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A Missile Guidance Law Based on Sontag's Formula to Intercept Maneuvering Targets

Chang-Kyung Ryoo, Yoon-Hwan Kim, Min-Jea Tahk, and Keeyoung Choi

Abstract: In this paper, we propose a nonlinear guidance law for missiles against maneuvering targets. First, we derive the equations of motion described in the line-of-sight reference frame and then we define the equilibrium subspace of the nonlinear system to guarantee target interception within a finite time. Using Sontag's formula, we derive a nonlinear guidance law that always delivers the state to the equilibrium subspace. If the speed of the missile is greater than that of the target, the proposed law has global capturability in that, under any initial launch conditions, the missile can intercept the maneuvering target. The proposed law also minimizes the integral cost of the control energy and the weighted square of the state. The performance of the proposed law is compared with the augmented proportional navigation guidance law by means of numerical simulations of various initial conditions and target maneuvers.

Keywords: Control Lyapunov function, missile guidance, nonlinear control, Sontag's formula.

1. INTRODUCTION

Recently, optimal control theory [1,2] has been widely used to derive new guidance laws [3-5]. Guidance laws based on modern control theory, such as receding horizon control [6] and sliding-mode control [7], have been occasionally found in the literature. In these studies, approximations such as a non-maneuvering target and linearized equations of motion are commonly used to derive closed-form solutions. Because these assumptions restrict the application of the devised laws to the real guidance loop represented by a highly nonlinear state-feedback system, a capturability analysis for the entire engagement scenario should be performed to specify the capture region and requirements of the guidance law parameters. In general, a capturability analysis is done by extensive 6-DOF simulations or by qualitatively inspecting the state behavior. However, because these kinds of approaches provide sufficient conditions only, the entire capture region may not be easily determined.

For decades there have been numerous studies on the closed-form solutions and capturability of proportional navigation (PN) laws and their variants [8-14]. In Pure PN (PPN), it has been proved that the missile always reaches a maneuvering target if the ratio of missile speed to target speed is larger than $\sqrt{2}$ and the navigation constant is greater than 1 [13]. Thus, if there are no limitations on initial heading angles and positions of the missile, PPN has global capturability. However, such global capturability is not meaningful for real applications because the guidance command from PPN tends to blow up as the missile approaches the maneuvering target. Augmented PN (APN) laws [15,16], namely PN laws with target acceleration as a bias term, produce a bounded control for maneuvering targets. Although APN laws are energy optimal and practical, they need to be implemented with special care because the capture region is not specified. Indeed, some nonlinear engagement simulation results for APN show that the capture region is limited. It is natural, therefore, to inquire about new guidance laws which, on the one hand, are subject to bounded control but, on the other hand, ensure global capturability against maneuvering targets.

We note the guidance law based on H_∞ control theory [17], in which target maneuvering is regarded as an unpredictable disturbance. The proposed law is obtained as a solution to the associated Hamilton-Jacobi partial differential inequality with the H_∞-norm for the measure of missile performance; it also

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Chang-Kyung Ryoo and Keeyoung Choi are with the Department of Aerospace Engineering, Inha University, 253, Yonghyun-dong, Nam-gu, Incheon 402-751, Korea (e-mails: {ckryoo, kchoi}@inha.ac.kr).

Yoon-Hwan Kim and Min-Jea Tahk are with the Department of Aerospace Engineering, Korea Advanced Institute of Science and Technology (KAIST), 373-1, Guseong-dong, Yuseong-gu, Daejon 305-701, Korea (e-mails: {yhkim, mjtahk}@fdcl.kaist.ac.kr).

provides bounded control for maneuvering targets. However, this H_∞ guidance law does not have global capturability because the storage function that guarantees the internal stability of the guidance law is not a Lyapunov function. Furthermore, the H_∞ guidance law is not practical for aerodynamically controlled missiles because its command direction is not confined in the normal direction to the missile velocity vector.

A nonlinear system with input is well known to be globally stabilizable by feedback control if there is a Control Lyapunov Function (CLF) that satisfies Artstein's inequality [18]. A CLF can be understood as the generalization of a Lyapunov function for a system with input. Based on the existence of a CLF, Sontag's formula generates a feedback control law that globally and asymptotically stabilizes the system if the system is affine in control [19]. Sontag's formula has an inverse optimality because it can be interpreted as a solution to the Hamilton-Jacobi partial differential equation, which is the necessary condition of an optimal control problem [20-22]. The first application of Sontag's formula to a guidance problem can be found in [23], where the guidance laws are defined in the line-of-sight (LOS) reference frame. For aerodynamically controlled missiles, the command component along the velocity direction is usually neglected because it is hard to implement. In this case, the capture region of the laws may be severely restricted even though the CLF method provides global capturability.

Fortunately, the equations of motion for a missile against a maneuvering target in the LOS reference frame can be represented as a nonlinear system with an input affine in control. The equilibrium subspace that ensures the maneuvering target interception within a finite time can also be specified. On the basis of these facts, we derived a new guidance law by using Sontag's formula to deliver the state to the equilibrium subspace. The proposed law always makes the missile state approach the equilibrium subspace in the Lyapunov sense, as long as the missile speed is greater than the speed of the target. Hence, the proposed law has global capturability in the sense that, under any initial launch conditions, the missile can intercept any maneuvering targets. Because Sontag's formula has inverse optimality, the proposed nonlinear guidance law can also optimally minimize the integral cost of the input energy and the range-varying weighted states. By including a proper choice of a gain in the proposed guidance law, we can ensure that maximum magnitude of the guidance command is smaller than that of APN. This property is useful in real applications because large miss distances are produced when the command is saturated.

In the following section, the basic notions of a CLF and Sontag's formula are briefly mentioned. In Section 3, we formulate the equations of motion for a missile pursuing a maneuvering target in the LOS frame and we specify the equilibrium subspace that corresponds to the state constraints that guarantee a maneuvering target intercept. Next, we discuss the derivation of the nonlinear guidance law, along with some of its properties, including the implementation aspect. In Section 5, we discuss the nonlinear simulations that were conducted to investigate the performance of the proposed law, and we compare the proposed law with APN for various launch conditions and target maneuvers. Finally, we present our conclusions in Section 6.

