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A PARAMETRIC BOUNDARY OF A PERIOD-2 COMPONENT IN THE DEGREE-3 BIFURCATION SET

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Abstract. The boundary of a typical period-2 component in the degree-3 bifurcation set is formulated by a parametrization of its image which is the unit circle under the multiplier map. Some properties on the geometry of the boundary are investigated including the root point, the cusp and the length as well as the area bounded by the boundary curve. The centroid of the area for the period-2 component was numerically found with high accuracy and compared with its center. An algorithm drawing the boundary curve with Mathematica codes is proposed and its implementation exhibits a good agreement with the analysis presented here.

1. Introduction

The degree-n bifurcation set denoted by $\boldsymbol{M}$ was introduced by Devaney [6] in 1986. It is the generalized Mandelbrot set [2, 3, 5-8, 13, 16] under the complex polynomial $P_c(z) = z^n + c$ for an integer $n \geq 2$ with $c, z \in \boldsymbol{C}$. Some useful properties of $\boldsymbol{M}$ including its geometry were investigated by a number of researchers [5, 11, 13, 16]. The various geometric shapes of $\boldsymbol{M}$ can be found in Geum and Kim [12]. An attracting period-k component in $\boldsymbol{M}$ is defined as a typical component

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[15] of the set:

$$\{c \in \boldsymbol{C} : there\, exists\, \xi \in \boldsymbol{C}\, such\, that\, P_c{}^k(\xi) = \xi, |\frac{d}{dz} P_c{}^k(z)|_{z=\xi} < 1\}. \tag{1.1}$$

and is denoted by $\boldsymbol{M}_k{}'$. When $k = 1$, the component is called the *main* component. A parametric boundary [11] of the main component in the degree-n bifurcation set is given by:

$$c = \frac{\alpha}{n}(n \cos\psi - \cos n\psi) + i\frac{\alpha}{n}(n \sin\psi - \sin n\psi), \tag{1.2}$$

with $\alpha = (1/n)^{1/(n-1)}$, $0 \leq \psi < 2\pi$. Let $\boldsymbol{W}$ be a component of some period in the Mandelbrot set(degree-2 bifurcation set) and $\boldsymbol{D}$ be an open unit disk in the complex plane. Douady and Hubbard [9] showed that the multiplier map $\lambda : \boldsymbol{W} \to \boldsymbol{D}$ is a conformal equivalence. It easily extends to a homeomorphism $\lambda : \boldsymbol{W} \bigcup \partial \boldsymbol{W} \to \boldsymbol{D} \bigcup \partial \boldsymbol{D}$ by Caratheodory [8]. In particular, the boundary [2, 7, 13, 16] of a period-2 component in the Mandelbrot set is found to be a circle of radius $\frac{1}{4}$ centered at $(-1, 0)$. It is a prime goal of this paper to establish the boundary of a period-2 component in the degree-3 bifurcation set [11, 12] whose geometric shape is shown in Figure 1.1. The component $\boldsymbol{M}_k{}'$ is identified by a number and shaded in different patterns.

On the boundary of the main component along the y-axis, two period-2 components of cardioid shapes are born symmetrically about the x-axis. Due to the symmetry of $\boldsymbol{M}$, the two components constitute a twin. Hence it suffices to investigate the properties of the period 2-component located above the x-axis. The boundary equation of this period-2 component will be formulated by a parametrization and some useful geometric properties of the boundary will be pursued. An algorithm and program codes written in Mathematica [17] are included to draw the boundary curve of the period-2 component. The computational boundary curve is illustrated with concluding remarks. The following notations and symbols will be used throughout the study.

