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G_δ-CONNECTEDNESS AND G_δ-DISCONNECTEDNESS IN FUZZY BITOPOLOGICAL SPACES

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Abstract. In this paper, the concepts of pairwise fuzzy G_δ-connected spaces and pairwise fuzzy G_δ-extremally disconnected spaces are introduced. The concept of pairwise fuzzy G_δ-basically disconnected spaces is defined. Characterizations of the above spaces are given besides giving several examples. Interrelations among the spaces introduced are discussed and some relevant counter examples are given.

1. Introduction and Preliminaries

Ever since the introduction of fuzzy set by L.A. Zadeh [8], the fuzzy concept has invaded almost all branches of Mathematics. The concept of fuzzy topological spaces was introduced in [4] by C.L. Chang. Since then many fuzzy toplogists [6 & 7] have extended various notions in classical toplogy to fuzzy topological spaces. In 1989, Kandil [5] introduced the concept of fuzzy bitopological spaces and since then many concepts in classical topology have been extended to fuzzy bitopological spaces. The purpose of this paper is to introduce pairwise fuzzy G_δ - connected spaces and pairwise fuzzy G_δ - disconnected spaces. Pairwise fuzzy connected and pairwise fuzzy extremally disconnected spaces were found in [3]. Pairwise fuzzy basically dissonnected space was found in [2].

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DEFINITION 1.0.1. Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a *fuzzy G_δ-set* [1] if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i \in T$.

DEFINITION 1.0.2. Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a *fuzzy F_σ-set* if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ where each $1 - \lambda_i \in T$.

DEFINITION 1.0.3. Let (X, T) be any fuzzy topological space. For any fuzzy set λ in X we define the *σ-closure* of λ, denoted by $\mathrm{cl}_\sigma \lambda$, to be the intersection of all fuzzy F_σ-sets containing λ. That is,

$$\mathrm{cl}_\sigma \lambda = \wedge\{\mu : \mu \text{ is a fuzzy } F_\sigma\text{-set and } \mu \geq \lambda\}.$$

DEFINITION 1.0.4. Let (X, T) be any fuzzy topological space. For any fuzzy set λ in X, we define the *δ-interior* of λ, denoted by $\mathrm{int}_\delta \lambda$, to be the union of all fuzzy G_δ-sets contained in λ. That is,

$$\mathrm{int}_\delta \lambda = \vee\{\mu : \mu \text{ is a fuzzy } G_\delta\text{-set and } \mu \leq \lambda\}.$$

DEFINITION 1.0.5. A *fuzzy bitopological space* [5] is a triple (X, T_1, T_2) where X is a set, T_1 and T_2 are any two fuzzy topologies on X.

NOTE 1. $G_\delta F_\sigma$ denotes the fuzzy set which is both fuzzy G_δ and fuzzy F_σ.

2. Main Results

2.1. Pairwise fuzzy G_δ-connected spaces

In this section, the concept of pairwise fuzzy G_δ-connected spaces is introduced. Besides giving examples, characterizations of pairwise fuzzy G_δ-connected spaces are also studied.

DEFINITION 2.1.1. A fuzzy bitopological space (X, T_1, T_2) is said to be pairwise fuzzy G_δ-connected iff (X, T_1, T_2) has no proper fuzzy sets λ_1 and λ_2 which are T_1-fuzzy G_δ and T_2-fuzzy G_δ respectively such that $\lambda_1 + \lambda_2 = 1$. A fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-disconnected if it is not pairwise fuzzy G_δ-connected.

REMARK 2.1.1. The pairwise fuzzy G_δ-connectedness of (X, T_1, T_2) is not governed by the fuzzy G_δ-connectedness of the spaces (X, T_1) and (X, T_2) as the following example shows.

EXAMPLE 2.1.1. Let $X = \{a, b\}$, T_1 be the discrete fuzzy topology, T_2 be the indiscrete fuzzy topology, $T_3 = \{0, 1, \lambda\}$, where $\lambda\colon X \to [0, 1]$ is such that

$$\lambda(a) = 1, \quad \lambda(b) = 0$$

and $T_4 = \{0, 1, \mu\}$, where $\mu\colon X \to [0, 1]$ is such that

$$\mu(a) = 0 \ \text{ and } \ \mu(b) = 1.$$

Then (X, T_1, T_2) is pairwise fuzzy G_δ-connected but (X, T_1) is not fuzzy G_δ-connected and (X, T_2) is fuzzy G_δ-connected. Also, (X, T_3, T_4) is not pairwise fuzzy G_δ-connected but (X, T_3) and (X, T_4) are both fuzzy G_δ-connected.

PROPOSITION 2.1.1. *The following statements are equivalent for a fuzzy bitopological space (X, T_1, T_2).*

(a) (X, T_1, T_2) *is pairwise fuzzy G_δ-connected.*
(b) *There exists no T_1-fuzzy G_δ-set $\lambda_1 \neq 0$ and T_2-fuzzy G_δ-set $\lambda_2 \neq 0$ such that $\lambda_1 + \lambda_2 = 1$.*
(c) *There exists no T_1-fuzzy F_σ-set $\lambda_1 \neq 1$ and T_2-fuzzy F_σ-set $\lambda_2 \neq 1$ such that $\lambda_1 + \lambda_2 = 1$.*
(d) (X, T_1, T_2) *contains no fuzzy set $\lambda \neq 0, 1$ and it is both T_1-fuzzy G_δ and T_2-fuzzy F_σ or both T_2-fuzzy G_δ and T_1-fuzzy F_σ.*

Proof. (a) $\Rightarrow$ (b). Assume that (a) is true. Then (b) follows from the Definition 2.1.1.