2. SONTAG'S FORMULA

Consider a nonlinear system with input given by

$$\dot{x} = f(x,u), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and assume that the system is continuous and satisfies $f(0,u)=0$ for a certain relaxed control u. In such a case, Artstein [18] has shown that the system in (1) is stabilizable by a closed-loop relaxed control if and only if there is a smooth, proper, and positive definite function of $V:\mathbb{R}^n \to \mathbb{R}_+$, such that $V(0)=0$, $V(x)>0$ if $x \neq 0$ and

$$\inf_u \left[\frac{\partial V(x)}{\partial x} f(x,u) \right] < 0 \quad \text{for all} \quad x \neq 0. \tag{2}$$

Moreover, the system is globally stabilizable by a closed-loop relaxed control if and only if V can be chosen in $\mathbb{R}^n$ with $V(x) \to \infty$ as $\|x\| \to \infty$. Equation (2) is called Artstein's inequality or the stabilizability condition. The function V is a CLF. Artstein's inequality implies that if it is possible to make the derivative negative at every point by an appropriate choice of u, then we can stabilize the system with a properly chosen V for the closed-loop system. The function V also implies that the existence of a CLF is equivalent to the existence of the asymptotically stabilizing feedback control $u=k(x)$, which is smooth everywhere except possibly at $x=0$, that is, $k(0)=0$ [19].

Suppose that the system is affine in control and can be expressed as

$$\dot{x} = f(x) + G(x)u. \tag{3}$$

Furthermore, if we assume the existence of a CLF, we can use the following equation to obtain a feedback control law that stabilizes the system:

$$k = \begin{cases} -\left[\dfrac{a+\sqrt{a^2+bq(b)}}{b}\right]\left(\dfrac{\partial V}{\partial x}G\right)^T & \text{for } \dfrac{\partial V}{\partial x}G \neq 0 \\ 0 & \text{for } \dfrac{\partial V}{\partial x}G = 0, \end{cases} \tag{4}$$

where

$$a = \frac{\partial V}{\partial x}f, \quad b = \frac{\partial V}{\partial x}GG^T\left(\frac{\partial V}{\partial x}\right)^T, \tag{5}$$

and $q(b)$ is a real function $q:\mathbb{R}\to\mathbb{R}$ such that $q(0)=0$ and $bq(b)>0$ for $b\neq 0$. Note that (4), which is called Sontag's formula [19], satisfies Artstein's inequality as follows:

$$\dot{V} = \frac{\partial V(x)}{\partial x}f(x) + \frac{\partial V(x)}{\partial x}G(x)k(x) < 0 \quad \text{for all } x \neq 0. \tag{6}$$

Now, we investigate the relationship between Sontag's formula and optimal control problems [21,22]. Let us consider the following optimal control problem: *Find* u *which minimizes*

$$J = \int_0^\infty \left[\bar{q}(x) + u^T(x)u(x)\right]dt \quad \textit{subject to } (3). \tag{7}$$

The solution of this optimal control problem is given by

$$u^* = -\frac{1}{2}G^T\left(\frac{\partial V^*}{\partial x}\right)^T. \tag{8}$$

Here V^* denotes the value function and is given by the following solution of the Hamilton-Jacobi-Bellman partial differential equation, which is a necessary condition of this optimal control problem:

$$\frac{\partial V^*}{\partial x}f - \frac{1}{4}\frac{\partial V^*}{\partial x}GG^T\left(\frac{\partial V^*}{\partial x}\right)^T + \bar{q}(x) = 0. \tag{9}$$

Assume that there exists a scalar function $\lambda(x)$ such that

$$\frac{\partial V^*}{\partial x} = \lambda(x)\frac{\partial V}{\partial x} \quad \text{for every } x. \tag{10}$$

Substituting (10) into (8) and (9), we then obtain

$$u^* = \begin{cases} -\left[\dfrac{a+\sqrt{a^2+b\bar{q}(x)}}{b}\right]\left(\dfrac{\partial V}{\partial x}G\right)^T & \text{for } \dfrac{\partial V}{\partial x}G \neq 0 \\ 0 & \text{for } \dfrac{\partial V}{\partial x}G = 0. \end{cases} \tag{11}$$

Equation (11) is exactly the same as (4) if

$$\bar{q}(x) = q(b(x)), \tag{12}$$

where $b(x)$ is given by (5). We conclude therefore that Sontag's formula represents the optimal control which minimizes

$$J = \int_0^\infty \left[q(b(x)) + u^T(x)u(x)\right]dt. \tag{13}$$

The optimality of Sontag's formula depends on the choice of V as well as $q(\cdot)$.

In general, an optimal guidance problem includes target interception conditions as a terminal cost or a terminal constraint. Because (13) does not contain any terminal states, the feedback control law given in (4) may not guarantee the interception of the target within a finite time. As explained in Section 4, this drawback can be overcome by considering the states that are inversely weighted by the square of a relative range for the choice of $q(\cdot)$.

3. MISSILE KINEMATICS AND EQUILIBRIUM SUBSPAPCE

3.1. Equations of motion

Three-dimensional engagement geometry between the missile and the maneuvering target is shown in Fig. 1. Here, $\vec{V}_m$ ($\vec{V}_t$) and $\vec{a}_m$ ($\vec{a}_t$) denote the velocity and acceleration vector of the missile (target), respectively. Four reference coordinate frames are used to define the equations of motion: the inertial reference frame (I); the LOS reference frame (L) with unit vector $[i_L, j_L, k_L]^T$, the missile velocity frame (M) with $[i_m, j_m, k_m]^T$, and the target velocity frame (T) with $[i_t, j_t, k_t]^T$. The angles ψ_L and θ_L denote the azimuth and elevation angles of the LOS to the inertial reference frame, respectively; and ψ_m (ψ_t) and θ_m (θ_t), the azimuth and elevation angles of the missile (target) velocity to the LOS