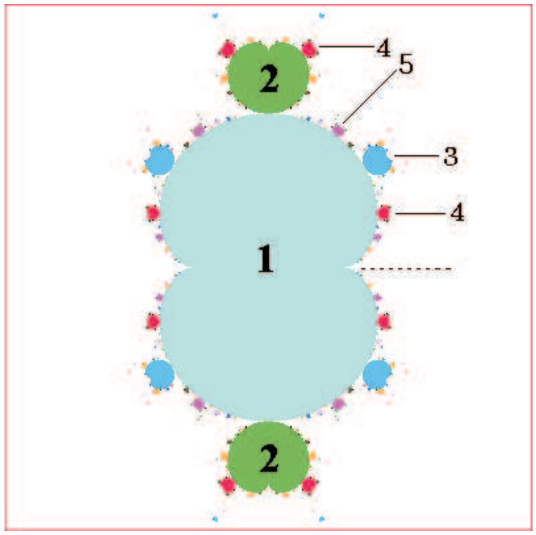


FIGURE 1. Bifurcation points on the boundary $\partial \boldsymbol{M}_k{}'$

$\boldsymbol{C}$: set of all complex numbers.
$\boldsymbol{R}$: set of all real numbers.
$\boldsymbol{N}$: set of all natural numbers.
$\partial \boldsymbol{H}$: boundary of the set $\boldsymbol{H}$.
$[a, b]$: interval $\{x : a \leq x \leq b\}$.
$f^{k+1}(z) = f \circ f^k(z)$: k-fold composite map of f at z with $f^0(z) = z$.
$f'(z)$: complex derivative of f evaluated at z.
$\bar{z}$: complex conjugate of z.
$i = \sqrt{-1}$: imaginary unit.

We will next introduce some definitions and investigate some theorems regarding the properties of the degree-n bifurcation set.

DEFINITION 1.1. Let $P_c(z) = z^n + c$ for $n \in \boldsymbol{N} - \{1\}$, with $c,\ z \in \boldsymbol{C}$. Then the *degree-n bifurcation set* is defined to be the set

$$\boldsymbol{M} = \left\{ c \in \boldsymbol{C} : \lim_{k \to \infty} P_c{}^k(0) \neq \infty \right\}.$$

If $n = 2$, then $\boldsymbol{M}$ reduces to the *Mandelbrot set*.

DEFINITION 1.2. The sets $\boldsymbol{P}_m = \{c \in \boldsymbol{C} : c = re^{i\phi_m}, r \geq 0, \phi_m = \frac{m\pi}{(n-1)}\}$ (for $m = 1, 2, \cdots, 2n-2$) are called the *rays of symmetry*. The set $\boldsymbol{P}_1$ is called the *principal ray of symmetry* and denoted by $\boldsymbol{P}$. The set $\boldsymbol{S}= \left\{c \in \boldsymbol{C} : c = re^{i\theta}, r \geq 0, 0 < \theta \leq \frac{\pi}{(n-1)}\right\}$ is called the *principal sector*.

The following theorem confirms the geometric symmetry of the degree-n bifurcation set with respect to rays of symmetry in the complex plane.

THEOREM 1.1. *The degree-n bifurcation set $\boldsymbol{M}$ is symmetric in the c-parameter plane about $\boldsymbol{P}_m$ for all $m \in \{1, 2, \cdots, 2n-2\}$.*

Proof. See p. 224, Geum and Kim [12]. □

DEFINITION 1.3. Let f be a complex-valued function on $\boldsymbol{C}$. The (*forward*) *orbit* of $z_0 \in \boldsymbol{C}$ under f is defined by the sequence $\{z_k \in \boldsymbol{C} : z_k = f^k(z_0), k = 0, 1, 2, \cdots\}$. If f is analytic[1] on $\boldsymbol{C}$ and $f\,'(\omega) = 0$, then $\omega \in \boldsymbol{C}$ is called a *critical point*. The orbit of ω is called the *critical orbit*. If there exists a smallest positive integer m satisfying $f^m(z_0) = z_0$ for $z_0 \in \boldsymbol{C}$, then z_0 is called the *period-m*(*m-periodic*) *point* and the orbit of z_0 is called the *m-cycle* or *period-m orbit* and m is the *period* of the orbit. The number $\lambda = \frac{d}{dz} f^m(z)|_{z=z_0}$ is called the *multiplier*(*eigenvalue*) of f^m at z_0. If $|\lambda| < 1$, the m-cycle is said to be *attracting*(*attractive*) and z_0 is called the *period-m attractor*. If $|\lambda| > 1$, the m-cycle is said to be *repelling*(*repulsive*) and z_0 is called the *period-m repeller*. If $|\lambda| = 1$, the m-cycle is said to be *indifferent*(*neutral*) and z_0 is called the *indifferent*(*neutral*) *period-m point*. The point z_0 is called *preperiodic*(*eventually periodic*) if z_k is periodic for some $k > 0$.