(b) $\Rightarrow$ (c). Assume that (b) is true. Let us suppose that there exists a T_1-fuzzy F_σ-set $\lambda_1 \neq 1$ and a T_2-fuzzy F_σ-set $\lambda_2 \neq 1$ such that $\lambda_1 + \lambda_2 = 1$. Then, $1 - \lambda_1 \neq 1 - 1 \neq 0$ is a non -zero T_1-fuzzy G_δ-set. Similarly, we get $1 - \lambda_2$ is a non-zero T_2-fuzzy G_δ-set. Now, $(1 - \lambda_1) + (1 - \lambda_2) = 2 - (\lambda_1 + \lambda_2) = 2 - 1 = 1$. Contradiction. Hence (c).

(c) $\Rightarrow$ (d). Assume that (c) is true. Suppose that (X, T_1, T_2) contains a fuzzy set $\lambda \neq 0, 1$ which is both T_1-fuzzy G_δ and T_2-fuzzy F_σ. Then $(1 - \lambda)$ is a proper T_1-fuzzy F_σ-set. Also by assumption, λ is T_2-fuzzy F_σ. Now, $(1 - \lambda) + \lambda = 1$. Contradiction. Hence (d).

(d) $\Rightarrow$ (a). Assume that (d) is true. Let us assume that (X, T_1, T_2) is not pairwise fuzzy G_δ-connected. Then (X, T_1, T_2) has proper fuzzy sets λ_1 and λ_2 where λ_1 is T_1-fuzzy G_δ-set and λ_2 is T_2-fuzzy G_δ-set respectively such that $\lambda_1 + \lambda_2 = 1$. Now, $\lambda_1 + \lambda_2 = 1$ implies $\lambda_1 = 1 - \lambda_2$. This implies that λ_1 is both T_2-fuzzy F_σ and $\lambda_1 \neq 1$ as λ_2 is non -zero T_2-fuzzy G_δ-set. Clearly, $\lambda_1 \neq 0, 1$ is in (X, T_1, T_2). Contradiction. Hence (a). □

2.2. Pairwise fuzzy super G_δ-connected spaces

In this section, the concept of pairwise fuzzy super G_δ-connected spaces is introduced. More examples are given to illustrate the concept introduced in this section. Characterizations of such spaces are also studied.

DEFINITION 2.2.1. Let (X, T_1, T_2) be any fuzzy bitopological space and let λ be any fuzzy set in (X, T_1, T_2). Then

1. λ is called $(1, 2)$ fuzzy regular G_δ if $\mathrm{int}_{\delta(T_1)} \mathrm{cl}_{\sigma(T_2)} \lambda = \lambda$.
2. λ is called $(2, 1)$ fuzzy regular G_δ if $\mathrm{int}_{\delta(T_2)} \mathrm{cl}_{\sigma(T_1)} \lambda = \lambda$ and
3. λ is called pairwise fuzzy regular G_δ if λ is both $(1, 2)$ fuzzy regular G_δ and $(2, 1)$ fuzzy regular G_δ.

DEFINITION 2.2.2. Let (X, T_1, T_2) be any fuzzy bitopological space. Then (X, T_1, T_2) is called pairwise fuzzy super G_δ-connected if it has no proper ($\neq 0, 1$) pairwise fuzzy regular G_δ-set.

EXAMPLE 2.2.1. Let $X = \{a, b\}$, $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$, where $\lambda\colon X \to [0, 1]$ is such that

$$\lambda(a) = 1, \quad \lambda(b) = 3/4,$$

and $\mu\colon X \to [0, 1]$ is such that

$$\mu(a) = 0 \quad \text{and} \quad \mu(b) = 1.$$

Then the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy super G_δ-connected and pairwise fuzzy G_δ-connected.

PROPOSITION 2.2.1. *If (X, T_1, T_2) is any fuzzy bitopological space, then (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c), where*

(a) *(X, T_1, T_2) is pairwise fuzzy super G_δ-connected space.*

(b) *the T_2-σ-closure or T_1-σ-closure of a pairwise fuzzy regular G_δ-set which is different from* 0 *is* 1.

(c) *the T_2-δ-interior or T_1-δ-interior of a pairwise fuzzy regular F_σ-set which is different from* 1 *is* 0.

Proof. (a) $\Rightarrow$ (b). Assume (a). Suppose there exists a pairwise fuzzy regular G_δ-set $\lambda \neq 0$ such that $\mathrm{cl}_{\sigma(T_2)} \lambda \neq 1$. Then

$$\mathrm{int}_{\delta(T_1)} \mathrm{cl}_{\sigma(T_2)} \lambda \neq 1. \tag{1}$$

But since λ is pairwise fuzzy regular G_δ-set,

$$\mathrm{int}_{\delta(T_1)} \mathrm{cl}_{\sigma(T_2)} \lambda = \lambda. \tag{2}$$

From (1) and (2) we get $\lambda \neq 1$. Thus we find that (X, T_1, T_2) has a proper pairwise fuzzy regular G_δ-set λ. Contradiction.

Similarly, we can show that T_1-σ-closure of a pairwise fuzzy regular G_δ-set which is different from 0 is one. Hence (b).

(b) $\Rightarrow$ (c). Assume (b). Suppose (c) is not true. This means that there exists a pairwise fuzzy regular F_σ-set $\lambda \neq 1$ such that $\mathrm{int}_{\delta(T_2)} \lambda \neq 0$. Now, $\mu = 1 - \lambda \neq 0$ and μ is a non-zero pairwise fuzzy regular G_δ-set. Then $\mathrm{cl}_{\sigma(T_2)} \mu = 1 - \mathrm{int}_{\delta(T_2)}(1 - \mu) = 1 - \mathrm{int}_{\delta(T_2)} \lambda \neq 1$ (since $\mathrm{int}_{\delta(T_2)} \lambda \neq 0$). Contradiction.