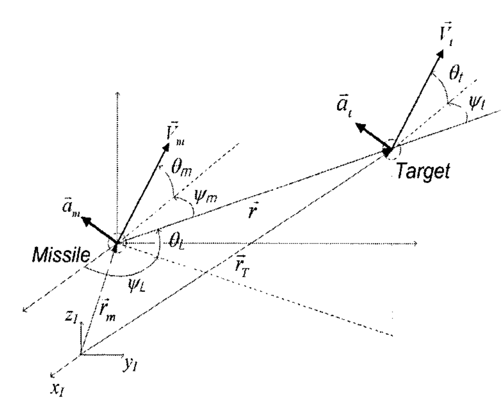


Fig. 1. 3-D interception geometry.

reference frame. A direction cosine matrix from frame 1 to frame 2 is given by

$$C_1^2 = T_y(-\theta_i)T_z(\psi_i) = \begin{bmatrix} c\theta_i c\psi_i & c\theta_i s\psi_i & s\theta_i \\ -s\psi_i & c\psi_i & 0 \\ -s\theta_i c\psi_i & -s\theta_i s\psi_i & c\theta_i \end{bmatrix}, \quad i = L,m,t. \tag{14}$$

Equations of motion of the engagement can be derived as follows: the relative range vector from the missile to the target $\vec{r}$ is given by

$$\vec{r} = r\,i_L = \vec{r}_t - \vec{r}_m, \tag{15}$$

where $\vec{r}_m$ and $\vec{r}_t$ denote the position vectors of the missile and the target, respectively.
The LOS rate to the I-frame $\vec{\omega}_L$ is defined as

$$\begin{aligned}\vec{\omega}_L &\triangleq \dot{\lambda}_x i_L + \dot{\lambda}_y j_L + \dot{\lambda}_z k_L \\ &= \dot{\psi}_L s\theta_L i_L - \dot{\theta}_L j_L + \dot{\psi}_L c\theta_L k_L.\end{aligned} \tag{16}$$

Let $\vec{\omega}_m$ and $\vec{\omega}_t$ denote the rate of the M-frame and the T-frame with respect to the L-frame, respectively. Then, we have

$$\vec{\omega}_m = \dot{\psi}_m s\theta_m i_m - \dot{\theta}_m j_m + \dot{\psi}_m c\theta_m k_m, \tag{17}$$

$$\vec{\omega}_t = \dot{\psi}_t s\theta_t i_t - \dot{\theta}_t j_t + \dot{\psi}_t c\theta_t k_t. \tag{18}$$

We assume that the missile and the target are aerodynamically controlled. For these assumptions to be confirmed, the acceleration vector of each vehicle must be normal to the velocity, as expressed in the following equations:

$$\vec{a}_m \triangleq a_{my} j_m + a_{mz} k_m, \tag{19}$$

$$\vec{a}_t \triangleq a_{ty} j_t + a_{tz} k_t. \tag{20}$$

The kinematics of the missile and the target is given by

$$\frac{d\vec{r}}{dt} = \vec{V}_t - \vec{V}_m = \frac{d}{dt}(r i_L) = \dot{r} i_L + \vec{\omega}_L \times \vec{r}, \tag{21}$$

$$\frac{d\vec{V}_m}{dt} = \vec{a}_m = \frac{d}{dt}(V_m i_m) = \dot{V}_m i_m + (\vec{\omega}_L + \vec{\omega}_m)\times \vec{V}_m, \tag{22}$$

$$\frac{d\vec{V}_t}{dt} = \vec{a}_t = \frac{d}{dt}(V_t i_t) = \dot{V}_t i_t + (\vec{\omega}_L + \vec{\omega}_t)\times \vec{V}_t. \tag{23}$$

Substituting (16)-(20) into (21)~(23) and rearranging them, we obtain the following equations of motion [24]:

$$\begin{aligned}
\dot{r} &= V_t c\theta_t c\psi_t - V_m c\theta_m c\psi_m, \\
\dot{\theta}_m &= \frac{a_{mz}}{V_m} + \frac{s\psi_m t\theta_L}{r}(V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m) \\
&\quad - \frac{c\psi_m}{r}(V_t s\theta_t - V_m s\theta_m), \\
\dot{\psi}_m &= \frac{a_{my}}{V_m c\theta_m} - \frac{1}{r}\left(\frac{s\theta_m c\psi_m t\theta_L}{c\theta_m} + 1\right)(V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m) \\
&\quad - \frac{s\theta_m s\psi_m}{r c\theta_m}(v_t s\theta_t - v_m s\theta_m), \\
\dot{\theta}_t &= \frac{a_{tz}}{V_t} + \frac{s\psi_t t\theta_L}{r}(V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m) \\
&\quad - \frac{c\psi_t}{r}(V_t s\theta_t - V_m s\theta_m), \\
\dot{\psi}_t &= \frac{a_{ty}}{V_t c\theta_t} - \frac{1}{r}\left(\frac{s\theta_t c\psi_t t\theta_L}{c\theta_t} + 1\right)(V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m) \\
&\quad - \frac{s\theta_t s\psi_t}{r c\theta_t}(V_t s\theta_t - V_m s\theta_m), \\
\dot{\theta}_L &= (V_t s\theta_t - V_m s\theta_m)/r, \\
\dot{\psi}_L &= (V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m)/r.
\end{aligned} \tag{24}$$

Let us consider the following relative velocity components in the L-frame:

$$\begin{aligned}
v_x &\triangleq V_t c\theta_t c\psi_t - V_m c\theta_m c\psi_m, \\
v_y &\triangleq V_t c\theta_t s\psi_t - V_m c\theta_m s\psi_m, \\
v_z &\triangleq V_t s\theta_t - V_m s\theta_m.
\end{aligned} \tag{25}$$

Because ψ_L does not affect the behavior of the other states, it can be ignored in the analysis. By using (25), the equations of motion given in (24) are greatly simplified as

$$x = \begin{bmatrix} r & v_x & v_y & v_z & \theta_L \end{bmatrix}^T, \tag{26}$$

$$f(x) = \begin{bmatrix} v_x \\ (v_y^2 + v_z^2)/r \\ (v_z t\theta_L - v_x)v_y/r \\ -(v_y^2 t\theta_L + v_x v_z)/r \\ v_z/r \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

and

$$u = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T, \tag{28}$$

where

$$\begin{aligned} u_x &= (-a_{tz} s\theta_t c\psi_t - a_{ty} s\psi_t) \\ &\quad + (a_{mz} s\theta_m c\psi_m + a_{my} s\psi_m), \end{aligned} \tag{29}$$

$$\begin{aligned} u_y &= (-a_{tz} s\theta_t s\psi_t + a_{ty} c\psi_t) \\ &\quad + (a_{mz} s\theta_m s\psi_m - a_{my} c\psi_m), \end{aligned} \tag{30}$$

$$u_z = (a_{tz} c\theta_t) - (a_{mz} c\theta_m). \tag{31}$$

The acceleration equations given in (29)-(31) clearly represent the relation between the guidance command and the missile acceleration command normal to the missile velocity.

