SPECTRAL AREA ESTIMATES FOR NORMS OF COMMUTATORS

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ABSTRACT. Let A and B be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when A is subnormal or p-hyponormal.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If T is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C. R. Putnam [7] proved that $\| T^*T - TT^* \| \le \mathrm{Area}(\sigma(T))/\pi$, where $\sigma(T)$ is the spectrum of T. The second named author [5] has proved that if T is a hyponormal operator and K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\|T^*K - KT^*\| \le 2\{\mathrm{Area}(\sigma(T))/\pi\}^{1/2}\|K\|.$$

We don't know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When T is a p-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A. Uchiyama [10] generalized the Putnam inequality, that is,

$$\|T^*T - TT^*\| \le \phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{\mathrm{Area}(\sigma(T))/\pi\}^p.$$

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\|T^*K - KT^*\|$ when $TK = KT$. H. Alexander [1] proved the following inequality for a uniform algebra A. If f is in A then

$$\mathrm{dist}(\bar{f}, A) \le \{\mathrm{Area}(\sigma(f))/\pi\}^{1/2}.$$

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\|T^*K - KT^*\|$ when T is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a p-hyponormal and we use it to estimate $\|T^*K - KT^*\|$.

In §4, we try to estimate $\|T^*K - KT^*\|$ for arbitrary contraction. In §5, we show a few results about area estimates for p-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, T is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, let $lat\mathcal{A}$ be the lattice of all $\mathcal{A}$-invariant projections. For a compact subset X in $\mathcal{C}$, rat(X) denotes the set of all rational functions on X.

2. Subnormal operators

In order to prove Theorem 1, we use the original Alexander inequality.

Theorem 1. *Let T be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and f a rational function on $\sigma(T)$ whose poles are not on it. Then*

$$\| T^*f(T) - f(T)T^* \| \leq \{\mathrm{Area}(\sigma(T))/\pi\}^{1/2}\{\mathrm{Area}(\sigma(f(T)))/\pi\}^{1/2}.$$

Proof. Suppose that $N \in \mathcal{B}(\mathcal{K})$ is a normal extension of $T \in \mathcal{B}(\mathcal{H})$ and P is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then $T = PN \mid \mathcal{H}$ and so

$$
\begin{aligned}
&T^*f(T) - f(T)T^* \\
=\ & PN^*Pf(N)P - Pf(N)PN^*P \\
=\ & PN^*f(N)P - Pf(N)PN^*P \\
=\ & Pf(N)N^*P - Pf(N)PN^*P \\
=\ & Pf(N)(1-P)N^*P \\
=\ & Pf(N)(1-P) \cdot (1-P)N^*P
\end{aligned}
$$

because $f(N)P = Pf(N)P$ and $f(N)N^* = N^*f(N)$.

Let F be a rational function in rat$(\sigma(T))$. Put $\mathcal{B}_F$ = the norm closure of $\{g(F(N)) \ ; \ g \in \mathrm{rat}(\sigma(F(N)))\}$ then P belongs to $lat\mathcal{B}_F$. Hence

$$
\begin{aligned}
&\| (1-P)F(N)^*P \| \\
\leq\ & \mathrm{dist}(F(N)^*, \mathcal{B}_F) \leq \mathrm{dist}(\bar{z}, \mathrm{rat}(\sigma(F(N)))) \\
\leq\ & \{\mathrm{Area}(\sigma(F(N)))/\pi\}^{1/2}
\end{aligned}
$$

by the Alexander's theorem [1]. Hence, applying F to $F = z$ or $F = f$

$$
\begin{aligned}
&\| T^*f(T) - f(T)T^* \| \\
\leq\ & \| (1-P)f(N)^*P \| \cdot \| (1-P)N^*P \| \\
\leq\ & \{\mathrm{Area}(\sigma(f(N)))/\pi\}^{1/2}\{\mathrm{Area}(\sigma(N))/\pi\}^{1/2} \\
\leq\ & \{\mathrm{Area}(\sigma(f(T)))/\pi\}^{1/2}\{\mathrm{Area}(\sigma(T))/\pi\}^{1/2}.
\end{aligned}
$$

$\square$

If T is a cyclic subnormal operator and $KT = TK$ then using a theorem of T. Yoshino [12] we can prove that

$$\|T^*K - KT^*\| \le \{\mathrm{Area}(\sigma(T))/\pi\}^{1/2}\{\mathrm{Area}(\sigma(K))/\pi\}^{1/2}.$$

The proof is almost same to one of Theorem 1.