Similarly, we can prove that T_1-δ-interior of a pairwise fuzzy regular F_σ-set which is different from 1 is 0. Hence (c). □

REMARK 2.2.1. The following examples give the relation between pairwise fuzzy super G_δ-connectedness, pairwise fuzzy G_δ-connectedness and pairwise fuzzy G_δ-disconnectedness.

EXAMPLE 2.2.2. Let $X = \{a, b\}$, $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$, where $\lambda\colon X \to [0, 1]$ is such that

$$\lambda(a) = 1/4, \quad \lambda(b) = 0$$

and $\mu\colon X \to [0, 1]$ is such that

$$\mu(a) = 0, \quad \mu(b) = 1.$$

Now, $\mathrm{int}_{\delta(T_1)} \mathrm{cl}_{\sigma(T_2)} \lambda = \lambda$, $\mathrm{int}_{\delta(T_2)} \mathrm{cl}_{\sigma(T_1)} \mu = \mu$, λ is $(1, 2)$ fuzzy regular and μ is $(2, 1)$ fuzzy regular. Therefore the fuzzy bitopological space (X, T_1, T_2) is not pairwise fuzzy super G_δ-connected. Also, $\lambda + \mu = 1$. Now the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-connected.

EXAMPLE 2.2.3. Let $X = \{a, b\}$, $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$, where $\lambda\colon X \to [0, 1]$ is such that

$$\lambda(a) = 1/4, \quad \lambda(b) = 0$$

and $\mu\colon X \to [0, 1]$ is such that

$$\mu(a) = 3/4, \quad \mu(b) = 1.$$

Then the fuzzy bitopological space (X, T_1, T_2) is not pairwise fuzzy super G_δ-connected and not pairwise fuzzy G_δ-connected.

2.3. Pairwise fuzzy strongly G_δ-connected spaces

In this section, the concept of pairwise fuzzy strongly G_δ-connected spaces is introduced. Interesting properties and characterizations are also discussed.

DEFINITION 2.3.1. A fuzzy bitopological space (X, T_1, T_2) is said to be *pairwise fuzzy strongly G_δ-connected* if it has no proper T_1-fuzzy F_σ-sets or T_2-fuzzy F_σ-sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 \leq 1$. If (X, T_1, T_2) is not pairwise fuzzy strongly G_δ-connected, then it will be called *pairwise fuzzy weakly G_δ-connected.*

PROPOSITION 2.3.1. *A fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy strongly G_δ-connected iff it has no proper T_1-fuzzy G_δ-sets or T_2-fuzzy G_δ-sets λ, μ such that $\lambda + \mu \geq 1$.*

Proof. (X, T_1, T_2) is pairwise fuzzy weakly G_δ-connected iff it has proper T_1-fuzzy F_σ-sets or T_2-fuzzy F_σ-sets f, k such that $f + k \leq 1 \Leftrightarrow$ it has proper T_1-fuzzy G_δ-sets or T_2-fuzzy G_δ-sets λ, μ where $\lambda = 1 - f,\ \mu = 1 - k$ such that $\lambda + \mu \geq 1$. □

REMARK 2.3.1. Pairwise fuzzy strongly G_δ-connectedness implies pairwise fuzzy G_δ-connectedness. However the converse is not true as shown in Example 2.3.1.

EXAMPLE 2.3.1. Let $X = [0, 1]$. Define $T_1 = \{0, 1, \lambda\}$ and $T_2 = \{0, 1, \mu\}$ where $\lambda\colon X \to [0, 1]$ is such that $\lambda(x) = 2/3$, for all $x \in X$ and $\mu\colon X \to [0, 1]$ is such that $\mu(x) = 3/4$, for all $x \in X$. Then the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-connected but not pairwise fuzzy strongly G_δ-connected.

PROPOSITION 2.3.2. *Let (X, T_1, T_2) be any fuzzy bitopological space and $A \subset X$ be any subset. Then the following statements are equivalent.*

(a) *$(A, T_1/A, T_2/A)$ is pairwise fuzzy strongly G_δ-connected subspace of (X, T_1, T_2).*
(b) *For any proper T_1-fuzzy G_δ-sets or T_2-fuzzy G_δ-sets λ_1, λ_2 such that $1_A \leq \lambda_1/A + \lambda_2/A$ implies either $1_A = \lambda_1/A$ or $1_A = \lambda_2/A$.*

Proof. (b) $\Rightarrow$ (a). Suppose A is not pairwise fuzzy strongly G_δ-connected subset of X. Then there exist proper T_1/A-fuzzy F_σ-set or T_2/A-fuzzy F_σ-sets f, k such that $f + k \leq 1_A$. Therefore , we can find proper T_1-fuzzy G_δ-sets or T_2-fuzzy G_δ-sets λ_1, λ_2 such that

$$\lambda_1/A = 1_A - f, \quad \lambda_2/A = 1_A - k.$$

Then

$$\lambda_1/A + \lambda_2/A = 1_A - f + 1_A - k = 2 - (f + k).$$

That is,

$$\lambda_1/A + \lambda_2/A \geq 1_A \quad (\text{since } f + k \leq 1_A). \tag{3}$$

Since

$$0 < \lambda_1/A < 1_A \tag{4}$$

and

$$0 < \lambda_2/A < 1_A, \tag{5}$$

we have from (3), (4) and (5) that $1_A \neq \lambda_1/A$ and $1_A \neq \lambda_2/A$. Contradiction. This proves (a).