3.2. Equilibrium subspace

Sontag's formula requires $f(0) = 0$ as the equilibrium point. Although $f(0) = 0$ is achieved by the equilibrium subspace $\{x \mid v_x = v_y = v_z = 0\}$, this equilibrium subspace does not guarantee target interception. Suppose that $v_y = 0$ and $v_z = 0$. Then, from (25), we obtain

$$\begin{aligned} v_z &= 0 \Rightarrow s\theta_m^{eq} = \frac{v_t s\theta_t}{v_m}, \\ v_y &= 0 \Rightarrow s\psi_m^{eq} = \frac{v_t c\theta_t s\psi_t}{v_m c\theta_m^{eq}} = \frac{v_t c\theta_t s\psi_t}{\sqrt{v_m^2 - v_t^2 s^2\theta_t}}. \end{aligned} \tag{32}$$

By using (32), we can easily show that for $v_y = v_z = 0$

$$\begin{aligned} c &\triangleq v_x \mid_{v_y = v_z = 0} \\ &= v_t c\theta_t c\psi_t - v_m c\theta_m^{eq} c\psi_m^{eq} \\ &= v_t c\theta_t c\psi_t - \sqrt{v_m^2 - v_t^2 \left(1 - c^2\theta_t c^2\psi_t\right)} \\ &< 0 \quad \text{for } v_m > v_t. \end{aligned} \tag{33}$$

Therefore, the equilibrium subspace of the system is defined as

$$x_{eq} = \left\{ x \mid v_y = v_z = 0 \text{ and } v_x = c \right\}. \tag{34}$$

The condition $v_y = v_z = 0$ implies that the normal components of the relative velocity to the LOS are zero. Because $\dot{r} < 0$, r approaches 0 as far as the equilibrium subspace given in (34) is satisfied. For a non-maneuvering target, c remains constant but it is time varying if the target maneuver is present. The equilibrium subspace given by (34) specifies the condition under which the missile intercepts the target within a finite time.

4. NONLINEAR GUIDANCE LAW

4.1. Guidance law formulation

We used Sontag's formula to derive the nonlinear guidance law that delivers the state to the equilibrium subspace. As mentioned in Section 2, the basic formulation of the guidance law depends largely on how we choose the Lyapunov function candidate, $V(x)$, and the state function, $q(b)$, that is included in the cost.

Suppose that a Lyapunov function candidate for the nonlinear system is given by

$$V = \frac{1}{2}\left[(v_x - c)^2 + v_y^2 + v_z^2 \right]. \tag{35}$$

If we find a feedback control that enables the time derivative of V to become negative for all t, then v_x, v_y, and v_z eventually approach the equilibrium subspace, thereby ensuring the target interception. Note that the storage function $U = \lambda v_x (v_y^2 + v_z^2)/r$, which is used by Yang and Chen [17], is not a Lyapunov function because U is possibly zero for $v_x = 0$ and for $v_y = v_z = 0$. Hence, we cannot guarantee the global capturability of the guidance law obtained by using U.

Each term included in Sontag's formula is calculated as

$$\frac{\partial V}{\partial x} = \begin{bmatrix} 0 & (v_x - c) & v_y & v_z & 0 \end{bmatrix}, \tag{36}$$

$$\frac{\partial V}{\partial x} G = \begin{bmatrix} (v_x - c) & v_y & v_z \end{bmatrix}, \tag{37}$$

$$\begin{aligned} a &= \frac{\partial V}{\partial x} f \\ &= (v_x - c)\frac{1}{r}\left(v_y^2 + v_z^2\right) + \frac{1}{r}\left(v_z t\theta_L - v_x\right) v_y^2 \\ &\quad - \frac{1}{r}\left(v_y^2 t\theta_L + v_x v_z\right) v_z \\ &= -\frac{c}{r}\left(v_y^2 + v_z^2\right), \end{aligned} \tag{38}$$

$$b = \frac{\partial V}{\partial x} G G^T \left(\frac{\partial V}{\partial x}\right)^T = (v_x - c)^2 + v_y^2 + v_z^2. \tag{39}$$

Next, we choose $q(b)$ in (13) as

$$q(b) = \frac{N'c^2}{r^2} b = \frac{N'c^2}{r^2}\left[(v_x - c)^2 + v_y^2 + v_z^2 \right], \tag{40}$$

where c is given by (33) and N' is an arbitrary chosen positive constant that tunes the weighting of the state function in the cost. The nonlinear guidance law is then calculated as

$$\begin{aligned} u &= \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T \\ &= \begin{cases} -N(x)\begin{bmatrix} v_x - c \\ v_y \\ v_z \end{bmatrix} & \text{for } (v_x - c)^2 + v_y^2 + v_z^2 \neq 0 \\ 0 & \text{for } (v_x - c)^2 + v_y^2 + v_z^2 = 0, \end{cases} \end{aligned} \tag{41}$$

where

$$N(x) = \frac{-\frac{c}{r}\left(v_y^2+v_z^2\right)+\left|\frac{c}{r}\right|\sqrt{\left(v_y^2+v_z^2\right)^2+N'\left[\left(v_x-c\right)^2+v_y^2+v_z^2\right]^2}}{\left(v_x-c\right)^2+v_y^2+v_z^2}. \tag{42}$$

From (13) and (40), the proposed law given by (41) with (42) minimizes

$$J=\int_0^\infty\left[\frac{2N'c^2}{r^2}V(x)+u^T(x)u(x)\right]dt \tag{43}$$

subject to (3) with (27).