3. p-hyponormal operators

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a p-hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W. Arveson [2, Lemma 2] and Lemma 2 is due to A. Uchiyama [11, Theorem 3].

We need the following notation to give Theorem 2 and Proposition 1. Let ϕ be a positive function on $(0, \infty)$ such that

$$\phi(t) = \begin{cases} t & \text{if } t \text{ is an integer} \\ t + 2 & \text{if } t \text{ is not an integer.} \end{cases}$$

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H})$ for each operator $T \in \mathcal{B}(\mathcal{H})$.

Lemma 1. *Let $\mathcal{A}$ be an arbitrary ultra-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, and let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathrm{dist}(T, \mathcal{A}) = \sup\{\|(1 - P)(1 \otimes T)P\| \; ; \; P \in lat(1 \otimes \mathcal{A})\}.$$

Lemma 2. *If T is a p-hyponormal operator, then*

$$\|T^*T - TT^*\| \le \phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{\mathrm{Area}(\sigma(T))/\pi\}^p.$$

Lemma 3. *If T is a p-hyponormal operator then*

$$\mathrm{dist}(T^*, \mathcal{A}) \le \sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{\mathrm{Area}(\sigma(T))/\pi\}^{p/2},$$

where $\mathcal{A}$ is the strong closure of $\{f(T) \; ; \; f \in \mathrm{rat}(\sigma(T))\}$.

Proof. Let $S = 1 \otimes T$. Then S is p-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup\{\|(1 - P)SP\| \; ; \; P \in lat(1 \otimes \mathcal{A})\}$. If $P \in lat(1 \otimes \mathcal{A})$ then $SP = PSP$ and so

$$\begin{aligned}
& \|(1 - P)SP\|^2 \\
= \;& \|PSS^*P - PSPS^*P\| \\
= \;& \|PSS^*P - PS^*SP + PS^*SP - PSPS^*P\| \\
\le \;& \|P(S^*S - SS^*)P\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\| \\
\le \;& \|S^*S - SS^*\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\|.
\end{aligned}$$

By [11, Lemma 4], PSP is p-hyponormal and so by Lemma 2 we have

$$\|PSS^*P - PSPS^*P\|^2$$

$$\leq \quad \phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{\text{Area}(\sigma(T))/\pi\}^p$$

$$+\phi\left(\frac{1}{p}\right)\|PSP\|^{2(1-p)}\{\text{Area}(\sigma(PSP))/\pi\}^p$$

$$\leq \quad 2\phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{\text{Area}(\sigma(T))/\pi\}^p$$

because $\|PSP\| \leq \|S\| = \|T\|$ and $\sigma(PSP) \subset \sigma(S) = \sigma(T)$. By Lemma 1,

$$\text{dist}(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}.$$

$\square$

Theorem 2. *If T is a p-hyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 2\sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{\text{Area}(\sigma(T))/\pi\}^{p/2}\|K\|.$$

Proof. When $\mathcal{A}$ is the strong closure of $\{f(T)\ ;\ f \in \text{rat}(\sigma(T))\}$, for any $A \in \mathcal{A}$

$$\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\|\|K\|.$$

Now Lemma 3 implies the theorem. $\square$

In Theorem 2, if $p = 1$, that is, T is hyponormal then $\|T^*K - KT^*\| \leq 2\sqrt{2}\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\|$. The constant $2\sqrt{2}$ is not best because the second author [5] proved that $\|T^*K - KT^*\| \leq 2\{\text{Area}(\sigma(T))/2\}^{1/2}\|K\|$. If $p = \frac{1}{2}$, that is, T is semi-hyponormal then

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|.$$

4. Norm estimates

In general, it is easy to see that $\|T^*T - TT^*\| \leq \|T\|^2$. By Theorem 1, if T is subnormal and f is an analytic polynomial then

$$\|T^*f(T) - f(T)T^*\| \leq \|T\|\|f(T)\|.$$

In this section, we will prove that $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$ for arbitrary T in $\mathcal{B}(\mathcal{H})$.

