On Realization of 2D Discrete Systems
by Fornasini-Marchesini Model

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Abstract: In this paper, we propose a constructive realization procedure for 2D systems which may lead to a Fornasini-Marchesini local state-space model with much lower order than the existing realization procedure given by Bisiacco et al. Nontrivial examples are illustrated and the conditions for minimal realization are also discussed.

Keywords: 2D systems, state-space model, realization.

1. INTRODUCTION

The field of two-dimensional (2D) filters and systems has received much attention over the past three decades with wide applications in image processing, geophysics, control systems and multipass processes (see e.g., [1-4]).

One of the fundamental issues in 2D filters and systems is the realization of a given transfer function or transfer matrix by a certain kind of 2D local state-space model, typically by Roesser model or Fornasini-Marchesini second (FM-II) model (see, e.g., [3,5-17]). It is well known that, unlike the one-dimensional (1D) case, it is very difficult to obtain a minimal state-space realization in the 2D case except for some special categories of 2D systems [7-17]. Hence, for a general 2D transfer function or transfer matrix it is desirable to obtain a state-space realization with as low order as possible. Note that by "a general system" we mean that there is not any restriction on the coefficients in the transfer function or transfer matrix of the system.

While state-space realization by Roesser model has

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been investigated by quite a number of researchers (see [9,15-17] and the references therein), less attention has been paid to state-space realization by FM-II model and few results are reported in the literature [8,13]. In this paper, we consider the realization of a given 2D multi-input multi-output (MIMO) system by FM-II model.

In the next section, we review an existing procedure, proposed by Bisiacco et al. [13], for state-space realization of a given MIMO 2D system by FM-II model. In Section 3, a new constructive realization procedure for FM-II model is presented which may give system matrices of much smaller sizes while preserving the desirable properties of the state-space realization by the procedure of [13]. Illustrative examples are given and the conditions for minimal realization are discussed in Section 4. Conclusions are briefly stated in Section 5.

2. PRELIMINARIES

The 2D FM-II model is described by

$$x(h+1,k+1) = A_1x(h,k+1) + A_2x(h+1,k)$$
$$+ B_1u(h,k+1) + B_2u(h+1,k), \quad (1a)$$
$$y(h,k) = Cx(h,k) + Du(h,k), \quad (1b)$$

where $x(h,k) \in \mathbf{R}^n$, $u(h,k) \in \mathbf{R}^l$ and $y(h,k) \in \mathbf{R}^m$ are respectively the local state, input and output vectors, and A_1, A_2, B_1, B_2, C and D are real system matrices of suitable sizes. The system is also conventionally denoted by $(A_1, A_2, B_1, B_2, C, D)$. The transfer matrix of (1) is

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D. \quad (2)$$

When $D = 0$, the system is said to be strictly causal. As $D = W(0,0)$ from (2), in this paper we will assume without loss of generality that the transfer

matrix $W(z_1, z_2)$ under investigation is strictly causal. Throughout the paper we order the 2D power products $z_1^h z_2^k$ by the total degree (lexico-graphic) order, i.e., $1 < z_1 < z_2 < z_1^2 < z_1 z_2 < z_2^2 < \cdots$. The degree of a 2D polynomial $f(z_1, z_2)$ is the degree of the maximal-order power product of the given polynomial, denoted by $\deg f$. We also denote the highest degree of z_1 or z_2 in $f(z_1, z_2)$ by $\deg_{z_1} f$ or $\deg_{z_2} f$.

Consider now an $m \times l$ strictly causal 2D transfer matrix $W(z_1, z_2)$ that has a right matrix fraction description (MFD) $N_R(z_1, z_2) D_R(z_1, z_2)^{-1}$ with $D_R(0,0) = I$ and $N_R(0,0) = 0$ [13]. Let $N(z_1, z_2) = N_R(z_1, z_2)$, $D(z_1, z_2) = I - D_R(z_1, z_2)$. Denote by k_i, $i = 1, \ldots, l$, the column degree of the i th column of

$$F(z_1, z_2) \triangleq \begin{bmatrix} N(z_1, z_2) \\ D(z_1, z_2) \end{bmatrix}, \tag{3}$$

that is, the degree of the polynomial with the maximal-order power product in the i th column. A constructive state-space realization procedure for FM-II model for a strictly causal $W(z_1, z_2)$ has been given in [13]. For comparison, we briefly outline this procedure below.