2. The boundary of a period-2 component in the degree-3 bifurcation set

We will formulate the boundary equation of a period-2 component in the degree-3 bifurcation set by a parametrization. The boundary equation is characterized as follows:

$$P_c{}^2(z) = z, \tag{2.1}$$

$$\mid \lambda \mid = 1, \tag{2.2}$$

where $P_c(w) = w^3 + c$, z is a period-2 point of P_c and $\lambda = \frac{d}{dw} P_c{}^2(w) \mid_{w=z}$ is a multiplier with $w, c \in \boldsymbol{C}$. The boundary curve is then described by c after eliminating z in the above two equations. Since Eq. (2.2) describes a circle in the image plane, we naturally introduce a parameter $\phi \in [0, 2\pi)$ such that $\lambda = e^{i\phi}$. Rewriting Eq.(2.1) leads to

$$(z^3 + c)^3 + c = z \tag{2.3}$$

From the elementary complex analysis [1], Eq. (2.2) can be rewritten as

$$z(z^3 + c) = (1/3)\ e^{i(\phi/2 + j\pi)} \text{ for } j = 0, 1. \tag{2.4}$$

Let $a = (1/3)\ e^{i(\psi + j\pi)}$ with $\psi = \phi/2$. Then Eq. (2.4) becomes

$$z(z^3 + c) = a. \tag{2.5}$$

Combining Eq. (2.5) with Eq. (2.3) immediately yields $c = z - (a/z)^3 = a/z - z^3$ and $(z - a/z)(z^2 + a^2/z^2 + a + 1) = 0$. Since $z - a/z = z - (z^3 + c) = 0$ implies a fixed point which is no longer of interest, we extract period-2 points z of P_c from the equation:

$$(z^2 + a^2/z^2 + a + 1) = 0 \tag{2.6}$$

Now we introduce Vièta's transformation $t = z + a/z$ to obtain

$$t = (a - 1)^{1/2} \tag{2.7}$$

and

$$\begin{aligned} c &= a/z - z^3 = a/z - z(z^2) = a/z - z(-a^2/z^2 - a - 1) \\ &= a/z + a^2/z + (a + 1)z = (a + 1)(z + a/z) = (a + 1)t \\ &= (a + 1)(a - 1)^{1/2}. \end{aligned} \tag{2.8}$$

Thus c has four branched values since a and $(a-1)^{1/2}$ have two branched values. It follows that the boundary can be parametrized by $c = c(\psi)$ with $a = \pm\, e^{i\psi}/3$. That is,

$$c = (a+1)(a-1)^{1/2} = \pm\, re^{i\theta}, \tag{2.9}$$

where $r = r_1\sqrt{r_2}$, $r_1 = \sqrt{10+6\cos\psi}/3$, $r_2 = \sqrt{10-6\cos\psi}/3$, $\theta = \psi_1 + \psi_2/2$,

$\psi_1 = \tan^{-1}\left(\frac{\sin\psi}{3+\cos\psi}\right)$, $\psi_2 = \tan^{-1}\left(\frac{\sin\psi}{-3+\cos\psi}\right)$ and $0 \le \psi < \pi$.

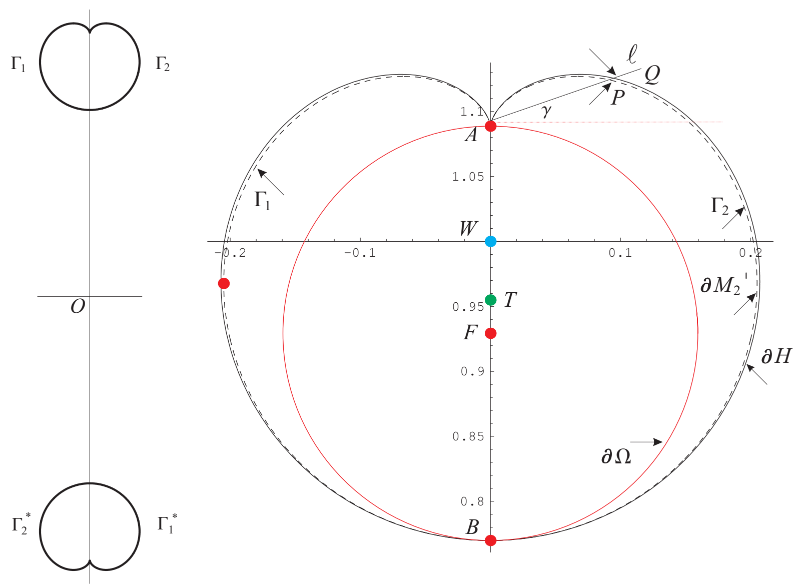
The points c represented by Eq. (2.9) trace four branch curves $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, ${\mathbf{\Gamma}_1}^*$ and ${\mathbf{\Gamma}_2}^*$ as shown in Figure 2.1-(a). Described in Section 3 is an algorithm drawing the boundary curve with Mathematica codes. Due to the symmetry [12] of $\boldsymbol{M}$ and in view of continuity of the boundary, it suffices to consider only one of the four branches. Consequently each branch is defined as follows:

$$\begin{gathered}
\mathbf{\Gamma}_1 = \left\{c \in \boldsymbol{C} : c = re^{i\theta},\ a = e^{i\psi}/3,\ 0 \le \psi < \pi\right\}, \\
{\mathbf{\Gamma}_1}^* = \left\{-c \in \boldsymbol{C} : c \in \mathbf{\Gamma}_1\right\}, \\
{\mathbf{\Gamma}_2}^* = \left\{\bar{c} \in \boldsymbol{C} : c \in \mathbf{\Gamma}_1\right\}, \\
\mathbf{\Gamma}_2 = \left\{\bar{c} \in \boldsymbol{C} : c \in {\mathbf{\Gamma}_1}^*\right\} = \left\{-\bar{c} \in \boldsymbol{C} : c \in \mathbf{\Gamma}_1\right\}.
\end{gathered} \tag{2.10}$$

The boundary of a period-2 component in the upper half plane is denoted by $\partial{\boldsymbol{M}_2}'$ and is given by:

$$\partial{\boldsymbol{M}_2}' = \mathbf{\Gamma}_1 \bigcup \mathbf{\Gamma}_2. \tag{2.11}$$

It forms a closed curve enclosing the point $W(0,1)$, which is the component center [12] of the period-2 component. We now pay attention to the parametric boundary of a period-2 component in the upper half plane and discuss some geometric properties of the boundary. The following Theorem 2.1 will reveal interesting facts on the boundary which looks like a cardioid.



(a) (b)

Figure 2.1: Boundaries of period-2 components

THEOREM 2.1. *Let* $r_s = \frac{4\sqrt{2}}{3\sqrt{3}}$, $r_e = \frac{r_s}{\sqrt{2}} = \frac{4}{3\sqrt{3}}$, $b = \frac{(r_s+r_e)}{2} = \frac{2\sqrt{3}(\sqrt{2}+1)}{9}$ *and* $\rho = (r_s - r_e)/2 = 2(\sqrt{2}-1)/3\sqrt{3}$ *be given. Let* $A(0, r_s)$, $B(0, r_e)$ *and* $F(0, b)$ *be three given points on the y-axis. Let* $\boldsymbol{H}$,$\boldsymbol{M}_2{}'$ *and* $\boldsymbol{\Omega}$ *respectively denote the interior bounded by a cardioid* $\partial\boldsymbol{H}$ *specified below, a period-2 component in the degree-3 bifurcation set in the upper half plane and a disk of radius* ρ *centered at* $F(0, b)$ *as shown in Figure 2.1-(b). Let* $\partial\boldsymbol{H}$, $\partial\boldsymbol{M}_2{}'$ *and* $\partial\boldsymbol{\Omega}$ *be represented by the parametric equations below:*

$\partial\boldsymbol{H} : x_1(t) = \rho \sin t(1 - \cos t),\ y_1(t) = \rho \cos t(1 - \cos t) + r_s,\ 0 \le t < 2\pi,$

$\partial\boldsymbol{M}_2{}' : x(\psi) = r\cos\theta,\ \ y(\psi) = r\sin\theta,$ *where* r *and* θ *are defined as in Eq. (2.9) and* $0 \le \psi < 2\pi,$

$\partial\boldsymbol{\Omega} : x_2(u) = \rho \sin u,\ y_2(u) = \rho\cos u + b,\ 0 \le u < 2\pi.$

Then the following hold:

(a) $\partial\boldsymbol{M}_2{}'$ *is inscribed in* $\partial\boldsymbol{H}$ *with two osculating points* A *and* B.