Note that the first term in the integrand of (43) is weighted by the inverse square of r. In the beginning of the engagement, r is so large that a small weighting is applied to the first term. This small weighting implies that the proposed law initially makes the missile follow a control energy minimization path. As the missile approaches the target, the weighting increases infinitely so that the guidance law enforces $V(x)\to 0$, which denotes the target interception condition. From (42) and (43), we see that N' plays a role in tuning the overall command history. As N' becomes larger, the magnitude of the guidance command grows to provide more agile maneuvers of the missile. In this case, however, the consumption of the control energy increases. By a proper choice of N', the maximum magnitude of the guidance command from the proposed law can be smaller than that of APN, which is regarded as a solution to the optimal control problem of minimizing the control energy only [15,16]. A direct comparison between the proposed law and APN in the formulation level is impossible because the optimal control problem given in (43) does not contain any terminal constraints that can be found in the optimal control problem for APN.

4.2. Implementation aspect

Since the guidance law given in (41) is defined in the L-frame, it cannot be directly used for missiles controlled by aerodynamic force and moment. By using (31), a_{mz} is uniquely determined as

$$a_{mz}=-\frac{1}{c\theta_m}\left[u_z-a_{tz}c\theta_t\right]. \tag{44}$$

From (29) and (30), we can eliminate a_{mz} to obtain

$$a_{my}=\bar{u}_x s\psi_m-\bar{u}_y c\psi_m, \tag{45}$$

where

$$\begin{aligned}\bar{u}_x &= u_x+a_{tz}s\theta_t c\psi_t+a_{ty}s\psi_t,\\ \bar{u}_y &= u_y+a_{tz}s\theta_t s\psi_t-a_{ty}c\psi_t.\end{aligned} \tag{46}$$

Some sensor systems and target tracking filters are required because the proposed guidance law is a full state–feedback control law. Suppose that the missile is equipped with an inertial measurement unit (IMU) and a seeker. Like other guidance laws associated with target maneuvering, a target-tracking filter is essential for implementing the proposed guidance law. Hence, we assume that the relative range and the relative target maneuver with respect to the missile can be estimated by target-tracking filters supplied by the IMU and the seeker. The following states are observable:

1) IMU
- $\vec{r}_m^I$, $\vec{V}_m^I$, $\vec{a}_m^I$: missile position, velocity, and acceleration in the I-frame
- Ψ, Θ, Φ: missile body Euler angles for the I-frame

2) Seeker
- ψ_g, θ_g, ϕ_g: seeker gimbal angles for the missile body
- $\dot{\lambda}_y(=-v_z/r)$, $\dot{\lambda}_z(=v_y/r)$: LOS rate along y and z axis of the L-frame

3) Target-tracking filter
- r, $\dot{r}=v_x$: relative range and time rate
- a_x, a_y, a_z: relative target acceleration to the missile in the L-frame.

By using the Euler angles from the IMU and the seeker, the direction cosine matrix from the I-frame to the L-frame is calculated as

$$C_I^L=C_B^L C_I^B, \tag{47}$$

where B denotes the body frame and

$$\begin{aligned}C_I^B &= T_x(\Phi)T_y(\Theta)T_z(\Psi),\\ C_B^L &= T_x(\phi_g)T_y(\theta_g)T_z(\psi_g).\end{aligned} \tag{48}$$

The missile velocity vector in the L-frame is then calculated as

$$\vec{V}_m^L=C_I^L\vec{V}_m^I. \tag{49}$$

By using (49), we can give the flight path angles for the L-frame by

$$\psi_m=\tan^{-1}\frac{V_{my}^L}{V_{mx}^L}, \tag{50}$$

$$\theta_m = \tan^{-1}\frac{V_{mz}^L}{\sqrt{\left(V_{mx}^L\right)^2+\left(V_{my}^L\right)^2}}.$$

On the other hand, the target velocity vector in the L-frame can be calculated by

$$\begin{aligned}\vec{V}_t^L &\triangleq \begin{bmatrix} V_{tx}^L & V_{ty}^L & V_{tz}^L \end{bmatrix}^T \\ &= \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T - C_I^L \vec{V}_m^I. \end{aligned} \tag{51}$$

By using (51), we obtain

$$\psi_t = \tan^{-1}\frac{V_{ty}^L}{V_{tx}^L},\ \theta_t = \tan^{-1}\frac{V_{tz}^L}{\sqrt{\left(V_{tx}^L\right)^2+\left(V_{ty}^L\right)^2}}. \tag{52}$$

Finally, we calculate the target acceleration described in the T-frame as

$$\begin{bmatrix} 0 \\ a_{ty} \\ a_{tz} \end{bmatrix} = C_L^T\left(\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} - C_I^L \vec{a}_m^I\right), \tag{53}$$

where

$$C_L^T = T_y(-\theta_t)T_z(\psi_t). \tag{54}$$

The target maneuver term included in APN is not a relative acceleration to the missile but an absolute one. For an autonomous mission, APN also requires an IMU to separate the absolute target acceleration from the estimated relative acceleration. Therefore, APN requires the same sensors for implementation as the proposed law.

5. NUMERICAL EXAMPLES

In this section, the proposed nonlinear guidance (NLG) law given by (44) and (45) is compared with PPN-based APN for several engagement scenarios. The capturability of three-dimensional PPN-based APN is thought to be global. Three-dimensional TPN and PPN-based APN are summarized in the appendix.

The initial conditions in the I-frame of the missile and the target for nonlinear simulation are given by

- For the missile: $x_m(0) = y_m(0) = z_m(0) = 0\text{m}$, $V_m = 1000\text{m/s}$.
- For the target: $x_t(0) = 5{,}000\text{m}$, $y_t(0) = 2{,}500\text{m}$, $z_t(0) = 0\text{m}$, $V_t = 500\text{m/s}$.

A circular maneuvering target with a 10g turn in the $y_I - z_I$ plane, which is shown in Fig. 2, was considered in all the simulations. This kind of target maneuver is known as an effective evasive maneuver policy for an aircraft against proportionally navigated missiles [25].

