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THE FINE SPECTRA OF THE RHALY OPERATORS ON c.

M. YILDIRIM

Abstract. In 1975, Wenger [4] determined the fine spectra of Cesàro operator C_1 on c, the space of convergent sequences. In [7], the spectrum of the Rhaly operators on c_0 and c, under the assumption that $\lim_{n\to\infty}(n+1)a_n = L \neq 0$, has been determined. In this paper the author determine the fine spectra of the Rhaly matrix R_a as an operator on the space c, with the same assumption.

1. Introduction

In this paper, s; c; ℓ^1; T^*; X^*; $B(X)$; A^t; $\pi_0(T,X)$; $\sigma(T,X)$; $O(1)$; will denote the set of all sequences; convergent sequences; sequences such that $\sum_k \mid x_k \mid < \infty$; the adjoint operator of T; the continuous dual of X; the linear space of all bounded linear operators on X, say, T on X into itself; the transposed matrix of A; the eigenvalues of T on X; the spectrum of T on X; capital order, that is, $x_n = O(1)$ if there exists $M \in R^+$ such that $\mid x_n \mid \leq M$ for all n, respectively.

In addition, we assume that; given a scalar sequence of $a = (a_n)$, a Rhaly matrix $R_a = (a_{nk})$ is the lower triangular matrix where $a_{nk} = a_n,\ k \leq n$ and $a_{nk} = 0$ otherwise. Let S denote the set $\{\ a_n\ :\ n = 0, 1, 2,\ ...\ \}$.

(a) $L = \lim_n (n+1)a_n$ exists, finite, and nonzero,

(b) $a_n > 0$ for all n, and

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(c) $a_i \neq a_j$ for $i \neq j$.
(d) $a = (a_n)$ is monoton decreasing.

In 1975, Wenger [4] determined the fine spectra of Cesàro operator C_1 on c, the space of convergent sequences. In [7], the spectrum of the Rhaly operators on c_0 and c, under the assumption that $\lim_{n\to\infty}(n+1)a_n = L \neq 0$ has been determined. Also in [3], [8], [9], [10], [11] and [12] the Spectrum of Rhaly Operator over some kinds of spaces has been determined.

Under the above conditions, the purpose of this study is to determine the fine spectra of Rhaly operator R_a as an operator on the Banach space c of convergent sequences normed by $\| x \| = \sup_{n\geq 0} | x_n |$.

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [1]).

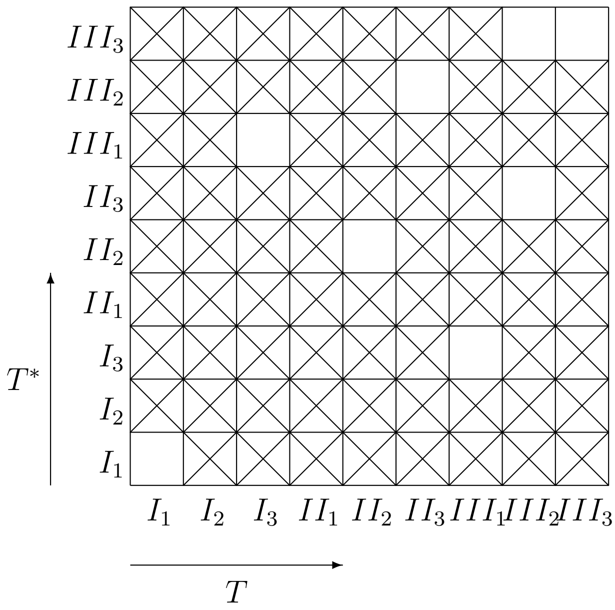

FIG.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If λ is a complex number such that $A = \lambda I - T \in I_1$ or $A = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $I_2\sigma(T, X)$, $I_3\sigma(T, X)$, $II_2\sigma(T, X)$, $II_3\sigma(T, X)$, $III_1\sigma(T, X)$, $III_2\sigma(T, X)$, $III_3\sigma(T, X)$. For example, if $A = \lambda I - T$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(T, X)$.