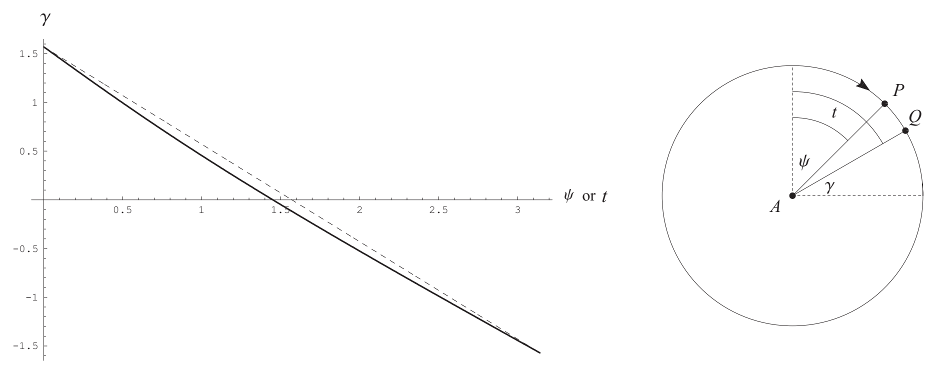
(b) $\partial\boldsymbol{\Omega}$ *is inscribed in* $\partial\boldsymbol{M}_2{}'$ *with two osculating points* A *and* B*.*
(c) The perimeter length of $\partial\boldsymbol{H}$ *is* $16(\sqrt{2}-1)/3\sqrt{3} \approx 1.275446995936534763$.
(d) The perimeter length of $\partial\boldsymbol{M}_2{}'$ *is approximately* 1.264033759241096936.
(e) The area bounded by $\partial\boldsymbol{H}$ *is* $2(3-2\sqrt{2})\pi/9 \approx 0.119780463211699382$.
(f) The area bounded by $\partial\boldsymbol{M}_2{}'$ *is approximately* 0.118042847838789250.
(g) The point A *is a cusp.*
(h) The centroid of the period-2 component is located at $(0, 0.954900554516503637 48)$.
(i) The component center of the period-2 component is located at $(0, 1)$.
(j) The point B *is a root point where the period-2 component buds from the boundary of the main component in the degree-3 bifurcation set.*

Proof. Due to the symmetry of $\boldsymbol{M}$, we only consider $\boldsymbol{\Gamma}_2$, the right half of $\partial\boldsymbol{M}_2{}'$ for our analysis. (a) Let ℓ be a ray passing through the point A with an inclined angle $-\pi/2 \le \gamma \le \pi/2$. Let $P(-x, y) = P(-r\cos\theta, r\sin\theta)$ and $Q(x_1, y_1) = Q(\rho\sin t(1-\cos t), \rho\cos t(1-\cos t)+ r_s)$ be the crossing points of ℓ with $\partial\boldsymbol{M}_2\prime$ and $\partial\boldsymbol{H}$, respectively. To show that Q lies outside of P, it suffices to show $\psi \le t$ in $[0, \pi]$. Since $\gamma = \tan^{-1}\left(\frac{y(\psi)-r_s}{x(\psi)}\right) = \tan^{-1}\left(\frac{y_1(t)-r_s}{x_1(t)}\right) = \pi/2 - t$, we treat γ as a monotone function of ψ or t. It is clear that $\gamma = \pi/2$ for $\psi = 0$ or $t = 0$ and $\gamma = -\pi/2$ for $\psi = \pi$ or $t = \pi$, which states $\psi = t$ at the end points of $[0, \pi]$.

Figure 2.2 shows γ as a function of ψ or t as well as the angle t and ψ with a circle centered at A.