(a) $\Rightarrow$ (b). Suppose there exist proper T_1-fuzzy G_δ-sets or T_2-fuzzy G_δ-sets λ_1, λ_2 such that $1_A \leq \lambda_1/A + \lambda_2/A$ but both $1_A \neq \lambda_1/A$ and $1_A \neq \lambda_2/A$. This shows by Proposition 2.3.1, A is not pairwise fuzzy strongly G_δ-connected. Contradiction. This proves (b). □

PROPOSITION 2.3.3. *Let (X, T_1, T_2) be any fuzzy bitopological space. Let $F \subset X$ be such that χ_F is T_1-fuzzy F_σ or T_2-fuzzy F_σ. Then (X, T_1, T_2) is pairwise fuzzy strongly G_δ-connected implies $(F, T_1/F, T_2/F)$ is pairwise fuzzy strongly G_δ-connected.*

Proof. Let $F \subset X$ be such that χ_F is T_1-fuzzy F_σ or T_2-fuzzy F_σ. We want to show that $(F, T_1/F, T_2/F)$ is pairwise fuzzy strongly G_δ-connected. Suppose $(F, T_1/F, T_2/F)$ is not pairwise fuzzy strongly G_δ-connected. This means there exist proper T_1/F-fuzzy F_σ-sets or T_2/F-fuzzy F_σ-sets f, k such that

$$f + k \leq 1. \tag{6}$$

Hence, we can find proper T_1-fuzzy F_σ or T_2-fuzzy F_σ sets λ_1, λ_2 such that $f = \lambda_1/F$, $k = \lambda_2/F$. Now, consider $(\lambda_1 \wedge \chi_F) + (\lambda_2 \wedge \chi_F)$. Since χ_F is T_1-fuzzy F_σ or T_2-fuzzy F_σ, $\lambda_1 \wedge \chi_F$ and $\lambda_2 \wedge \chi_F$ are T_1-fuzzy F_σ or T_2-fuzzy F_σ. From (6) we find that

$$(\lambda_1 \wedge \chi_F) + (\lambda_2 \wedge \chi_F) \leq 1_X.$$

This shows (X, T_1, T_2) is not pairwise fuzzy strongly G_δ-connected, which is a contradiction. Hence the proposition. □

2.4. Pairwise fuzzy G_δ-extremally disconnected spaces

In this section, the concept of pairwise fuzzy G_δ-extremally disconnected spaces is introduced. Characterizations and some interesting properties are also given with necessary examples.

DEFINITION 2.4.1. A fuzzy bitopological space (X, T_1, T_2) is said to be pairwise fuzzy G_δ-extremally disconnected if T_1-σ-closure of each T_2-fuzzy G_δ-set is T_2-fuzzy G_δ and T_2-σ-closure of each T_1-fuzzy G_δ-set is T_1-fuzzy G_δ.

EXAMPLE 2.4.1. Let $X = \{a, b, c, d\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3, \mu_4 : X \to [0, 1]$ as follows:

$$\begin{aligned}
&\lambda_1(a) = \lambda_1(c) = \lambda_1(d) = 0, \quad \lambda_1(b) = 1,\\
&\lambda_2(a) = \lambda_2(b) = 1, \quad \lambda_2(c) = \lambda_2(d) = 0,\\
&\lambda_3(b) = \lambda_3(d) = 1, \quad \lambda_3(a) = \lambda_3(c) = 0,\\
&\lambda_4(a) = \lambda_4(b) = \lambda_4(d) = 1, \quad \lambda_4(c) = 0,\\
&\mu_1(a) = \mu_1(b) = \mu_1(d) = 0, \quad \mu_1(c) = 1,\\
&\mu_2(a) = \mu_2(c) = 1, \quad \mu_2(b) = \mu_2(d) = 0,\\
&\mu_3(a) = \mu_3(b) = 0, \quad \mu_3(c) = \mu_3(d) = 1,\\
&\mu_4(a) = \mu_4(c) = \mu_4(d) = 1, \quad \mu_4(b) = 0.
\end{aligned}$$

Clearly, $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $T_2 = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4\}$ are fuzzy topologies on X. Then we can easily see that (X, T_1, T_2) is pairwise fuzzy G_δ-extremally disconnected eventhough both (X, T_1) and (X, T_2) are fuzzy G_δ-connected spaces.

PROPOSITION 2.4.1. *For any fuzzy bitopological space (X, T_1, T_2), the following are equivalent.*

(a) *(X, T_1, T_2) is pairwise fuzzy G_δ-extremally disconnected.*

(b) *Whenever λ is a T_1 fuzzy F_σ-set, $\operatorname{int}_{\delta(T_2)} \lambda$ is a T_1-fuzzy F_σ-set. Similarly, whenever μ is a T_2-fuzzy F_σ-set, $\operatorname{int}_{\delta(T_1)} \mu$ is a T_2-fuzzy F_σ-set.*

(c) *Whenever λ is a T_1-fuzzy G_δ-set, we have*

$$\operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda) = 1 - \operatorname{cl}_{\sigma(T_2)} \lambda.$$

Similarly, whenever λ is a T_2-fuzzy G_δ-set, we have

$$\operatorname{cl}_{\sigma(T_2)}(1 - \operatorname{cl}_{\sigma(T_1)} \lambda) = 1 - \operatorname{cl}_{\sigma(T_1)} \lambda.$$