First, we investigate the performance changes for NLG with different N' s. In this simulation, we

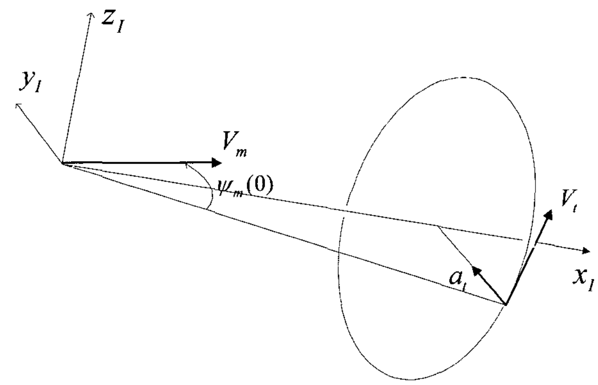


Fig. 2. Target maneuver plane for nonlinear simulations.

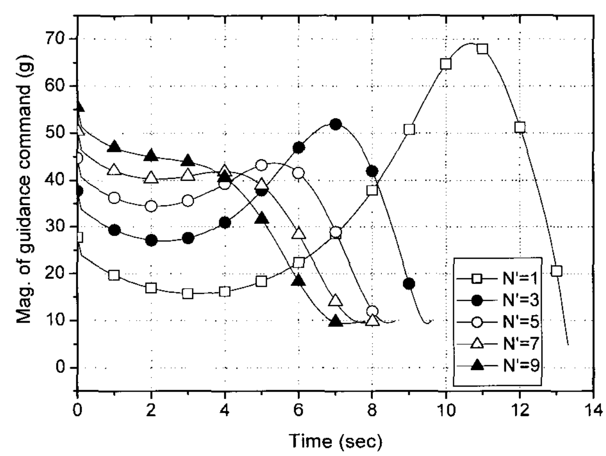


Fig. 3. Guidance command histories for NLG with different N' values.

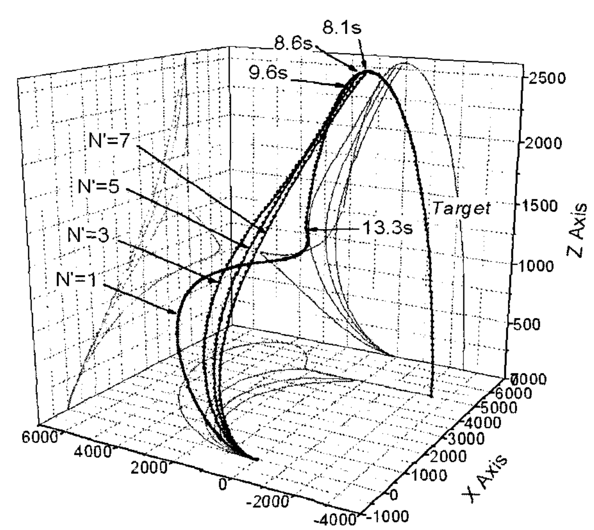
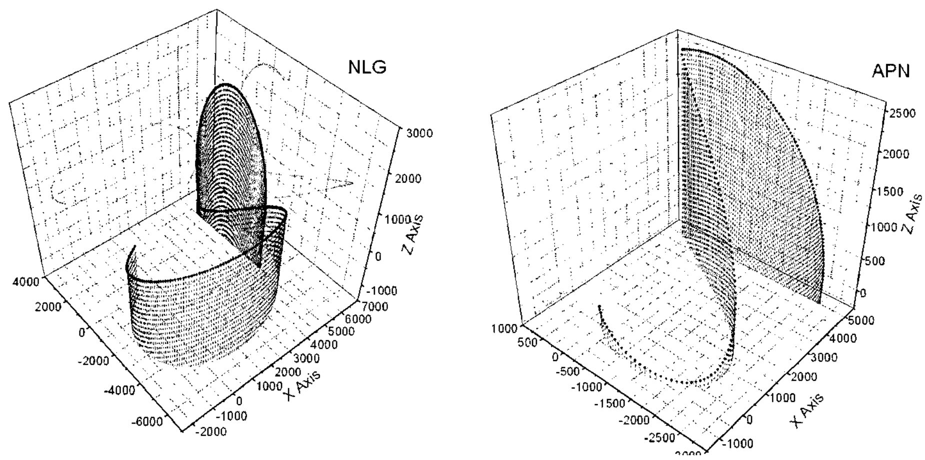
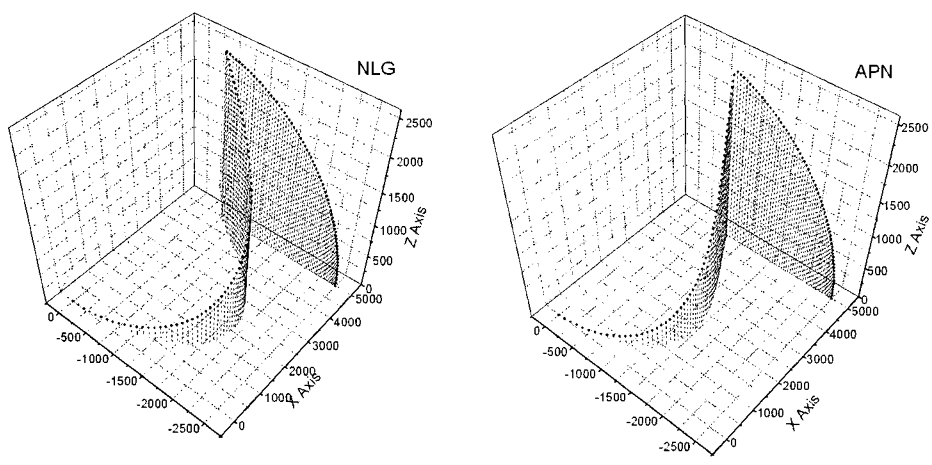