THEOREM 1.1. $L = \lim_{n\to\infty}(n+1)a_n = 0$ then $\pi_0(R_a, c) = S$. (see [6])

THEOREM 1.2. If $0 < L < \infty$ then $S \cap (2L, \infty) \subseteq \pi_0(R_a, c) \subseteq S \cap [2L, \infty)$. (see [7])

THEOREM 1.3. If $L = 0$ then $\pi_0(R_a^*, c^* \cong \ell^1) = S \cup \{0\}$. (see [6])

THEOREM 1.4. If $0 < L < \infty$ then

$$\left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | < \ \frac{L}{2} \right\} \ \cup \ S \ \cup \ \{L\} \ \subseteq \pi_0(R_a^*, c \cong \ell^1)$$
$$\subseteq \left(\left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | \leq \ \frac{L}{2} \right\} - \{0\} \right) \ \cup \ S.$$

(see [7])

Theorem 1.5. If $L = 0$, then $\sigma(M, c_0) \ = \ S \ \cup \ \{0\}$. (see [6])

Theorem 1.6. If $0 < L < \infty$ then

$$\sigma(R_a, c) = \left\{ \lambda \ : | \ \lambda - \frac{L}{2} \ | \leq \ \frac{L}{2} \right\} \ \cup \ S.$$

(see [7])

Theorem 1.7. If $L = 0$, then $0 \in III_2\sigma(M, c)$ and $\lambda \in III_3\sigma(M, c)$ for $\lambda = a_m,$ ($m = 0, 1, 2, ...$). (see [6])

2. Main Results

Leibovitz showed in [2] that R_a is a bounded operator on c iff $\{ (n+1)a_n \}$ converges. Also, it is shown that if $R_a : c \to c$ and $L = \lim_{n\to\infty} (n+1)a_n$, then $R_a^* \in B(\ell^1)$ and

$$R_a^* \ = \ \begin{pmatrix} L & 0 \\ 0 & R_a^t \end{pmatrix} \tag{1}$$

[6].

Theorem 2.1. Let $0 < L < \infty$. If $\lambda \notin S$ and $\alpha L > 1$, then $\lambda \in III_1\sigma(R_a, c)$.

Proof. Since $\lambda \notin S$, $T_\lambda = \lambda I - R_a$ is a normal matrix. Hence the matrix T_λ^{-1} exists. Since

$$T_\lambda^* = \begin{pmatrix} \lambda - L & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda - a_0 & -a_1 & -a_2 & -a_3 & \cdots \\ 0 & 0 & \lambda - a_1 & -a_2 & -a_3 & \cdots \\ 0 & 0 & 0 & \lambda - a_2 & -a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2}$$

$T_\lambda^* x = (\lambda I - R_a^*)x = \theta$ implies the following;

$$\begin{aligned}
(\lambda - L)x_0 & = 0 \\
(\lambda - a_0)x_1 - \sum_{i=1}^{\infty} a_i x_{i+1} & = 0 \\
(\lambda - a_1)x_2 - \sum_{i=2}^{\infty} a_i x_{i+1} & = 0 \\
& \vdots
\end{aligned}$$

Thus $x_0 = 0$ and

$$x_n = \prod_{k=0}^{n-2} (1 - \frac{a_k}{\lambda}) x_1$$

for $n \geq 1$. Since $\alpha L > 1$, $x \in \ell^1$[6, Lemma 2.3]. Therefore it is not required $\forall n,\ x_n = 0$ for $x = (0, x_1, x_2, x_3, \ldots) \in \ell^1$. That is to say $x = \theta$ does not need to be satisfied; i.e. $T_\lambda^* = \lambda I - R_a^*$ is not one-to-one. Thus T_λ does not have a dense range [1, II.3.7.Theorem]. That is; $\overline{R(T_\lambda)} \neq c$. So that, $T_\lambda \in III$.