(d) *For every pair of T_1-fuzzy G_δ-set λ and T_2-fuzzy G_δ-set μ in (X, T_1, T_2) with $\operatorname{cl}_{\sigma(T_2)} \lambda + \mu = 1$, we have $\operatorname{cl}_{\sigma(T_2)} \lambda + \operatorname{cl}_{\sigma(T_1)} \mu = 1$. Similarly, for every pair of T_2-fuzzy G_δ-set λ and T_1-fuzzy G_δ-set μ in (X, T_1, T_2) with $\operatorname{cl}_{\sigma(T_1)} \lambda + \mu = 1$, we have $\operatorname{cl}_{\sigma(T_2)} \lambda + \operatorname{cl}_{\sigma(T_1)} \mu = 1$.*

Proof. (a) $\Rightarrow$ (b). Suppose (a) is true. Let λ be any T_1-fuzzy F_σ-set. Then $1 - \lambda$ is a T_1-fuzzy G_δ-set. Then from (a), $\operatorname{cl}_{\sigma(T_2)}(1 - \lambda)$ is a T_1-fuzzy G_δ-set. Clearly, $1 - \operatorname{cl}_{\sigma(T_2)}(1 - \lambda)$ is T_1-fuzzy F_σ-set. But

$$1 - \operatorname{cl}_{\sigma(T_2)}(1 - \lambda) = \operatorname{int}_{\delta(T_2)} \lambda$$

and so $\operatorname{int}_{\delta(T_2)} \lambda$ is T_1-fuzzy F_σ-set. Similar statement holds for T_2-fuzzy F_σ-sets. Thus (b) is proved.

(b) $\Rightarrow$ (c). Assume that (b) is true. Suppose λ is a T_1-fuzzy G_δ-set. Then $1 - \lambda$ is a T_1-fuzzy F_σ-set. Now, $\operatorname{cl}_{\sigma(T_2)} \lambda$ is T_1-fuzzy G_δ and therefore $1 - \operatorname{cl}_{\sigma(T_2)} \lambda$ is T_1-fuzzy F_σ. Therefore,

$$\operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda) = 1 - \operatorname{cl}_{\sigma(T_2)} \lambda.$$

Similarly, we can show that $\operatorname{cl}_{\sigma(T_2)}(1 - \operatorname{cl}_{\sigma(T_1)} \lambda) = 1 - \operatorname{cl}_{\sigma(T_1)} \lambda$ when λ is a T_2-fuzzy G_δ-set. Hence (c).

(c) $\Rightarrow$ (d). Assume for every T_1-fuzzy G_δ-set λ, we have

$$\mathrm{cl}_{\sigma(T_1)}(1-\mathrm{cl}_{\sigma(T_2)}\lambda)=1-\mathrm{cl}_{\sigma(T_2)}\lambda,$$

and for every T_2-fuzzy G_δ-set λ,

$$\mathrm{cl}_{\sigma(T_2)}(1-\mathrm{cl}_{\sigma(T_1)}\lambda)=1-\mathrm{cl}_{\sigma(T_1)}\lambda.$$

Suppose that λ is T_1-fuzzy G_δ and μ is T_2-fuzzy G_δ-set such that

$$\mathrm{cl}_{\sigma(T_2)}\lambda+\mu=1. \tag{7}$$

Then,

$$\mathrm{cl}_{\sigma(T_2)}\lambda+\mathrm{cl}_{\sigma(T_1)}(1-\mathrm{cl}_{\sigma(T_2)}\lambda)=1=\mathrm{cl}_{\sigma(T_2)}\lambda+\mu$$

implies $\mu=\mathrm{cl}_{\sigma(T_1)}(1-\mathrm{cl}_{\sigma(T_2)}\lambda)$ and so $1-\mathrm{cl}_{\sigma(T_2)}\lambda=\mathrm{cl}_{\sigma(T_1)}(1-\mathrm{cl}_{\sigma(T_2)}\lambda)$. That is, $1-\mathrm{cl}_{\sigma(T_2)}\lambda$ is T_1-fuzzy F_σ and hence $\mathrm{cl}_{\sigma(T_1)}\mu=1-\mathrm{cl}_{\sigma(T_2)}\lambda$. That is, $\mathrm{cl}_{\sigma(T_1)}\mu+\mathrm{cl}_{\sigma(T_2)}\lambda=1$. Similarly, we can prove the other part. Hence (d).

(d) $\Rightarrow$ (a). Assume that (d) is true. Let λ be any T_1-fuzzy G_δ-set. Put $\mathrm{cl}_{\sigma(T_2)}\lambda+\mu=1$. That is, $\mu=1-\mathrm{cl}_{\sigma(T_2)}\lambda$. By (d), $\mathrm{cl}_{\sigma(T_2)}\lambda+\mathrm{cl}_{\sigma(T_1)}\mu=1$ and hence $\mathrm{cl}_{\sigma(T_2)}\lambda$ is T_1-fuzzy G_δ in (X,T_1,T_2). Similarly, we can show that T_1-σ-closure of a T_2-fuzzy G_δ-set is T_2-fuzzy G_δ. Therefore, (X,T_1,T_2) is pairwise fuzzy G_δ-extremally disconnected. □

PROPOSITION 2.4.2. *Let (X,T_1,T_2) be a pairwise fuzzy G_δ-extremally disconnected space. If $A\subset X$ is such that χ_A is T_1-fuzzy G_δ and T_2-fuzzy G_δ, then the fuzzy subspace $(A,T_1/A,T_2/A)$ is pairwise fuzzy G_δ-extremally disconnected.*

