzy knowledge spaces and fuzzy surmise functions. In future, we will study some properties of fuzzy knowledge spaces which are well-graded, compatibility, the maximal mesh and entailment, etc.

References

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