Step 1: Construct the matrix $\Psi = \text{diag}\{\Psi_1, \ldots, \Psi_l\}$ where Ψ_i is the column vector defined as

$$\Psi_i = [z_2^{k_i} \ z_2^{k_i-1} z_1 \cdots z_2 z_1^{k_i-1} \ z_1^{k_i} \cdots z_2 z_1 \ z_1^2 \ z_2 \ z_1]^T. \tag{4}$$

Step 2: Write $D(z_1, z_2)$ and $N(z_1, z_2)$ in the form of

$$D(z_1, z_2) = D_{HT} \Psi, \tag{5}$$
$$N(z_1, z_2) = N_{HT} \Psi, \tag{6}$$

where D_{HT} and N_{HT} are real matrices with sizes conformable to Ψ.

Step 3: Construct matrices A_{i0}, B_i, $i = 1, 2$, such that

$$(I - A_{10} z_1 - A_{20} z_2)^{-1} (B_1 z_1 + B_2 z_2) = \Psi. \tag{7}$$

Step 4: The realization is finally obtained as $A_i = A_{i0} + B_i D_{HT}$, B_i, $i = 1, 2$, and $C = N_{HT}$.

The above realization procedure produces a state-space description (A_1, A_2, B_1, B_2, C) which have the following desirable properties.

(i) $\begin{bmatrix} I - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$ is full rank in $\mathbb{C}^2$;

(ii) $\det (I - A_1 z_1 - A_2 z_2) = \det D_R(z_1, z_2)$.

If $D_R(z_1, z_2)$ and $N_R(z_1, z_2)$ are right factor coprime (r.f.c.) the resultant state-space realization will be free of hidden modes [13]. These properties are very important in 2D control systems and signal processing [4,13]. However, a disadvantage of this procedure is that the sizes of the resultant system matrices are quite large in general, mainly due to the large size of matrix Ψ. From the above realization procedure, it can be easily seen that the size of column vector Ψ_i is $n_i' = \sum_{k=2}^{k_i+1} k$ while the size of matrix Ψ is $n' \times l$ where $n' = \sum_{i=1}^{l} n_i$ is usually a rather large number. When n' is large, the sizes of the resultant system matrices will also be large as their sizes all depend on n' in the following manner, $D_{HT} : l \times n'$; $N_{HT}, C : m \times n'$; $A_i : n' \times n'$; and $B_i : n' \times l$, $i = 1, 2$.

In the next section, we will propose a new procedure for construction of Ψ, A_i, B_i and C whose sizes may be much smaller than those obtained by the procedure of [13]. Moreover, the desirable properties (i), (ii) as well as other nice properties reported in [13] are also preserved.

3. MAIN RESULTS

As discussed in the previous section, the main factor contributing to the large sizes of system matrices in the realization procedure of [13] is the way in which $\Psi = \text{diag}\{\Psi_1, \ldots, \Psi_l\}$ is constructed. In fact, the vector Ψ_i given in (3) is constructed by using all the power products whose total degree order are equal to or below the column degree k_i, $i = 1, \ldots, l$ [13]. However, it is easy to see that, to realize a 1D transfer function $W(z)$ with degree n, one has to use all the power products z^h with $0 < h \le n$ even when some of them are absent, i.e., have zero coefficients in $W(z)$, while for a 2D transfer function $W(z_1, z_2)$ with degree n, even in the case that some power products $z_1^h z_2^k$ with $0 < h + k \le n$ are absent in $W(z_1, z_2)$, it is still possible to construct a 2D state-space realization (see the examples given in next section). This is essentially due to the properties that, in the 2D case, the number of power products $z_1^h z_2^k$ with $0 < h + k \le n$ is $n' = \sum_{j=2}^{n+1} j$ which is usually much larger than the degree n of $W(z_1, z_2)$, while in the 1D case there are only n power products z^h with $0 < h \le n$ which is just the same as the degree n of $W(z)$. This fact means that Ψ constructed by the procedure of [13] may contain some redundant power

products that do not contribute to the resultant realization. Hence, a critical point is to construct a new Ψ containing only the power products necessary for the realization.