For any given $-\pi/2 \le \gamma \le \pi/2$, the graphical analysis immediately suggests that $\psi <= t \in [0, \pi]$. It is easy to show that for $\psi, t \in (0, \pi)$ the condition $\psi < t$ is equivalent to

$$\gamma = \tan^{-1}\left(\frac{y(\psi) - r_s}{x(\psi)}\right) < \pi/2 - \psi. \tag{2.12}$$



(a) (b)

Figure 2.2: The inclined angle γ as a function of ψ or t

By the monotonicity of $\tan\gamma$, it is immediately equivalent to the condition that

$$\left(\frac{y(\psi)-r_s}{x(\psi)}\right) < \frac{\cos\psi}{\sin\psi},$$

from which it remains to show, for $\psi \in (0,\pi)$, that the newly defined continuous function f satisfies

$$f(\psi) = \sin\psi\ (y(\psi)-r_s) + x(\psi)\ \cos\psi = r_s\ \sin\psi - r\ \cos(\psi-\theta) > 0. \tag{2.13}$$

By a more detailed analysis, it can be shown that

$$\frac{df(\psi)}{d\psi} = \frac{g(\psi)}{12\cdot 2^{1/4}\cdot\sqrt{3}(5-3\cos\psi)^{3/4}\sqrt{5+3\cos\psi}}, \tag{2.14}$$

where $g(\psi) = 16(10-6\cos\psi)^{3/4}\cdot\cos\psi\cdot\sqrt{5+3\cos\psi} + 161\sin(\psi-\theta)$
$+\ 3\,(\sin(2\psi-\theta) - 6\sin(3\psi-\theta) + 9\sin\theta - 3\sin(\psi+\theta))$.

An elementary calculus confirms that g has in $(0,\pi)$ only one real zero ψ^* which is approximately 1.94304509978953403743. Since g is positive for $\psi \in (0,\psi^*)$ and non-positive for $\psi \in (\psi^*,\pi)$, the monotonicity in each interval leads to the fact that $f > 0 = \lim_{\psi->0^+}\ f(\psi)$

for $\psi \in (0, \psi^*)$ and $f > 0 = \lim_{\psi->\pi^-} f(\psi)$ for $\psi \in (\psi^*, \pi)$. Consequently $f > 0 \in (0, \pi)$. The points A and B are the osculating points since $x_1(t)|_{t=0} = -x(\psi)|_{\psi=0} = 0$, $x_1(t)|_{t=\pi} = -x(\psi)|_{\psi=\pi} = 0$, $y_1(t)|_{t=0} = y(\psi)|_{\psi=0} = r_s$ and $y_1(t)|_{t=\pi} = y(\psi)|_{\psi=\pi} = r_e$ are satisfied. This completes the proof.
(b) Let $h(\psi) = (-x(\psi))^2 + y(\psi)^2 = x(\psi)^2 + y(\psi)^2$, which measures the distance between $(-x(\psi), y(\psi)) \in \mathbf{\Gamma}_2$ and the point $F(0, b)$. Then the derivative $h'(\psi)$ is given by

$$h'(\psi) = \frac{\sin[\psi/2]\sqrt{2/\gamma}}{9} \{w_1(\psi) + w_2(\psi)\}, \tag{2.15}$$

where $w_1(\psi) = -5\cos(2\theta - \psi/2) + 9/2 \cdot \cos(2\theta + \psi/2)$
$+ 9/2 \cdot \cos(2\theta + 3\psi/2) + 3 \cdot \cos(\psi/2) + 3 \cdot \cos(3\psi/2)$,
$w_2(\psi) = \frac{(2\sqrt{2}-2+2^{3/4}\gamma^{1/4}\sqrt{\beta}\sin\theta)\cdot(9\sin(\theta+3\psi/2)-\sin(\theta-\psi/2))}{2^{3/4}\gamma^{1/4}\sqrt{\beta}}$,
$\gamma = 5 - 3\cos\psi$ and $\beta = 5 + 3\cos\psi$.