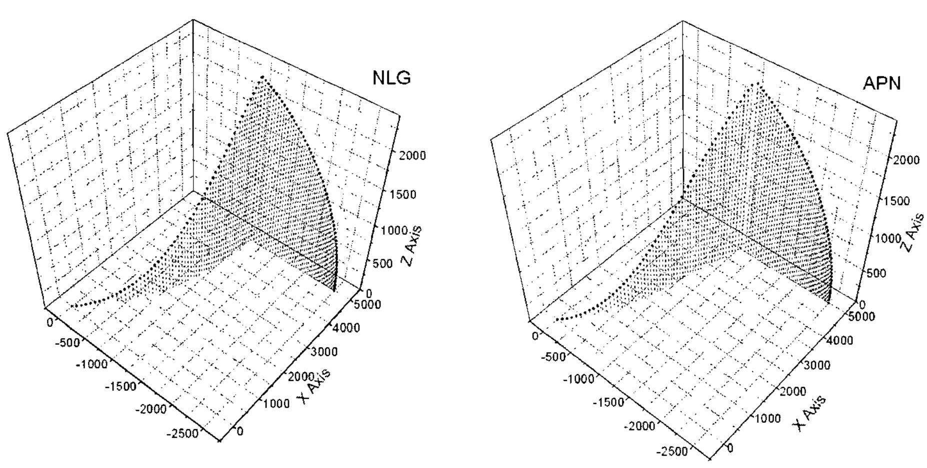
Fig. 4. Trajectories for NLG with different N' values.



(a) $\psi_m(0) = -120^o, \theta_m(0) = 0^o$.



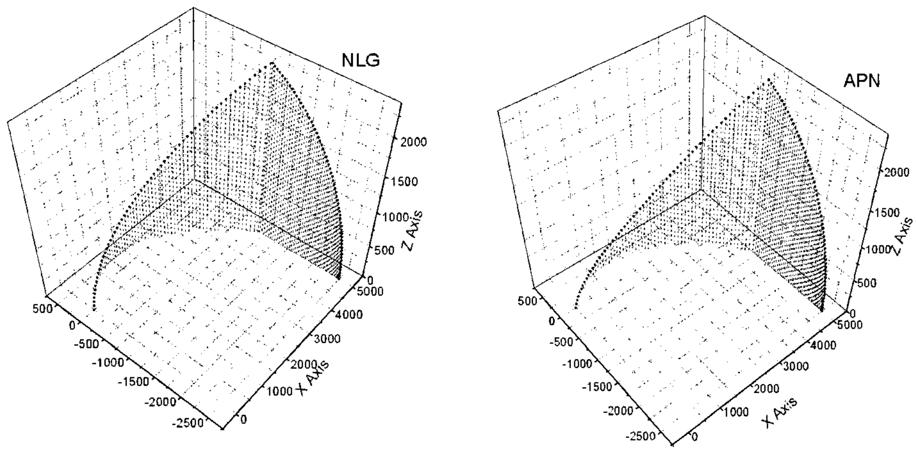
(b) $\psi_m(0) = -60^o, \theta_m(0) = 0^o$.



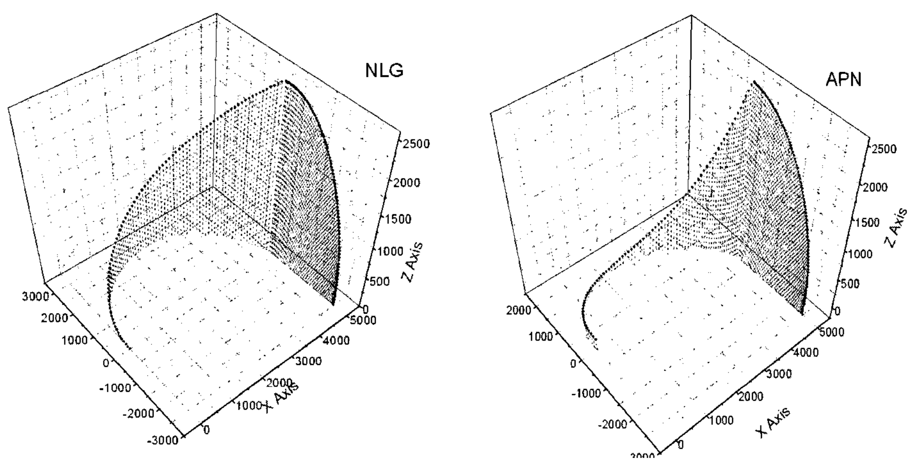
(c) $\psi_m(0) = 0^o, \theta_m(0) = 0^o$.

Fig. 5. Trajectory comparisons of NLG with $N' = 6$ and APN with $N = 4$ (continue).

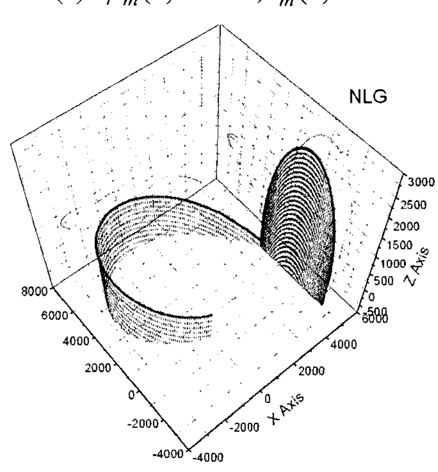
assume $\psi_m(0)=120^\circ$ and $\theta_m(0)=0^\circ$. As shown in Fig. 3, we observed that the guidance command in the beginning of the flight becomes large as N' increases, whereas the maximum guidance command for a small N' value is large in the terminal flight phase. For $N' > 7$, the maximum guidance command occurs in the beginning of the flight. Because the maximum guidance command is minimized near $N' = 7$, $6 \le N' \le 8$ is recommended for practical application. As shown in Fig. 4, the resultant trajectory approaches a straight line as N' increases. Indeed, for an infinitely large N', the trajectory



(d) $\psi_m(0) = 60^o, \theta_m(0) = 0^o$.



(e) $\psi_m(0) = 120^o, \theta_m(0) = 0^o$.