By thoroughly investigating the structural properties of FM-II model (see Example 1 shown in the next section), we found that, to meet the relations specified in (5), (6) and (7) so that a realization can be obtained, Ψ has to satisfy the following conditions:

(a) Ψ_i contains all the power products occurring in the polynomial entries of the i th column of $F(z_1, z_2)$.

(b) Ψ_i contains z_1 or z_2, or both z_1 and z_2.

(c) Let n_i ($i = 1, ..., l$) be the dimension of Ψ_i. For every entry $\Psi_i(j)$ ($j \in \{1, ..., n_i\}$) except for the entry that is either z_1 or z_2, there exists another entry $\Psi_i(h)$ ($h \in \{1, ..., n_i\}$) such that

$$\Psi_i(j) = z_1 \Psi_i(h) \quad \text{or} \quad \Psi_i(j) = z_2 \Psi_i(h). \tag{8}$$

It is shown in the sequel that, based on these conditions, it is possible to identify the power products that are necessary for construction of a realization whereas discarding those redundant power products.

Consider again the 2D polynomial matrix $F(z_1, z_2)$ defined in (3). Let n_i ($i = 1, ..., l$) denote the number of all the power products $z_1^h z_2^k$ occurring in any of the polynomial entries of the i th column of $F(z_1, z_2)$ with non-zero coefficients. Construct column vector Ψ_i using these ordered power products with the first entry $\Psi_i(1)$ having the maximal-order while the last entry $\Psi_i(n_i)$ having the minimal-order. Obviously, the initial column vectors Ψ_i ($i = 1, ..., l$) of dimension n_i constructed in the above manner satisfies the condition of (a). Starting from the initial Ψ_i ($i = 1, ..., l$), an algorithm is given below for constructing the final Ψ_i ($i = 1, ..., l$) which satisfy the conditions of (b) and (c). Note that, for Ψ_i with its entries ordered in the above manner, the condition of (c) can be modified to the following form:

(c) For every entry $\Psi_i(j)$ ($j \in \{1, ..., n_i\}$) except for the entry that is either z_1 or z_2, there exists another entry $\Psi_i(h)$ ($j < h \leq n_i$) such that

$$\Psi_i(j) = z_1 \Psi_i(h) \quad \text{or} \quad \Psi_i(j) = z_2 \Psi_i(h). \tag{9}$$

The main idea in the following algorithm is to insert appropriate power products into Ψ_i until Ψ_i finally satisfies the conditions of (b) and (c). Note that the dimension (n_i) of the final column vector Ψ_i

may be larger than that of the initial Ψ_i.

Algorithm 1

Step 1: $i = 0$.

Step 2: $i = i + 1$; $j = 0$. If $I > l$, exit. Otherwise, proceed to Step 3.

Step 3: $j = j + 1$, $r = 1$. Check whether there exists an entry $\Psi_i(h)$ ($j < h \leq n_i$) such that condition (9) is satisfied. If yes, repeat Step 3. Otherwise, proceed to Step 4.

Step 4: If $j = n_i - 1$, check whether $\Psi_i(j) = z_2$ (which implies $\Psi_i(j+1) = \Psi_i(n_i) = z_1$). If yes, return to Step 2. Otherwise proceed to Step 5.

Step 5: If $j = n_i$, check whether $\Psi_i(j) = z_1$ or $\Psi_i(j) = z_2$. If the answer is yes, return to Step 2, and if the answer is no, go to Step 7. If $j \neq n_i$, proceed to Step 6.

Step 6: $r = r + 1$. Check whether there exists an entry $\Psi_i(k)$ for $j < k \leq n_i$ that satisfies either $\Psi_i(j) = z_1^r \Psi_i(k)$ or $\Psi_i(j) = z_2^r \Psi_i(k)$.
If the answer is yes, insert either $z_1^{(r-s)} \Psi_i(k)$ or $z_2^{(r-s)} \Psi_i(k)$ at an appropriate position according to the descending order of power products in Ψ_i for $s = 1, ..., r - 1$. Let $n_i = n_i + (r - 1)$, and return to Step 3. In the case that the answer is no, check whether $r < \deg \Psi_i(j) - 2$. If yes, repeat Step 6. Otherwise, proceed to Step 7.