At this point, we have to show that T_λ^{-1} is continuous; i.e. $R(T_\lambda^*) = c^* = \ell^1$ [1, II.3.11 Theorem].

Let $y = (y_n) \in \ell^1$. If $T_\lambda^* x = y$ and $\exists x = (x_n) \in \ell^1$, then the following are satisfied.

$$(\lambda - L)x_0 = y_0,$$

$$(\lambda - a_{n-1})x_n - \sum_{k=n+1}^{\infty} a_{k-1} x_k = y_n$$

for $n \geq 1$. If we choose $x_1 = 0$, then we have

$$
\begin{aligned}
x_2 &= \frac{1}{\lambda}\left[y_2 - y_1\right] \\
x_3 &= \frac{1}{\lambda}\left[y_3 - \frac{a_1}{\lambda}y_2 - (1-\frac{a_1}{\lambda})y_1\right] \\
x_4 &= \frac{1}{\lambda}\left[y_4 - \frac{a_2}{\lambda}y_3 - \frac{a_1}{\lambda}(1-\frac{a_2}{\lambda})y_2 - (1-\frac{a_2}{\lambda})(1-\frac{a_1}{\lambda})y_1\right] \\
&\vdots \\
x_n &= \frac{1}{\lambda}\Big\{y_n - \frac{a_{n-2}}{\lambda}y_{n-1} - \frac{a_{n-3}}{\lambda}(1-\frac{a_{n-2}}{\lambda})y_{n-2} \\
&- \frac{a_{n-4}}{\lambda}(1-\frac{a_{n-2}}{\lambda})(1-\frac{a_{n-3}}{\lambda})y_{n-3} \\
&- \ldots - \frac{a_1}{\lambda}(1-\frac{a_{n-2}}{\lambda})(1-\frac{a_{n-3}}{\lambda})\ldots(1-\frac{a_2}{\lambda})y_2 \\
&- (1-\frac{a_{n-2}}{\lambda})(1-\frac{a_{n-3}}{\lambda})\ldots(1-\frac{a_2}{\lambda})(1-\frac{a_1}{\lambda})y_1\Big\} \\
&\vdots
\end{aligned}
$$

This defines the matrix transformation

$$a_{00} = \frac{1}{\lambda - L};\ a_{0k} = 0;\ a_{1k} = 0,;\ a_{no} = 0 \tag{3}$$

$$a_{21} = -\frac{1}{\lambda} \tag{4}$$

$$a_{n1} = -\frac{1}{\lambda}\prod_{j=1}^{n-2}(1-\frac{a_j}{\lambda}),\ n > 2 \tag{5}$$

$$a_{n,n-1} = -\frac{a_{n-2}}{\lambda},\ n > 2 \tag{6}$$

$$a_{n1} = -\frac{1}{\lambda^2}a_{k-1}\prod_{j=k}^{n-2}(1-\frac{a_j}{\lambda}),\ 1 < k < n-1 \tag{7}$$

$$a_{nn} = \frac{1}{\lambda} \tag{8}$$

$$a_{nk} = 0, \ k > n > 1. \tag{9}$$

From (3), we obtain

$$\sum_{n=0}^{\infty} | \ a_{n0} \ |= O(1). \tag{10}$$

Using [[7], Lemma 4.1.3] where $\alpha L > 1$, from [5] and [6] we have

$$\begin{aligned} \sum_{n=0}^{\infty} | \ a_{n1} \ | &= | \ a_{11} \ | + | \ a_{21} \ | + \sum_{n=3}^{\infty} | \ a_{n1} \ | \\ &= \frac{1}{| \ \lambda \ |} + \sum_{n=3}^{\infty} \frac{1}{| \ \lambda \ |} \prod_{j=1}^{n-2} | \ 1 - \frac{a_j}{| \ \lambda \ |} \ | \\ &= \frac{1}{| \ \lambda \ |} \left(1 + \sum_{n=3}^{\infty} \frac{O(1)}{(n-2)^{\alpha L}} \right) \ = \ O(1). \end{aligned} \tag{11}$$