Theorem 3. *If T is a contraction on $\mathcal{H}$ and f is an analytic function on the closed unit disc $\bar{D}$ then $\| T^*f(T) - f(T)T^* \| \leq \sup_{z \in D} | f(z) |.$*

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator U on $\mathcal{K}$ such that $\mathcal{K}$ is a Hilbert space with $\mathcal{K} \supseteq \mathcal{H}$ and $T^n = PU^n \mid \mathcal{K}$ for $n \geq 0$, where P is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then it is known that $U^*P = PU^*P$ and $f(T) = Pf(U) \mid \mathcal{H}$. Hence

$$
\begin{aligned}
& T^* f(T) - f(T)T^* \\
= \ & PU^*Pf(U)P - Pf(U)PU^*P \\
= \ & PU^*Pf(U)P - Pf(U)U^*P \\
= \ & PU^*(I - P)f(U)P
\end{aligned}
$$

because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore

$$
\begin{aligned}
& \| \ T^* f(T) - f(T)T^* \ \| \\
= \ & \| \ PU^*(I - P)f(U)P \ \| \leq \sup_{z \in D} | \ f(z) \ | \ .
\end{aligned}
$$

$\square$

Corollary 1. *If T is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$ $\| \ T^*T^n - T^nT^* \ \| \leq \| \ T \ \|^{n+1}$.*

Proof. Put $A = T/\|T\|$ then A is a contraction and so by Theorem 2 $\|A^*A^n - A^nA^*\| \leq 1$ and so $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$. $\square$

5. Remarks

In this section, we give spectral area estimates for p-quasihyponomal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1, T$ is said to be p-quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. A 1-quasihyponormal operator is called quasihyponormal.

Lemma 4. *Let T be p-quasihyponormal and P be a projection such that $TP = PTP$. Then PTP is also p-quasihyponormal.*

Proof. Since T is p-quaihyponormal, $T^*(T^*T)^pT \geq T^*(TT^*)^pT$. Hence, we have

$$
PT^*(T^*T)^pTP \geq PT^*(TT^*)^pTP.
$$

Since by the Hansen's inequality [4]

$$
\begin{aligned}
& PT^*(T^*T)^pTP = (PTP)^*P(T^*T)^pP(PTP) \\
\leq \ & (PTP)^*(PT^*TP)^p(PTP) \\
= \ & (PTP)^*\{(PTP)^*(PTP)\}^p(PTP)
\end{aligned}
$$

and by $0 < p < 1$

$$
\begin{aligned}
& PT^*(TT^*)^pTP \geq (PT^*P)(TPT^*)^p(PTP) \\
= \ & (PTP)^*\{(PTP)(PTP)^*\}^p(PTP),
\end{aligned}
$$

we have

$$
(PTP)^*\{(PTP)^*(PTP)\}^p \geq (PTP)^*\{(PTP)(PTP)^*\}^p(PTP).
$$

Hence, PTP is p-quasihyponormal. $\square$

Proposition 1. *If T is a p-quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 4\left[\phi\left(\frac{1}{p}\right)\right]^{1/4} \|T\|^{1-p/2}\{\text{Area}(\sigma(T))/\pi\}^{p/4}\|K\|.$$

In particular, if T is quasihyponormal then

$$\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{\text{Area}(\sigma(T))/\pi\}^{1/4}\|K\|.$$

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6], $\|T^*T - TT^*\| \leq 2\|T\|^{2-p}\sqrt{\phi(\frac{1}{p})}\{\text{Area}(\sigma(T))/\pi\}^{p/2}$. Hence by Lemma 4

$$\text{dist}(T^*, \mathcal{A}) \leq 2\|T\|^{1-\frac{p}{2}}\phi\left(\frac{1}{p}\right)^{\frac{1}{4}}\{\text{Area}(\sigma(T))/\pi\}^{p/4}.$$

This implies the proposition.   □

Let H^2 and H^∞ be the usual Hardy spaces on the unit circle and z the coordinate function. M denotes an invariant subspace of H^2 under the multiplication by z. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose N is the orthogonal complement of M in H^2. For a function ϕ in H^∞, S_ϕ is an operator on N such that $S_\phi f = P(\phi f)$ $(f \in N)$, where P is the orthogonal projection from H^2 to N. For a symbol ϕ in L^∞, T_ϕ denotes the usual Toeplitz operator on H^2.