Step 7: Check whether $\deg_{z_1} \Psi_i(j) \geq \deg_{z_2} \Psi_i(j)$. If yes, insert $z_1^{-1} \Psi_i(j)$, while if no, insert $z_2^{-1} \Psi_i(j)$, into Ψ_i according to the descending order of power products. $n_i = n_i + 1$ Return to Step 3.

Once $\Psi_1, ..., \Psi_l$ are constructed by Algorithm 1, $\Psi = \text{diag}\{\Psi_1, ..., \Psi_l\}$ can be readily obtained. Ψ is now of size $n \times l$, where $n = \sum_{i=1}^{l} n_i$. Note that n will be smaller than n' if some power products are absent in the polynomial entries of $F(z_1, z_2)$. This will be illustrated by a non-trivial example in the next section.

Next, express $D(z_1, z_2)$ and $N(z_1, z_2)$ in the form

$$D(z_1, z_2) = D_{HT} \Psi, \quad N(z_1, z_2) = N_{HT} \Psi, \tag{10}$$

where

$$D_{HT} = \begin{bmatrix} D_{11} & \cdots & D_{1l} \\ \vdots & \ddots & \vdots \\ D_{l1} & \cdots & D_{ll} \end{bmatrix}, \quad N_{HT} = \begin{bmatrix} N_{11} & \cdots & N_{1l} \\ \vdots & \ddots & \vdots \\ N_{m1} & \cdots & N_{ml} \end{bmatrix}$$

and $D_{ij} \in \mathbf{R}^{1 \times n_j}$ and $N_{ij} \in \mathbf{R}^{1 \times n_j}$ are row vectors whose entries are the coefficients of the (i,j) - indexed polynomial in $D(z_1, z_2)$ and $N(z_1, z_2)$, respectively. The system matrices A_1, A_2, B_1, B_2, C can then be constructed by the following algorithm.

Algorithm 2

Step 1: Introduce $n_i \times n_i$ matrices $A_{10}^{(i)}$ and $A_{20}^{(i)}$, $i = 1, \cdots, l$ which are determined in the following way. Set initially all entries of $A_{10}^{(i)}$ and $A_{20}^{(i)}$ to zero. For $k = 1, \ldots, n_i$, let only $A_{10}^{(i)}(k, h_k) = 1$ if there exists certain h_k $(k < h_k \le n_i)$ such that

$$\Psi_i(k) = z_1 \Psi_i(h_k), \tag{11}$$

and let only $A_{20}^{(i)}(k, m_k) = 1$ if condition (10) does not hold and there exists certain m_k $(k < m_k \le n_i)$ such that

$$\Psi_i(k) = z_2 \Psi_i(m_k). \tag{12}$$

It is obvious that $A_{10}^{(i)}$ and $A_{20}^{(i)}$ are upper triangular matrices and have at most only one entry equal to 1 in the same row of $A_{10}^{(i)}$ and $A_{20}^{(i)}$.

Step 2: For $k=1,2$, construct column vector $B_k^{(i)} \in \mathbf{R}^{n_i}$, $i = 1, \ldots, l$, by setting initially all the entries of $B_k^{(i)}$ to zero. If there exists some h such that $\Psi_i(h) = z_k$, change the h th entry of $B_k^{(i)}$ to 1.

Step 3: Construct the $n \times n$ matrices A_{10}, A_{20} and the $n \times l$ matrices B_1, B_2 as follows.

$$A_{k0} = \text{diag} \{A_{10}^{(1)}, A_{10}^{(2)}, \cdots, A_{10}^{(l)}\},$$
$$B_k = \text{diag} \{B_i^{(1)}, B_i^{(2)}, \cdots, B_i^{(l)}\}, \quad k = 1, 2. \tag{13}$$

Step 4: It is easy to see that

$$(I - A_{10}z_1 - A_{20}z_2)\Psi = B_1 z_1 + B_2 z_2 \tag{14}$$

and it can be shown, in the same way of [13], that

$$\{I - (A_{10} + B_1 D_{HT})z_1 - (A_{20} + B_2 D_{HT})z_2\}^{-1}$$
$$\cdot (B_1 z_1 + B_2 z_2) = \Psi D_R^{-1}(z_1, z_2).$$