From (6), (7), (8) and (9) we have

$$\begin{aligned} \sum_{n=0}^{\infty} | \ a_{nk} \ | = | \ a_{kk} \ | + | \ a_{k+1,k} \ | + \sum_{n=k+2}^{\infty} | \ a_{nk} \ | \\ = \frac{1}{| \ \lambda \ |} + \frac{a_{k-1}}{| \ \lambda \ |^2} \sum_{n=k+2}^{\infty} \frac{a_{k-1}}{\lambda^2} \prod_{j=k}^{n-2} | \ 1 - \frac{a_j}{\lambda} \ | \\ = \frac{1}{| \ \lambda \ |} + \frac{a_{k-1}}{| \ \lambda \ |^2} \left[1 + \sum_{n=k+2}^{\infty} \frac{\prod_{j=0}^{n-2} | \ 1 - \frac{a_j}{\lambda} \ |}{\prod_{j=0}^{k-1} | \ 1 - \frac{a_j}{\lambda} \ |} \right] \end{aligned}$$

$$\leq \frac{1}{|\lambda|} + \frac{a_{k-1}}{|\lambda|^2}\left[1 + (k-1)^{\alpha L} \sum_{n=k+2}^{\infty} \frac{1}{(n-1)^{\alpha L}}\right]$$

$$\leq \frac{1}{|\lambda|} + \frac{a_{k-1}}{|\lambda|^2}\left[1 + (k-1)^{\alpha L} \int_{k+1}^{\infty} \frac{dx}{(x-1)^{\alpha L}}\right]$$

$$\leq \frac{1}{|\lambda|} + \frac{a_{k-1}}{|\lambda|^2}\left[1 + (k-1)^{\alpha L} \frac{k^{1-\alpha L}}{\alpha L - 1}\right] \tag{12}$$

$$\leq \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\left[a_{k-1} + k a_{k-1} k^{\alpha L - 1} \frac{k^{1-\alpha L}}{\alpha L - 1}\right]$$

$$\leq \frac{1}{|\lambda|}\left[1 + \frac{a_{k-1}}{|\lambda|} + \frac{O(1)}{|\lambda|(\alpha L - 1)}\right] = O(1)$$

for $k > 2$.

Therefore, with (10), (11) and (12), we have $\sup_k \sum_n | a_{nk} |< \infty$; i.e. $A \in B(\ell^1)$. This allows us to write $Ay = x \in \ell^1$, so that T_λ^* is shown to be onto. As a result, $T_\lambda \in III_1$ and $\lambda \in III_1\sigma(R_a, c)$. □

Theorem 2.2. Let $0 < L < \infty$. If $\lambda \notin S$, $\lambda \neq L$ and $\alpha L = 1$, then $\lambda \in II_2\sigma(R_a, c)$.

Proof. Since $\lambda \notin S$, then the matrix that corresponds to operator $T_\lambda = \lambda I - R_a$ is a normal matrix. So, $T_\lambda = \lambda I - R_a$ is one-to-one; i.e. $T_\lambda = \lambda I - R_a \in (1)$ or $T_\lambda = \lambda I - R_a \in (2)$.

Consider the adjoint operator $T_\lambda^* = \lambda I - R_a^*$. Then if $T_\lambda^* x = \theta$, then $x_0 = 0$ and

$$\prod_{k=0}^{n-2}(1 - \frac{a_k}{\lambda})x_1 = 0$$

for $n \geq 1$. Since $\alpha L = 1$, we have

$$x = (0, x_1, x_2, x_3, \ldots) \in \ell^1 \iff x_1 = 0 \iff x = \theta.$$

Hence T_λ^* is one-to-one. Using this result and [[1], II.3.7 Theorem] we have proved that T_λ has dense range;i.e. $\overline{R(T_\lambda)} = c$ and therefore we conclude that $T_\lambda \in II$. Since $\lambda \in \sigma(R_a, c)$, we get that $\lambda \in II_2\sigma(R_a, c)$;i.e. $\lambda \in II_2\sigma(R_a, c)$. □

Theorem 2.3. Let $0 < L < \infty$. If for all $\lambda \neq a_m$, then $L \in III_2\sigma(R_a, c)$.