It has three real zeros 0, $\bar{\psi}$ and π with $\bar{\psi} \approx 1.456050346953239800468$, found by a numerical method of high precision. Hence h assumes its minima ρ^2 at $A(0, r_s)$ when $\psi = 0$ or at $B(0, r_e)$ when $\psi = \pi$. The points A and B are the osculating points since $x_2(u)|_{u=0} = -x(\psi)|_{\psi=0} = 0$, $x_2(u)|_{u=\pi} = -x(\psi)|_{\psi=\pi} = 0, y_2(u)|_{u=0} = y(\psi)|_{\psi=0} = r_s$ and $y_2(u)|_{u=\pi} = y(\psi)|_{\psi=\pi} = r_e$ are satisfied. This proves the assertion.
(c) The perimeter length of $\partial\boldsymbol{H}$ is from a basic calculus easily given by
$\int_0^\pi \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2} dt = 8\rho = 16(\sqrt{2}-1)/3\sqrt{3} \approx 1.275446995936534763 9$.
(d) The perimeter length of $\partial\boldsymbol{M}_2{}'$ is from a basic calculus easily given by
$\int_0^\pi \sqrt{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2} d\psi \approx 1.264033759241096936$, which was numerically found.
(e) Note that $r(t) = \sqrt{x^2 + (y - r_s)^2} = \rho(1 - \cos t)$ defines a cardioid

with cusp point at $A(0, r_s)$. The area bounded by $\partial\boldsymbol{H}$ is thus from a basic calculus easily given by
$2\cdot\frac{1}{2}\int_0^\pi \rho^2(1-\cos t)^2 dt = \frac{3}{2}\rho^2\pi = 2(3-2\sqrt{2})\pi/9 \approx 0.119780463211699382.$
(f) The area bounded by $\partial{\boldsymbol{M}_2}'$ is from a basic calculus easily given by $2(A_1-A_2)$, where $A_1 = \int_0^{\psi^\dagger} |y(\psi)x'(\psi)|\, d\psi$ and $A_2 = \int_{\psi^\dagger}^{\pi} |y(\psi)x'(\psi)|\, d\psi$, with $\psi^\dagger \approx 2.00817189752764 35150$ as a parameter of the rightmost point on $\partial{\boldsymbol{M}_2}'$ found by numerical method of high precision. As a result, the area is approximately 0.118042847838789250.
(g) It can be shown that $\frac{x'(\psi)}{y'(\psi)} = \frac{dx}{dy} = 0$ at $\psi = 0$ and $\frac{x'(\psi)}{y'(\psi)} = \frac{dx}{dy} > 0$ for $\psi = \epsilon > 0$ with ϵ as a sufficiently small positive number. The symmetry of $\boldsymbol{M}$ confirms that A is a cusp.
(h) The x-coordinate of the centroid T is obviously 0 due to the symmetry of ${\boldsymbol{M}_2}'$. Let $\bar{y}$ be the y-coordinate of the centroid and S be the area bounded by $\partial{\boldsymbol{M}_2}'$. Then we have

$$\bar{y} = \frac{1}{S}\int_{{\boldsymbol{M}_2}'} y\; dS. \tag{2.16}$$

Divide the region ${\boldsymbol{M}_2}'$ into two subregions by the horizontal line passing the point A. In each subregion, appropriate expression for dS is considered to obtain the integral (2.16). A direct computation shows that $\bar{y} = 0.954900554516503637 48$.
(i) See p. 230, Geum and Kim [10] for computing the component center W.
(j) The equation [8] $\lambda^2 = 1$ with $\lambda = \frac{d}{dw}P_c(w)|_{w=z}$ and z as a fixed point gives the corresponding root point satisfying Eq. (1.2). □

3. Algorithm and implementation drawing $\partial{\boldsymbol{M}_2}'$

In this section we list an algorithm and Mathematica codes drawing all the boundaries of period-2 components in the degree-3 bifurcation set. The symmetry of $\boldsymbol{M}$ is effectively used in drawing the boundaries.

The Mathematica[17] command *ParametricPlot* plays a critical role for plotting the boundaries. Figure 2.1-(a) shows an implementation result.

Algorithm 3.1

Step 1. Set variables $r, r_1, r_2, \psi_1, \psi_2$ and θ according to Eq. (2.9).

Step 2. Let (x, y) represent a point on $\mathbf{\Gamma}_2$, with $x = r\cos\theta$ and $y = r\sin\theta$.

Step 3. Plot $(x(\psi), y(\psi))$, $(-x(\psi), y(\psi))$, $(x(\psi), -y(\psi))$, $(-x(\psi), -y(\psi))$ simultaneously, with a parameter $0 \le \phi \le \pi$, using the symmetry of the degree-3 bifurcation set.