Therefore, we have

$$N_R(z_1, z_2)D_R^{-1}(z_1, z_2) = N_{HT} \Psi D_R^{-1}(z_1, z_2) =$$
$$N_{HT}\{I - (A_{10} + B_1 D_{HT})z_1 - (A_{20} + B_2 D_{HT})z_2\}^{-1}$$
$$\cdot (B_1 z_1 + B_2 z_2)$$

and thus the realization for $W = N_R D_R^{-1}$ with

$$A_1 \triangleq A_{10} + B_1 D_{HT}, \quad A_2 \triangleq A_{20} + B_2 D_{HT},$$
$$B_1, B_2, C \triangleq N_{HT}.$$

Remark 1: It is ready to show that the obtained realization has properties (i) and (ii). Hence, when the MFD of $W(z_1, z_2)$ is r.f.c., the state-space realization by the proposed algorithms will be free of hidden modes.

Remark 2: It should be noted that Algorithms 1 and 2 are constructive and applicable to a general 2D system. That is, for any arbitrarily given strictly causal 2D transfer matrix, the matrix Ψ and the corresponding realization (A_1, A_2, B_1, B_2, C) can be definitely constructed. Then the order of the resultant realization is immediately determined by the dimension of Ψ constructed for the specified transfer matrix, which is the lowest possible one in the sense that if any power product in Ψ_i $(i = 1, \ldots, l)$ is removed, then conditions (a), (b) and (c) will be violated. Obviously, if no power product is absent in the polynomial entries of $F(z_1, z_2)$, the realization by our procedure will have the same order with that of [13], which may be viewed as an upper bound of the order due to our procedure. Otherwise, our procedure may give a realization with lower order according to the specified transfer matrix. Generally, the more the absent power products are, the lower the order of the resultant realization is. In the best case, as shown in the next section, a minimal realization, or more precisely, an absolutely minimal realization [11,16], can be obtained, which may be viewed as a lower bound of the order due to our procedure.

4. EXAMPLES AND DISCUSSIONS

Though the procedure described above seems complicated, the basic idea adopted is in fact very simple. To help the reader to grasp the essence of the idea, we first illustrate the procedure by a simple SISO (single-input single-output) example.

Example 1: Consider a strictly causal 2D system given by the transfer function $G(z_1, z_2) = b(z_1, z_2) / a(z_1, z_2)$ where

$$a(z_1, z_2) = 1 + a_{10}z_1 + a_{01}z_2 + a_{11}z_1 z_2 + a_{02}z_2^2,$$
$$b(z_1, z_2) = b_{10}z_1 + b_{01}z_2 + b_{11}z_1 z_2.$$

Let $\tilde{G}(z_1,z_2) = 1/a(z_1,z_2) = X(z_1,z_2)/U(z_1,z_2)$,
$\tilde{a}(z_1,z_2) = 1 - a(z_1,z_2)$. Then, we have

$$X(z_1,z_2) = \tilde{a}(z_1,z_2)X(z_1,z_2) + U(z_1,z_2)$$

$$= [-a_{02} \ -a_{11} \ -a_{01} \ -a_{10}]\begin{bmatrix} z_2^2 \\ z_1 z_2 \\ z_2 \\ z_1 \end{bmatrix} X(z_1,z_2)$$

$$+ U(z_1,z_2)$$

$$\triangleq D_{HT}\Psi X(z_1,z_2) + U(z_1,z_2).$$

Note that Ψ defined above has already satisfied conditions (a), (b) and (c) stated previously, and it is ready to verify that, if any of the conditions is not satisfied, the operations shown below for construction of a realization will never be possible.

As $z_i X = D_{HT} z_i \Psi X + z_i U$, $i = 1, 2$, it is easy to see that the following relation holds.

$$\begin{bmatrix} z_2^2 \\ z_1 z_2 \\ z_2 \\ z_1 \end{bmatrix} X(z_1,z_2) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -a_{02} & -a_{11} & -a_{01} & -a_{10} \end{bmatrix} z_1 \right.$$

$$+ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -a_{02} & -a_{11} & -a_{01} & -a_{10} \\ 0 & 0 & 0 & 0 \end{bmatrix} z_2 \left. \begin{bmatrix} z_2^2 \\ z_1 z_2 \\ z_2 \\ z_1 \end{bmatrix} X(z_1,z_2) \right.$$

$$+ \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} z_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} z_2 \right\} U(z_1,z_2)$$

$$\triangleq (A_1 z_1 + A_2 z_2)\Psi X(z_1,z_2) + (B_1 z_1 + B_2 z_2)U(z_1,z_2).$$

Note that A_i can be expressed as $A_i = A_{i0} + B_i D_{HT}$, $i = 1, 2$, where

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which are just the ones defined in the algorithms and can be constructed by the method described there.