Proof. Since $T_L = LI - R_a$ is a triangular matrix, therefore T_L is one-to-one. Now consider the adjoint operator $T_L^* = LI - R_a^*$. Since $T_L^* e_1 = \theta$ (where $e_1 = (1, 0, 0, \ldots)$), T_L^* is not one-to-one. From [1, II.3.7 Theorem], T_L does not have a dense range;i.e. $\overline{R(T_\lambda)} \neq c$;i.e. $T_L \in III$. On the other hand, since T_L is one-to-one we have $L \in III_1\sigma(R_a, c) \cup III_2\sigma(R_a, c)$. In order to show that T_L^{-1} is continuous we must have that from [[1] ,II.3.11 Theorem] that $R(T_L^*) = c^* \cong \ell^1$.

Let $e_1 = (1, 0, 0, \ldots) \in \ell^1$ then we conclude that there is no $x \in \ell^1$ such that $T_L^* x = e_1$. As a result T_L does not have a bounded inverse;i.e. $L \in III_2\sigma(R_a, c)$. □

Theorem 2.4. Let $0 < L < \infty$. If $\lambda = a_m \neq L$ for at least one m ($m = 0, 1, \ldots$), then $\lambda = a_m \in III_3\sigma(R_a, c)$ for $L < a_m$; $\lambda = a_m \in II_1\sigma(R_a, c)$ for $L > a_m$.

Proof. Then the system $(a_m I - R_a)x = 0$ implies $x_k = 0$ for $k = 0, 1, \ldots, m-1$, and for $n \geq 1$

$$\begin{aligned} x_{m+n} &= \frac{a_{m+n} a_m^{n-1}}{(a_m - a_{m+n})(a_m - a_{m+n-1}) \ldots (a_m - a_{m+1})} x_m \\ &= \frac{\frac{a_{m+n}}{a_m}}{\left(1 - \frac{a_{m+n}}{a_m}\right)\left(1 - \frac{a_{m+n-1}}{a_m}\right) \ldots \left(\frac{a_{m+1}}{a_m}\right)} x_m, \ n = 1, 2, \ldots \end{aligned}$$

Let $b_k := \dfrac{a_{m+k}}{a_m}$ and

$\lim\limits_{n\to\infty} (n+1) b_n = \lim\limits_{n\to\infty} (n+1) \dfrac{a_{m+n}}{a_m} = \dfrac{1}{a_m} \lim\limits_{n\to\infty} (n+1) a_{m+n} = \dfrac{L}{a_m} =: L_0.$

Hence

$$\begin{aligned}
\lim_{n\to\infty} x_{m+n} &= \lim_{n\to\infty} \frac{b_n}{(1-b_n)(1-b_{n-1})\dots(1-b_1)} x_m \\
&= \lim_{n\to\infty} \frac{(n+1)b_n}{(n+1)(1-b_n)(1-b_{n-1})\dots(1-b_1)} x_m \\
&= x_m \frac{L}{a_m} \lim_{n\to\infty} \frac{1}{(n+1)(1-b_n)(1-b_{n-1})\dots(1-b_1)} x_m \\
&= x_m \frac{L}{a_m} \lim_{n\to\infty} \frac{(1-\frac{1}{2})(1-\frac{1}{3})\dots(1-\frac{1}{n+1})}{(1-b_n)(1-b_{n-1})\dots(1-b_1)} x_m \\
&= x_m \frac{L}{a_m} \lim_{n\to\infty} \prod_{k=1}^{n} \frac{(1-\frac{1}{k+1})}{1-b_k}
\end{aligned}$$

Let $C_k := \dfrac{1-\frac{1}{k+1}}{1-b_k} = 1 + \dfrac{(k+1)b_k}{(1-b_k)(k+1)}$ and $D_k := \dfrac{(k+1)b_k}{(1-b_k)(k+1)}$. Then we have