(*Mathematica Codes for Algorithm 3.1: The boundary* $\partial \boldsymbol{M}_2{}'$)

```
r1 = 1/3*Sqrt[10 + 6*Cos[ψ]];
r2 = 1/3*Sqrt[10 - 6*Cos[ψ]];
ψ1 = ArcTan[3 + Cos[ψ], Sin[ψ]];
ψ2 = ArcTan[-3 + Cos[ψ], Sin[ψ]];
r = r1*Sqrt[r2]; θ = ψ1 + ψ2 / 2;
x = r * Cos[θ]; y = r * Sin[θ];
ParametricPlot[{{x, y}, {-x, y}, {x, -y}, {-x, -y}}, {ψ, 0, Pi},
PlotPoints → 400, PlotStyle → Thickness[0.02],
AspectRatio → Automatic, PlotRange → All, Ticks → None];
```

4. Results and Discussion

The approach in the proof of Theorem 2.1-(i) promisingly determines an inclusion relation whether a parametric boundary is inscribed in a known curve. The numerical results in Section 2 are found from the *FindRoot* command of *Mathematica* 4.0 [17], which is a basically a Newton's method [4, 10]. The command was employed with appropriate options such as *AccuracyGoal* → 20, *DampingFactor* → 1, *MaxIterations* → 25 and *WorkingPrecisions* → 48. In order to obtain more accurate results, the computation was carried out with 48 significant digits of precision and the final numerical results are expected to be accurate up to approximately 20 significant digits.

The symmetry of $\boldsymbol{M}_2{}'$ was extensively used in the construction of the boundary $\partial \boldsymbol{M}_2{}'$. The first branch as well as other branches was drawn with *ParametricPlot* command of *Mathematica* using symmetry and reflection.

In view of the results of Theorem 2.1, the boundary $\partial \boldsymbol{M}_2{}'$ in the degree-3 bifurcation set can be considered as a deformed cardioid whose area and length are slightly smaller (approximately 1.5 % and 0.9 % less) than those of the actual cardioid ∂ . The perimeter length of $\partial \boldsymbol{M}_2{}'$ and the area bounded by $\partial \boldsymbol{M}_2{}'$ are accurately obtained from the *NIntegrate* command of *Mathematica* with options such as *AccuracyGoal* $\rightarrow$ 20 and *WorkingPrecisions* $\rightarrow$ 48. The total area bounded by $\partial \boldsymbol{M}_2{}'$ is found to be approximately 0.2360856956775785 as a result of Theorem 2.1-(f), which occupies about 13.1 % of the estimated gross area of the degree-3 bifurcation set. The author of this paper estimated that the gross area is around 1.79 by the pixel counting method[14] with a common *escape-time algorithm* [2, 16] for the grid size of 15000 X 15000 in the region $\{c \in \boldsymbol{C} : |\text{Re}(c)| \leq 1.3515625,\ |\text{Im}(c)| \leq 1.3515625\}$ with an iteration limit of 8192. The better estimation requires a higher iteration limit as well as a larger grid size. The boundary of the period-2 component $\partial \boldsymbol{M}_2{}'$ in the degree-3 bifurcation set is precisely inscribed in $\partial \boldsymbol{H}$. The circle $\partial \boldsymbol{\Omega}$ is also inscribed in both $\partial \boldsymbol{M}_2{}'$ and $\partial \boldsymbol{H}$. As a result of Theorem 2.1, we observe that $\boldsymbol{\Omega} \subset \boldsymbol{M}_2{}' \subset \boldsymbol{H}$.

The computation of the centroid requires the topmost point (0.06718663, 1.1269745) and the point $(0.155766139, r_s)$ where the horizontal line passing through A meets $\boldsymbol{\Gamma}_2$. Observe that the centroid is located below the component center (0,1) as expected.

Since the boundary equations of period-2 components in the degree-3 bifurcation set are explicitly known by a parametrization as shown in Eq. (2.9), the escape-time algorithm constructing the whole degree-3 bifurcation set can be updated to bypass the points in period-2 components and to reduce the construction time. The current analysis shown in this study can be easily extended to the case when $n \geq 4$ and $k = 2$, despite the expected complexity of the algebraic manipulation.

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