Now, it is easy to see that

$$b(z_1,z_2) = [0 \ b_{11} \ b_{01} \ b_{10}]\Psi \triangleq N_{HT}\Psi$$

and

$$\Psi \frac{X(z_1,z_2)}{U(z_1,z_2)} = \Psi \frac{1}{a(z_1,z_2)}$$

$$= (I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2).$$

Letting $C = N_{HT}$, we have that

$$G(z_1,z_2) = b(z_1,z_2)\frac{1}{a(z_1,z_2)} = C\Psi \frac{1}{a(z_1,z_2)}$$

$$= C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2).$$

The algorithms shown in the previous section are in fact a kind of extension and implementation of the method described in the above example, by further considering all possible cases where the initial Ψ constructed from the power products in the given transfer function or transfer matrix does not satisfy the conditions of (b) and (c).

To verify and illustrate the proposed algorithms, consider now the following example.

Example 2: Consider the 2D transfer matrix:

$$W(z_1,z_2) =$$

$$\begin{bmatrix} \dfrac{b_{11}z_1 + b_{12}z_2}{1 + a_{11}z_1 + a_{12}z_1 z_2} & \dfrac{b_{21}z_2}{1 + a_{21}z_2 + a_{22}z_1^2 z_2} \\ \dfrac{b_{31}z_2 + b_{32}z_1 z_2}{1 + a_{11}z_1 + a_{12}z_1 z_2} & \dfrac{b_{41}z_2}{1 + a_{21}z_2 + a_{22}z_1^2 z_2} \end{bmatrix}.$$

A right MFD $W = N_R D_R^{-1}$ can be given by

$$D_R(z_1,z_2) =$$

$$\begin{bmatrix} 1 + a_{11}z_1 + a_{12}z_1 z_2 & 0 \\ 0 & 1 + a_{21}z_2 + a_{22}z_1^2 z_2 \end{bmatrix},$$

$$N_R(z_1,z_2) = \begin{bmatrix} b_{11}z_1 + b_{12}z_2 & b_{21}z_2 \\ b_{31}z_2 + b_{32}z_1 z_2 & b_{41}z_2 \end{bmatrix}.$$

With $N = N_R$ and $D = I - D_R$, we have

$$F(z_1,z_2) = \begin{bmatrix} N(z_1,z_2) \\ D(z_1,z_2) \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}z_1 + b_{12}z_2 & b_{21}z_2 \\ b_{31}z_2 + b_{32}z_1 z_2 & b_{41}z_2 \\ -a_{11}z_1 - a_{12}z_1 z_2 & 0 \\ 0 & -a_{21}z_2 - a_{22}z_1^2 z_2 \end{bmatrix}.$$

Thus the column degrees of columns 1 and 2 in $F(z_1,z_2)$ are $k_1 = 2$ and $k_2 = 3$, and the power products occurring in the polynomial entries of the two columns with non-zero coefficients are $\{z_1, z_2, z_1 z_2\}$ and $\{z_2, z_1^2 z_2\}$, respectively. The initial

column vectors Ψ_1 and Ψ_2 satisfying condition (a) can be constructed as follows.

$$\Psi_1 = [z_1 z_2 \quad z_2 \quad z_1]^T, \quad \Psi_2 = [z_1^2 z_2 \quad z_2]^T.$$

Apply now Algorithm 1. As $\Psi_1(1) = z_2 \Psi_1(3)$, $\Psi_1(2) = z_2$, $\Psi_1(3) = z_1$, Ψ_1 already satisfies conditions (b) and (c) and the calculation for $i = 1$ will end at Step 4 and return to Step 2. However, as $\Psi_2(1) \neq z_1 \Psi_2(2)$ and $\Psi_2(1) \neq z_2 \Psi_2(2)$, a new term has to be inserted into Ψ_2. Since $\Psi_2(1) = z_1^2 \Psi_2(2)$, $z_1^{-1} \Psi_2(1) = z_1 z_2$ will be chosen and inserted into Ψ_2 to produce the new vector

$$\Psi_2 = [z_1^2 z_2 \quad z_1 z_2 \quad z_2]^T$$

at Step 6. This new Ψ_2 now satisfies conditions (b) and (c), and the calculation for $i = 2$ will finally end at Step 5. Therefore, we have

$$\Psi = \text{diag}\{\Psi_1, \Psi_2\} = \begin{bmatrix} z_1 z_2 & z_2 & z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1^2 z_2 & z_1 z_2 & z_2 \end{bmatrix}^T.$$