Proof. Let $A\subset X$ be such that χ_A is T_1-fuzzy G_δ and T_2 fuzzy G_δ. Let λ_1 be T_1/A-fuzzy G_δ and let λ_2 be T_2/A-fuzzy G_δ in A such that $\mathrm{cl}_{\sigma(T_2/A)}\lambda_1+\lambda_2=1$. Then, there exist T_1-fuzzy G_δ-set μ_1 and T_2-fuzzy G_δ-set μ_2 in (X,T_1,T_2) such that $\mu_1/A=\lambda_1$ and $\mu_2/A=\lambda_2$. That is, $\mu_1\wedge\chi_A=\lambda_1$ and $\mu_2\wedge\chi_A=\lambda_2$. Since χ_A is T_1-fuzzy G_δ and T_2-fuzzy G_δ, $\lambda_1\wedge\chi_A$ is T_1-fuzzy G_δ and $\lambda_2\wedge\chi_A$ is T_2-fuzzy G_δ. That is, λ_1 is T_1-fuzzy G_δ and λ_2 is T_2-fuzzy G_δ in (X,T_1,T_2). Since (X,T_1,T_2) is pairwise fuzzy G_δ-extremally disconnected, $\mathrm{cl}_{\sigma(T_2)}\lambda_1+\mathrm{cl}_{\sigma(T_1)}\lambda_2=1$ in (X,T_1,T_2) and therefore in $(A,T_1/A,T_2/A)$. Thus, $(A,T_1/A,T_2/A)$ is pairwise fuzzy G_δ-extremally disconnected. □

2.5. Pairwise fuzzy G_δ-basically disconnected spaces

In this section, the concept of pairwise fuzzy G_δ-basically disconnected spaces is introduced. Characterizations and properties are discussed with examples.

DEFINITION 2.5.1. A fuzzy bitopological space (X, T_1, T_2) is said to be *pairwise fuzzy G_δ-basically disconnected* if the T_1-σ-closure of each T_2-fuzzy G_δ, T_2-fuzzy F_σ-set is T_2-fuzzy G_δ and T_2-σ-closure of each T_1-fuzzy G_δ, T_1-fuzzy F_σ-set is T_1-fuzzy G_δ.

EXAMPLE 2.5.1. Let $X = \{a, b, c, d\}$, $T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $T_2 = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_4\}$, where $\lambda_i : X \to [0, 1]$, $i = 1, 2, 3, 4$ and $\mu_j : X \to [0, 1]$, $j = 1, 2, 3, 4$ are defined as follows:

$$\begin{aligned}
&\lambda_1(a) = \lambda_1(c) = \lambda_1(d) = 0, \quad \lambda_1(b) = 1,\\
&\lambda_2(a) = \lambda_2(b) = 1, \quad \lambda_2(c) = \lambda_2(d) = 0,\\
&\lambda_3(b) = \lambda_3(d) = 1, \quad \lambda_3(a) = \lambda_3(c) = 0,\\
&\lambda_4(a) = \lambda_4(b) = \lambda_4(d) = 1, \quad \lambda_4(c) = 0,\\
&\mu_1(a) = \mu_1(b) = \mu_1(d) = 0, \quad \mu_1(c) = 1,\\
&\mu_2(a) = \mu_2(c) = 1, \quad \mu_2(b) = \mu_2(d) = 0,\\
&\mu_3(a) = \mu_3(b) = 0, \quad \mu_3(c) = \mu_3(d) = 1,\\
&\mu_4(a) = \mu_4(c) = \mu_4(d) = 1, \quad \mu_4(b) = 0.
\end{aligned}$$

Clearly, (X, T_1) and (X, T_2) are fuzzy topological spaces. Also, we can easily see that they are both fuzzy G_δ-connected spaces (since both (X, T_1) and (X, T_2) have no proper fuzzy $G_\delta F_\sigma$-sets).

Also, in fuzzy topological space (X, T_1), there is no such T_1-fuzzy G_δ, T_1-fuzzy F_σ-set and also there is no such T_2-fuzzy G_δ, T_2-fuzzy F_σ-set in (X, T_2). Therefore , the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-basically disconnected even though both (X, T_1) and (X, T_2) are fuzzy G_δ-connected.

EXAMPLE 2.5.2. Let $X = \{a, b, c\}$. Suppose

$$T_1 = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$$

and $T_2 = \{0,1\}$, where $\lambda_i : X \to [0,1]$, $i = 1$ to 6 are defined as follows:

$$\begin{aligned} \lambda_1(a) = \lambda_1(b) = 1, \quad \lambda_1(c) = 0, \\ \lambda_2(b) = \lambda_2(c) = 1, \quad \lambda_2(a) = 0, \\ \lambda_3(a) = \lambda_3(c) = 1, \quad \lambda_3(b) = 0, \\ \lambda_4(a) = 1, \quad \lambda_4(b) = \lambda_4(c) = 0, \\ \lambda_5(a) = \lambda_5(c) = 0, \quad \lambda_5(b) = 1, \\ \lambda_6(a) = \lambda_6(b) = 0, \quad \lambda_6(c) = 1. \end{aligned}$$

Clearly , (X, T_1) is a fuzzy topological space and (X, T_2) is the indiscrete fuzzy topological space. Clearly, (X, T_1) is a fuzzy G_δ-disconnected space and (X, T_2) is a fuzzy G_δ-connected space.

We claim the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy G_δ-basically disconnected space.

Let λ be any non-zero T_1-fuzzy G_δ , T_1-fuzzy F_σ-set. Then $\mathrm{cl}_{\sigma(T_2)}\,\lambda = 1$ which is clearly T_1-fuzzy G_δ. Similarly, we can see that $\mathrm{cl}_{\sigma(T_1)}\,\mu = 1$ whenever μ is a non-zero T_2-fuzzy G_δ, fuzzy F_σ-set and clearly $\mathrm{cl}_{\sigma(T_1)}\,\mu$ is T_2-fuzzy G_δ.

Therefore, the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-basically disconnected space.