$\ln(C_k) = \ln(1+D_k) = D_k - \frac{1}{2}D_k^2 + O(D_k^2)$. Hence

Case I. If $L > a_m$ (i.e, $L_0 > 1$), then $\sum_k \ln(1+D_k) = +\infty$. This implies $\lim_{n\to\infty} x_n = +\infty$. Then $x \in c$ iff $x_m = 0$ (i.e, $x = \theta$. Then $a_m I - R_a$ is a triangle and is therefore injective, so that $a_m I - R_a \in (1) \cup (2)$.

Case II. If $L < a_m$ (i.e; $L_0 < 1$), then $\sum_k \ln(1+D_k) = -\infty$. So $\lim_{n\to\infty} x_n = 0$. Hence $(a_m I - R_a)^{-1}$ does not exist;i.e. $a_m I - R_a \in (3)$.

Now we consider the operator $a_mI - R_a^* : \ell^1 \to \ell^1$ and let $x \in \ell^1$. Therefore we have

$$(a_mI - R_a^*)x = \Bigg((a_m - L)x_0, (a_m - a_0)x_1 - \sum_{i=1}^{\infty} a_i x_{i+1}, (a_m - a_1)x_2 - \sum_{i=2}^{\infty} a_i x_{i+1}, \ldots, (a_m - a_{m-1})x_m - \sum_{i=m}^{\infty} a_i x_{i+1}, - \sum_{i=m+1}^{\infty} a_i x_{i+1}, (a_m - a_{m+1})x_{m+2} - \sum_{i=m+2}^{\infty} a_i x_{i+1}, (a_m - a_{m+2})x_{m+3} - \sum_{i=m+3}^{\infty} a_i x_{i+1}, \ldots\Bigg).$$

It is seen from the image sequence that, non zero sequences are mapped into the zero sequence.

For if $0 = x_{m+2}, x_{m+3}, \ldots$, then the contditions

$$\begin{aligned}
(a_m - L)x_0 &= 0 \\
(a_m - a_0)x_1 - \sum_{i=1}^{m} a_i x_{i+1} &= 0 \\
(a_m - a_1)x_2 - \sum_{i=2}^{m} a_i x_{i+1} &= 0 \\
&\ddots \\
(a_m - a_{m-2})x_{m-1} - a_{m-1}x_m - a_m x_{m+1} &= 0 \\
(a_m - a_{m-1})x_m - a_m x_{m+1} &= 0
\end{aligned}$$

together with $x_0 = 0$ will ensure that the sequence x will be mapped into zero. Therefore, we obtain an equation for a homogeneous system of m unknown with m equations and $\{x_1, x_2, \ldots, x_{m+1}\}$ with $m+1$ unknown which obviously is the solution of this homogeneous system.

We shall turn back to case II. Since $a_mI - R_a \in (3)$ and $a_mI - R_a^* \in (3)$, according to diagram $a_mI - R_a \in III_3$;i.e, $a_m \in III_3\sigma(R_a, c)$.

Now we can consider the case I. It is Clearly seen that $a_m I - R_a \in III$. It remains to show that $a_m I - R_a^*$ is surjective. Let us take any $y \in \ell^1$. If $(a_m I - R_a^*)x = y$, then

$$\begin{aligned}
(a_m - L)x_0 &= y_0 \\
(a_m - a_0)x_1 - \sum_{i=1}^{\infty} a_i x_{i+1} &= y_1 \\
(a_m - a_1)x_2 - \sum_{i=2}^{\infty} a_i x_{i+1} &= y_2 \\
\cdots \ \cdots \ \cdots \ \cdots &\quad \cdots \\
(a_m - a_{m-1})x_m - \sum_{i=m}^{\infty} a_i x_{i+1} &= y_m \\
- \sum_{i=m+1}^{\infty} a_i x_{i+1} &= y_{m+1} \\
(a_m - a_{m+1})x_{m+2} - \sum_{i=m+2}^{\infty} a_i x_{i+1} &= y_{m+2}
\end{aligned}$$