For Ψ constructed above, it is ready to calculate that

$$D(z_1, z_2) = D_{HT} \Psi, \quad N(z_1, z_2) = N_{HT} \Psi,$$

where

$$D_{HT} = \begin{bmatrix} -a_{12} & 0 & -a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{22} & 0 & -a_{21} \end{bmatrix},$$

$$N_{HT} = \begin{bmatrix} 0 & b_{12} & b_{11} & 0 & 0 & b_{21} \\ b_{32} & b_{31} & 0 & 0 & 0 & b_{41} \end{bmatrix}.$$

Since only $\Psi_1(1) = z_1 \Psi_1(2)$ holds for Ψ_1, it follows from Algorithm 2 that

$$A_{10}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{20}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, for Ψ_2, as $\Psi_2(1) = z_1 \Psi_2(2)$ and $\Psi_2(2) = z_1 \Psi_2(3)$, Algorithm 2 produces

$$A_{10}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{20}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is also easy to obtain $B_1^{(1)} = [0\ 0\ 1]^T$, $B_2^{(1)} = [0\ 1\ 0]^T$, $B_1^{(2)} = [0\ 0\ 0]^T$ and $B_2^{(2)} = [0\ 0\ 1]^T$, since $\Psi_1(3) = z_1$, $\Psi_1(2) = z_2$ in Ψ_1 and $\Psi_2(3) = z_2$ in Ψ_2. Let

$$A_{10} = \text{diag}\{A_{10}^{(1)}, A_{10}^{(2)}\}, \quad A_{20} = \text{diag}\{A_{20}^{(1)}, A_{20}^{(2)}\},$$

we can readily construct the following system matrices:

$$A_1 = A_{10} + B_1 D_{HT}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{12} & 0 & -a_{11} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = A_{20} + B_2 D_{HT}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{12} & 0 & -a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0, & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{22} & 0 & -a_{21} \end{bmatrix},$$

$$B_1 = \text{diag}\{B_1^{(1)}, B_1^{(2)}\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$B_2 = \text{diag}\{B_2^{(1)}, B_2^{(2)}\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$C = N_{HT}.$$

Remark 3: The order of the realization for Example 2 produced by the procedure of [13] would be 14 while the one by our procedure is only 6 as shown above.

It is natural to ask if it is possible to get a minimal realization of FM-II model by the proposed method. For simplicity, consider only the SISO case here. Let $W(z_1,z_2) = q(z_1,z_2)/p(z_1,z_2)$ and $n=\max(n_q,n_p)$ where n_q and n_p are the degrees of the polynomials $q(z_1,z_2)$ and $p(z_1,z_2)$, respectively. It is then ready to see that if the dimension of the vector Ψ associated with $W(z_1,z_2)$ under conditions (a), (b) and (c) is just equal to n, which implies that Ψ must be in the form of

$$\Psi^T = [z_1^{k_n} z_2^{h_n} \quad z_1^{k_{n-1}} z_2^{h_{n-1}} \quad \cdots \quad z_1^{k_2} z_2^{h_2} \quad z_1^{k_1} z_2^{h_1}] \quad (15)$$

with $k_i + h_i = i$, $i = 1, 2, \ldots, n$, then a minimal realization of FM-II model can be obtained by our method. For example, when $n=3$, a minimal realization is possible if the associated Ψ for a given $W(z_1,z_2)$ is in one of the following forms:

$$\begin{bmatrix} z_1^3 \\ z_1^2 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} z_2^3 \\ z_2^2 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} z_1^2 z_2 \\ z_1^2 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} z_1 z_2^2 \\ z_2^2 \\ z_2 \end{bmatrix},$$

$$\begin{bmatrix} z_1^2 z_2 \\ z_1 z_2 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} z_1^2 z_2 \\ z_1 z_2 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} z_1 z_2^2 \\ z_1 z_2 \\ z_1 \end{bmatrix}, \quad \begin{bmatrix} z_1 z_2^2 \\ z_1 z_2 \\ z_2 \end{bmatrix}.$$

The first two cases just correspond to some 1D transfer functions in z_1 or z_2 where our method will produce the 1D minimal realizations in controllable canonical form. This fact means that our procedure includes the minimal realization of 1D systems as a special case.