REMARK 2.5.1. Every pairwise fuzzy G_δ-extremally disconnected space is pairwise fuzzy G_δ-basically disconnected, but the converse is not true as shown in Example 2.5.3.

EXAMPLE 2.5.3. Let $X = \{a, b\}$. Suppose $T_1 = \{0, 1, \lambda_1, \lambda_2\}$ and $T_2 = \{0, 1, \mu_1, \mu_2\}$, where $\lambda_i : X \to [0,1]$, $i = 1, 2$ and $\mu_j : X \to [0,1]$, $j = 1, 2$ are defined as follows:

$$\begin{aligned} \lambda_1(a) = 1/2, \quad \lambda_1(b) = 1, \\ \lambda_2(a) = 0, \quad \lambda_2(b) = 1/3, \\ \mu_1(a) = 1/2, \quad \mu_1(b) = 1/3, \\ \mu_2(a) = 1/4, \quad \mu_2(b) = 1/3. \end{aligned}$$

Then clearly, (X, T_1) and (X, T_2) are fuzzy topological spaces. Also, we can easily say in the fuzzy topological space (X, T_1) there is no such T_1-fuzzy G_δ, T_1-fuzzy F_σ-set and also there is no such T_2-fuzzy

G_δ, T_2-fuzzy F_σ-set in (X, T_2). Therefore, the fuzzy bitopological space (X, T_1, T_2) is pairwise fuzzy G_δ-basically disconnected but not pairwise fuzzy G_δ-extremally disconnected.

PROPOSITION 2.5.1. *For any fuzzy bitopological space (X, T_1, T_2), the following are equivalent.*

(a) *(X, T_1, T_2) is pairwise fuzzy G_δ-basically disconnected.*
(b) *Whenever λ is a T_1-fuzzy G_δ and T_1-fuzzy F_σ-set,* $\operatorname{int}_\delta(T_1) \operatorname{cl}_{\sigma(T_2)} \lambda$ *is T_2-fuzzy F_σ. Similar statement holds when λ is a T_2-fuzzy G_δ and T_2-fuzzy F_σ-set.*
(c) *Whenever λ is a T_1-fuzzy G_δ and T_1-fuzzy F_σ-set, we have* $\operatorname{cl}_{\sigma(T_2)} \lambda \le 1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda)$. *Similar statement holds when λ is a T_2-fuzzy G_δ and T_2-fuzzy F_σ-set.*
(d) *Whenever λ is a T_1-fuzzy G_δ-set and μ is a T_2-fuzzy G_δ-set such that $\lambda+\mu \le 1$ and λ being a T_1-fuzzy F_σ-set or μ being a T_2-fuzzy F_σ-set, we have* $\operatorname{cl}_{\sigma(T_2)} \lambda + \operatorname{cl}_{\sigma(T_1)} \mu \le 1$.

Proof. (a) $\Rightarrow$ (b). Let λ be any T_1-fuzzy G_δ and T_1-fuzzy F_σ-set. Now,

$$\operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda = 1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda). \tag{8}$$

By (a), $\operatorname{cl}_{\sigma(T_2)} \lambda$ is T_1-fuzzy G_δ and therefore from (8) it follows that $\operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda$ is T_2-fuzzy F_σ. Similar argument holds when λ is a T_2-fuzzy G_δ and T_2-fuzzy F_σ-set.

(b) $\Rightarrow$ (c). Let λ be any T_1-fuzzy G_δ and T_1-fuzzy F_σ-set and suppose that $\operatorname{cl}_{\sigma(T_2)} \lambda \not\le 1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda)$. Then there exists an $x \in X$ such that $\operatorname{cl}_{\sigma(T_2)} \lambda(x) \not\le (1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda))(x)$. Now by (b), $\operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda$ is T_2-fuzzy F_σ. Also, $\operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda) = 1 - \operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda$. Hence it follows that

$$\operatorname{cl}_{\sigma(T_2)} \lambda(x) \not\le (1 - (1 - \operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda))(x) \not\le \operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda(x)$$

which is not possible; for by (b), $\operatorname{int}_{\delta(T_1)} \operatorname{cl}_{\sigma(T_2)} \lambda$ is T_2-fuzzy F_σ containing λ. Hence, $\operatorname{cl}_{\sigma(T_2)} \lambda \le 1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda)$. Similar proof holds when λ is a T_2-fuzzy G_δ and T_2-fuzzy F_σ-set.

(c) $\Rightarrow$ (d). Let λ be any T_1-fuzzy G_δ, T_1-fuzzy F_σ-set and μ be any T_2-fuzzy G_δ-set such that $\lambda+\mu \le 1$. We know that $\mu \le 1 - \operatorname{cl}_{\sigma(T_2)} \lambda$ and $\lambda \le 1 - \operatorname{cl}_{\sigma(T_1)} \mu$. But by hypothesis, $\operatorname{cl}_{\sigma(T_2)} \lambda \le 1 - \operatorname{cl}_{\sigma(T_1)}(1 - \operatorname{cl}_{\sigma(T_2)} \lambda)$

and therefore $\mu \leq 1 - \mathrm{cl}_{\sigma(T_2)}\,\lambda$. Since $\mathrm{cl}_{\sigma(T_1)}\,\mu$ is the smallest T_1-fuzzy F_σ-set containing μ, we have

$$\mathrm{cl}_{\sigma(T_1)}\,\mu \leq \mathrm{cl}_{\sigma(T_1)}(1 - \mathrm{cl}_{\sigma(T_2)}\,\lambda). \tag{9}$$

Also, since $\mathrm{cl}_{\sigma(T_2)}\,\lambda + \mathrm{cl}_{\sigma(T_1)}(1 - \mathrm{cl}_{\sigma(T_2)}\,\lambda) \leq 1$, it follows from (9) that $\mathrm{cl}_{\sigma(T_2)}\,\lambda + \mathrm{cl}_{\sigma(T_1)}\,\mu \leq 1$.