By choosing $x_{m+1} = 0$, we can solve for $x_1, x_2, \ldots, x_m$ in terms of $y_1, y_2, \ldots, y_{m+1}$. The remaining equations can be written in the from $x = By$, where the nonzero entries of B are

$$\begin{aligned}
b_{m+2,m+1} &= -\frac{1}{a_m} \\
b_{m+n,m+n} &= \frac{1}{a_m}, \ n \geq 2 \\
b_{m+n,m+n-1} &= -\frac{a_{m+n-1}}{a_m^2}, \ 3 \leq n \leq \infty \qquad (13) \\
b_{m+n,m+j} &= -\frac{a_{m+j-1}}{a_m^{n-j+1}} \prod_{k=j}^{n-2} (a_m - a_{m+k}), \ 2 \leq j \leq n-2, \ n > 4 \\
b_{m+n,m+1} &= -\frac{1}{a_m^{n-1}} \prod_{j=1}^{n-2} (a_m - a_{m+j}), \ n \geq 4
\end{aligned}$$

Hence if Ratio test and raabe test are used respectively, we have

$$\sum_{n=2}^{\infty} | b_{m+n,m+1} | =| b_{m+2,m+1} | + \sum_{n=3}^{\infty} | b_{m+n,m+1} |$$

$$= \frac{1}{a_m} + \sum_{n=3}^{\infty} \frac{1}{a_m^{n-1}} \prod_{j=1}^{n-2} | a_m - a_{m+j} | = O(1)$$

and for $j > 1$,

$$\sum_{n=j}^{\infty} |b_{m+n,m+j} |$$

$$=| b_{m+j,m+j} | + | b_{m+j+1,m+j} | \sum_{n=j+2}^{\infty} | b_{m+n,m+j} |$$

$$= \frac{1}{a_m} + \frac{a_{mj-1}}{a_m^2} \sum_{n=j+2}^{\infty} \frac{a_{m+j-1}}{a_m^{n-j+1}} \prod_{k=j}^{n-2} | a_m - a_{m+k} | = O(1).$$

Therefore, we have $\sup_k \sum_n | b_{nk} |< \infty$; i.e. $B \in B(\ell^1)$. This allows us to write $By = x \in \ell^1$, so that T_λ^* is shown to be onto. As result, $T_\lambda \in III_1$ for $\lambda > a_m$ and $a_m \in III_1\sigma(R_a, c)$. □

THEOREM 2.5. Let R_a be a regular transformation.

(a) If $Re\alpha > 0$ and $\lim(\alpha x + (1-\alpha)R_a x) = t$, then $\lim x = t$,

(b) Let $Re\alpha < 0$ and $\alpha \neq a_m$, ($m = 0, , 1, 2, ...$). If $\lim(\alpha x + (1-\alpha)R_a x) = t$, then $\lim x = t$ or x is unbounded,

(c) If $Re\alpha = 0$ ($\alpha \neq 0$), then the operator $\alpha I + (1-\alpha)R_a$ sums bounded divergent sequences.

Proof. From [[2], Propesition 3.3.(a)] R_a Rhaly matrix is regular iff R_a is asymptotic to Cesàro matrix. (i.e. $\lim_n (n+1)a_n = 1$)

(i) Let $Re\alpha > 0$ and $\lim(\alpha x + (1-\alpha)R_a x) = t$. Since

$$\alpha I + (1-\alpha)R_a = I$$

for $\alpha = 1$, $\lim x = t$. Lets suppose that $\alpha \neq 1$. Using the second part of Theorem 1.4 $\lambda := \dfrac{\alpha}{\alpha - 1} \in \rho(R_a, c) \Longleftrightarrow \dfrac{\alpha-1}{\alpha} < 1 \Longleftrightarrow Re\alpha > 0$.

From [[4],Lemma 5.2] $\lim(\alpha x+(1-\alpha)R_a x)=t$ which requires $\lim x = t$.