Here, let us see an example for the other 2D cases.

Example 3: Let $W(z_1, z_2) = (b_{01}z_2 + b_{11}z_1z_2 +$ $b_{12}z_1z_2^2)/ (1 + a_{01}z_2 + a_{11}z_1z_2 + a_{12}z_1z_2^2)$. Then, it is ready to obtain $\Psi = [z_1 z_2^2 \ z_1 z_2 \ z_2]^T$ and a minimal realization (A_1, A_2, B_1, B_2, C) with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -a_{12} & -a_{11} & -a_{01} \end{bmatrix},$$

$$B_1 = [0 \ 0 \ 0]^T, \quad B_2 = [0 \ 0 \ 1]^T, \quad C = [b_{12} \ b_{11} \ b_{01}].$$

Remark 4: The order of the realization for Example 3 obtained by the procedure of [13] is 9 while the order of the realization obtained here is only 3 and it is obviously the minimal one.

Remark 5: Once a state-space realization is obtained, it is usually straightforward to get a circuit realization by the well-known techniques (see, e.g., [3,16,17]). For instance, from the state-space realization of Example 3, we have that

$$\begin{cases} x_1(h,k) = x_2(h,k-1), \\ x_2(h,k) = x_3(h-1,k), \\ x_3(h,k) = -a_{12}x_1(h,k-1) - a_{11}x_2(h,k-1) \\ \qquad\qquad -a_{01}x_3(h,k-1) + u(h,k-1). \end{cases}$$

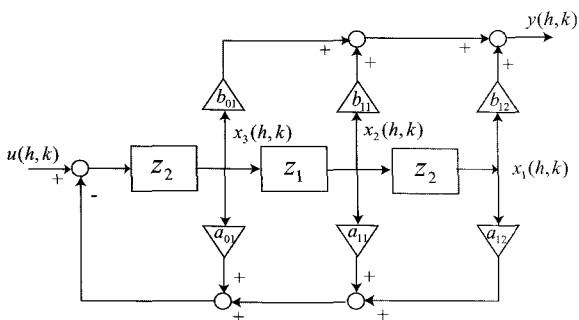It is easy to see that a circuit realization corresponding to the above result can be given as follows.

Fig. 1. A circuit realization for Example 3.

5. CONCLUSIONS

A constructive state-space realization procedure has been proposed for 2D systems which may produce FM-II local state-space model realization with much lower order than those by the procedure given in [13]. The main idea adopted in the proposed procedure is to identify the power products that are necessary for construction of a realization whereas discarding those redundant power products so that a realization with lowest possible order can be achieved for the specified transfer function or transfer matrix. The condition for minimal realization has also been given in (15), by which one can explicitly identify all the classes of the systems for a certain order n that can be minimally realized by the proposed procedure, even before actually conducting the realization. Nontrivial examples have been given to illustrate the effectiveness and the advantages of the proposed realization procedure.

Finally, it is worth pointing out that when we get a realization with a certain order, say r, which is not the absolutely minimal one, we can neither infer that the considered system has no an absolutely minimal realization, nor tell if r is the minimal one among all the possible realizations. Therefore, we face the same difficulty as in the case for Reosser model realization [3,9,11,12,14-17]). To establish a necessary and sufficient condition to verify the minimality of an n D ($n \geq 2$) realization is still a very challenging open problem.

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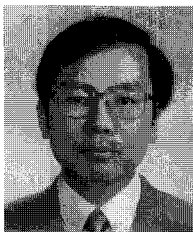
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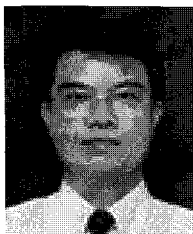
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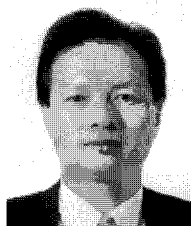
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