(d) $\Rightarrow$ (a). Let λ be any T_1-fuzzy G_δ, T_1-fuzzy F_σ-set. We shall show that $\mathrm{cl}_{\sigma(T_2)}\,\lambda$ is T_1-fuzzy G_δ. Let $\mu = 1 - \mathrm{cl}_{\sigma(T_2)}\,\lambda$. Clearly, μ is T_2-fuzzy G_δ and $\mu + \lambda \leq 1$. Hence by (d), we have $\mathrm{cl}_{\sigma(T_2)}\,\lambda + \mathrm{cl}_{\sigma(T_1)}\,\mu \leq 1$ and therefore by construction of μ, we have $1 - \mathrm{cl}_{\sigma(T_1)}\,\mu = \mathrm{cl}_{\sigma(T_2)}\,\lambda$. This shows $\mathrm{cl}_{\sigma(T_2)}\,\lambda$ is T_1-fuzzy G_δ. Similarly, we can show for any T_2-fuzzy G_δ and T_2-fuzzy F_σ-set λ, $\mathrm{cl}_{\sigma(T_1)}\,\lambda$ is T_2-fuzzy G_δ. □

PROPOSITION 2.5.2. *Let (X, T_1, T_2) be a pairwise fuzzy G_δ-basically disconnected space and let $(Y, T_1/Y, T_2/Y)$ be any pairwise fuzzy subspace of (X, T_1, T_2). Then $(Y, T_1/Y, T_2/Y)$ is pairwise fuzzy G_δ-basically disconnected.*

Proof. Let λ_1 and λ_2 be T_1/Y-fuzzy G_d-set and T_2/Y- fuzzy G_δ-set in $(Y, T_1/Y, T_2/Y)$ respectively such that $\lambda_1 + \lambda_2 \leq 1$ and suppose that λ_1 is T_1/Y- fuzzy F_σ-set. Define $\lambda_1^1 \colon X \to [0,1]$ and $\lambda_2^2 \colon X \to [0,1]$ on X as follows:

$$\lambda_1^1(x) = \begin{cases} \lambda_1(x), & \text{if } x \in X, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\lambda_2^2 = \begin{cases} \lambda_2(x), & \text{if } x \in X, \\ 0, & \text{otherwise.} \end{cases}$$

We know that λ_1^1 and λ_2^2 are T_1-fuzzy G_δ-set and T_2-fuzzy G_δ-set respectively such that $\lambda_1^1 + \lambda_2^2 \leq 1$ and that λ_1^1 is T_1-fuzzy F_σ-set. Since (X, T_1, T_2) is pairwise fuzzy G_δ-basically disconnected, it follows that $\mathrm{cl}_{\sigma(T_2)}(\lambda_1^1) + \mathrm{cl}_{\sigma(T_1)}(\lambda_2^2) \leq 1$ and this in turn implies

$$\mathrm{cl}_{\sigma(T_2/Y)}(\lambda_1) + \mathrm{cl}_{\sigma(T_1/Y)}(\lambda_2) \leq 1.$$

We arrive at the same conclusion when we assume λ_2 is T_2/Y-fuzzy F_σ-set. Hence the proposition holds. □

PROPOSITION 2.5.3. *The fuzzy bitopological sum of a family of disjoint pairwise fuzzy G_δ-basically disconnected spaces is pairwise fuzzy G_δ-basically disconnected.*

Proof. Let $\{(X_\alpha, T_\alpha, T_\alpha^*) : \alpha \in \Delta\}$ be a family of disjoint pairwise fuzzy G_δ-basically disconnected spaces. Let $(X, \oplus_{\alpha\in\Delta}T_\alpha, \oplus_{\alpha\in\Delta}T_\alpha^*)$ be the fuzzy bitopological sum of these spaces. Let λ_1 and λ_2 be $\oplus_{\alpha\in\Delta}T_\alpha$-fuzzy G_δ and $\oplus_{\alpha\in\Delta}T_\alpha^*$-fuzzy G_δ-sets in X respectively such that $\lambda_1 + \lambda_2 \leq 1$. Also, we shall assume that λ_1 is $\oplus_{\alpha\in\Delta}T_\alpha$-fuzzy F_σ-set. Now, from the assumptions, it is clear that λ_1/X_α and λ_2/X_α are T_α-fuzzy G_δ-set and T_α^*-fuzzy G_δ-set in X_α respectively for each $\alpha \in \Delta$. Also, $\lambda_1/X_\alpha + \lambda_2/X_\alpha \leq 1$ and λ_1/X_α is T_α-fuzzy F_σ-set in X_α. Since $(X_\alpha, T_\alpha, T_\alpha^*)$ is pairwise fuzzy G_δ-basically disconnected, we have

$$\mathrm{cl}_{\sigma(T_\alpha^*)}(\lambda_1/X_\alpha) + \mathrm{cl}_{\sigma(T_\alpha)}(\lambda_2/X_\alpha) \leq 1, \quad \alpha \in \Delta.$$

Hence,

$$\mathrm{cl}_{\sigma(\oplus_{\alpha\in\Delta}T_\alpha^*)}(\lambda_1) + \mathrm{cl}_{\sigma(\oplus_{\alpha\in\Delta}T_\alpha)}(\lambda_2) \leq 1.$$

This proves that the fuzzy bitopological sum is a pairwise fuzzy G_δ-basically disconnected space. □

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