(ii) Let $Re\alpha<0$ and $\alpha\neq a_m$, $m = 0, , 1, 2, \ldots$. In this case $Re\alpha<0 \iff \frac{1}{\lambda}=: \frac{\alpha-1}{\alpha}$ such that $Re\frac{1}{\lambda} = Re\frac{\alpha-1}{\alpha}<1$. Hence from Theorem 2.1., $\lambda=\frac{\alpha}{\alpha-1}\in III_1\sigma(M,c)$ Since $[\alpha I+(1-\alpha)R_a]^{-1}$ exists and is bounded, $R(\alpha I+(1-\alpha)R_a)$ is closed in c. Then using [5], $\alpha I+(1-\alpha)R_a$ sums no bounded divergent sequence. Hence if $\lim \alpha x+(1-\alpha)R_a x=t$, which requires other $x\in c$ or x is not bounded. Since if $x\in c$, then $\lim R_a x=\lim x$, therefore $\lim\alpha x+(1-\alpha)R_a x = \alpha\lim x+(1-\alpha)\lim R_a x$ So, $\lim x=t$ or $x\notin \ell_\infty$.

(iii) Let $Re\alpha=0$ for $\alpha\neq 0$. Hence

$$Re\alpha=0 \iff Re\frac{1}{\lambda} = Re\frac{\alpha-1}{\alpha}=1.$$

By Theorem 2.2., we have $\lambda:=\frac{\alpha}{\alpha-1}\in II_2\sigma(R_a,c)$. So $[\alpha I+(1-\alpha)R_a]^{-1}$ exist but not continuous and $Re(\alpha I+(1-\alpha)R_a)\neq c$ but $\overline{Re(\alpha I+(1-\alpha)R_a)}=c$. That is, $Re(\alpha I+(1-\alpha)R_a)$ is not closed in c. Hence, Using [[5], Theorem 17], $\alpha I+(1-\alpha)R_a$ sums a bounded divergent sequence. □

REFERENCES

[1] S. Goldberg,*Unbounded Linear Operators*, (Mc Graw-Hill Book Comp, 1966).

[2] G. Leibowitz, 'Rhaly Matrices', *J. Math. Analysis and Applications*, **128** (1987), 272–286.

[3] H. C. Rhaly JR., ' Terraced Matrices', *Bull. London Math. Soc.*, **21** (1989), 399–406.

[4] R. B. Wenger, 'The Fine Spectra of Hölder Summability Operators', *Indian J. Pure. Appl. Math.* **6** (1965), 695–712.

[5] A. Wilansky, *Topological divsors of zero and Tauberian Theorems.*, *Trans. Amer. Math. Soc.* **113** (1964), 240–251.

[6] M. Yildirim, *On the spectrum and fine spectrum of the compact Rhaly operators*, Indian J. Pure Appl. Math. **27** (1996), 779–784.

[7] M. Yildirim, *On the spectrum of the Rhaly operators on* c_0 *and* c, Indian J. Pure Appl. Math. **29**(1998), 1301–1309.

[8] M. Yildirim, *On the spectrum of the Rhaly operators on* ℓ^p, Indian J. Pure Appl. Math. **32** (2001), 191–198.

[9] M. Yildirim, *The Fine Spectra of the Rhaly operators on* c_0, Turkish J. Math. **26** (2002)3, 273–282.
[10] M. Yildirim, *On the spectrum of the Rhaly operators on bv*, East Asian Math. J. **18** (2002)1, 21–41.
[11] M. Yildirim, *On the spectrum of the Rhaly operators on* bv_0, Comm. Korean Math. Soc. **18** (2003)4, 669–674.
[12] M. Yildirim, *On the spectrum and Fine Spectrum of the Compact Rhaly operators*, Indian J. Pure Appl. Math. **34** (2003)10, 1443–1452.

Department of Mathematics, Faculty of Science
Cumhuriyet University
Sivas 58410, Turkey
E-mail: yildirim@cumhuriyet.edu.tr