Proposition 2. *Suppose $\Phi = q\bar{\phi}$ belongs to H^∞. Then*
  (1) $\| S_\phi^*S_\phi - S_\phi S_\phi^* \| \leq \text{Area}(\overline{\Phi(D)})/\pi$;
  (2) $\| S_\phi^*S_\phi^n - S_\phi^n S_\phi^* \| \leq \{\text{Area}(\overline{\Phi(D)})/\pi\}^{n+1}$ *for $n \geq 0$.*

Proof. By a well known theorem of Sarason [8],

$$\| S_\phi \| = \| \phi + qH^\infty \| = \| \bar{q}\phi + H^\infty \| = \| \bar{\Phi} + H^\infty \|.$$

By Nehari's theorem [6], $\| \bar{\Phi} + H^\infty \| = \| H_{\bar{\Phi}} \|$, where $H_{\bar{\Phi}}$ denotes a Hankel operator from H^2 to $\bar{z}\bar{H}^2$. Since $\| H_{\bar{\Phi}} \|^2 = \| T_\Phi^*T_\Phi - T_\Phi T_\Phi^* \|$, where T_Φ denotes a Toeplitz operator on H^2, by the Putnam inequality

$$\| T_\Phi^*T_\Phi - T_\Phi T_\Phi^* \| \leq \text{Area}(\sigma(T_\Phi))/\pi = \text{Area}(\overline{\Phi(D)})/\pi.$$

Now since $\| S_\phi^*S_\phi - S_\phi S_\phi^* \| \leq \| S_\phi \|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.   □

Proposition 3. *Suppose f and g are in H^∞. Then*

$$\|T_f^*T_g - T_g T_f^*\| \leq \{\text{Area}(\overline{f(D)})/\pi\}^{1/2}\{\text{Area}(\overline{g(D)})/\pi\}^{1/2}.$$

Proof. It is easy to see that $T_f^* T_g - T_g T_f^* = H_{\bar{g}}^* H_{\bar{f}}$. Hence

$$\|T_f^* T_g - T_g T_f^*\| \leq \|H_{\bar{g}}\| \cdot \|H_{\bar{f}}\|.$$

Since $H_{\bar{f}}^* H_{\bar{f}} = T_f^* T_f - T_f T_f^*$, by the Putnam inequality

$$\|T_f^* T_g - T_g T_f^*\| \leq \{\mathrm{Area}(\overline{f(D)})/\pi\}^{1/2} \{\mathrm{Area}(\overline{g(D)})/\pi\}^{1/2}.$$

$\square$

References

[1] H. Alexander, *Projections of polynomial hulls*, J. Funct. Anal. **13** (1973), 13–19.

[2] W. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), no. 3, 208–233.

[3] M. Chō and M. Itoh, *Putnam's inequality for p-hyponormal operators*, Proc. Amer. Math. Soc. **123** (1995), no. 8, 2435–2440.

[4] F. Hansen, *An operator inequality*, Math. Ann. **246** (1979/80), no. 3, 249–250.

[5] T. Nakazi, *Complete spectral area estimates and self-commutators*, Michigan Math. J. **35** (1988), no. 3, 435–441.

[6] Z. Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162.

[7] C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **116** (1970), 323–330.

[8] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

[9] B. Sz.-Nagy and C. Foias, *Harmonic Analysis Of Operators On Hilbert Space*, American Elsevier, New York, 1970.

[10] A. Uchiyama, *Berger-Shaw's theorem for p-hyponormal operators*, Integral Equations Operator Theory **33** (1999), 221–230.

[11] _____, *Inequalities of Putnam and Berger-Shaw for p-quasihyponormal operators*, Integral Equations Operator Theory **34** (1999), no. 2, 91–106.

[12] T. Yoshino, *Subnormal operators with a cyclic vector*, Tohoku Math. (2) **21** (1969), 47–55.

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