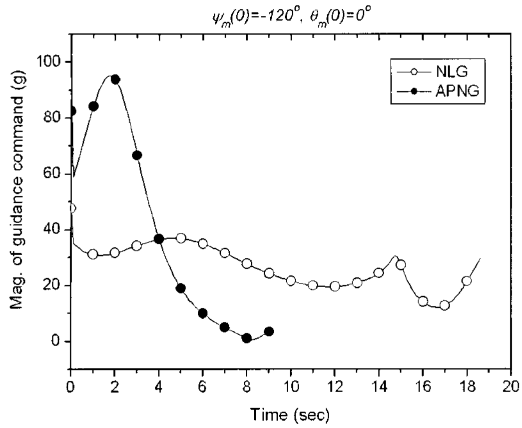


(f) $\psi_m(0) = 180^o, \theta_m(0) = 0^o$.

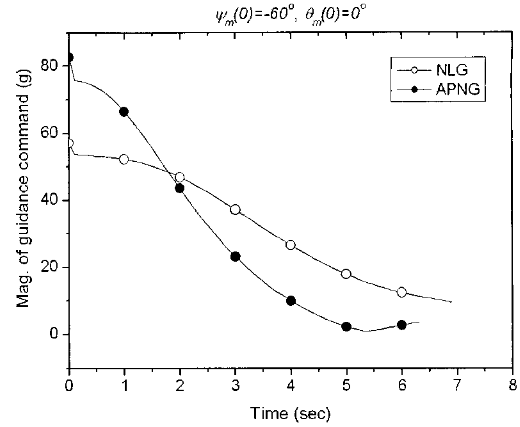
Fig. 5. Trajectory comparisons of NLG with $N' = 6$ and APN with $N = 4$ (continue).

converges to a straight collision path.

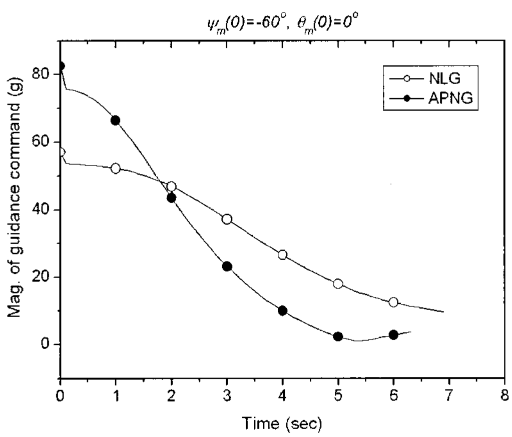
The three-dimensional trajectory and guidance command history comparisons between NLG with $N' = 6$ and APN with $N = 4$ for various initial heading angles of the missile are shown in Figs. 5 and 6, respectively. Note that for each initial heading angle the APN trajectories are less curved than the NLG trajectories. Fig. 6 also confirms that the magnitude of the initial guidance command of APN is greater than that of NLG. This result implies that faster heading changes are produced by APN. As shown in Figs. 6(b) to 6(d), for small heading angles the command



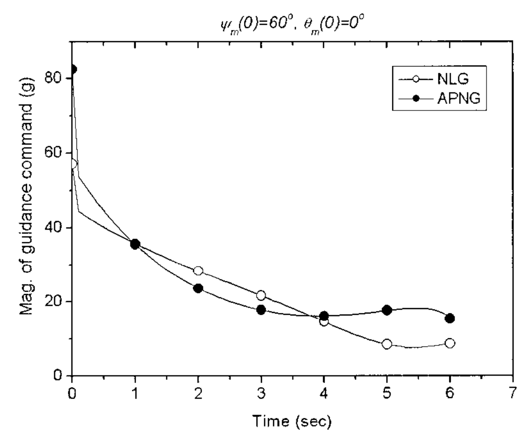
(a) $\psi_m(0) = -120^o, \theta_m(0) = 0^o$.



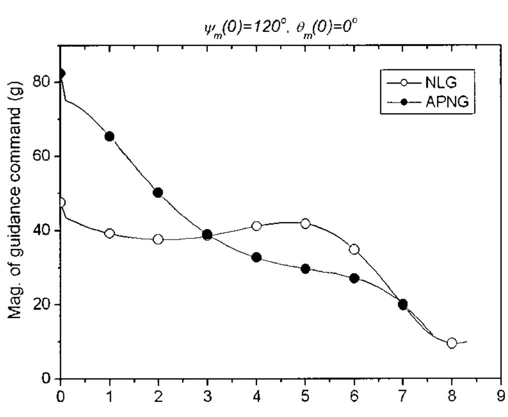
(b) $\psi_m(0) = -60^o, \theta_m(0) = 0^o$.



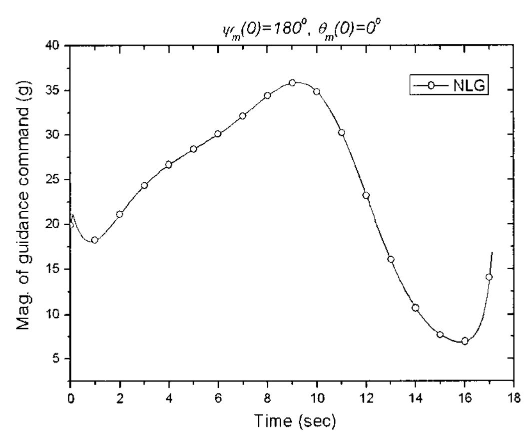
(c) $\psi_m(0) = 0^o, \theta_m(0) = 0^o$.



(d) $\psi_m(0) = 60^o, \theta_m(0) = 0^o$.



(e) $\psi_m(0) = 120^o, \theta_m(0) = 0^o$.



(f) $\psi_m(0) = 180^o, \theta_m(0) = 0^o$.

Fig. 6. Guidance command comparisons of NLG with $N' = 6$ and APN with $N = 4$.

magnitude of NLG in the terminal homing phase is also smaller than that of APN. In the case of $\psi_m(0) = 180^o$ and $\theta_m(0) = 0^o$, the missile with APN does not capture the target but NLG can reach the target with a very curved trajectory, as shown in Fig. 5(f). From these simulations, we can see that the guidance command of NLG needs to be of a smaller magnitude than that of APN. Moreover, NLG can capture an evasively maneuvering target for the entire domain of the initial heading angles whereas APN has a bounded capture region for the initial heading angles.

Figs. 7 and 8 show a capture region of NLG with $N' = 6$ and APN with $N = 4$ in the $\psi_m(0) - \theta_m(0)$ plane under missile command limitations of 50 g. In

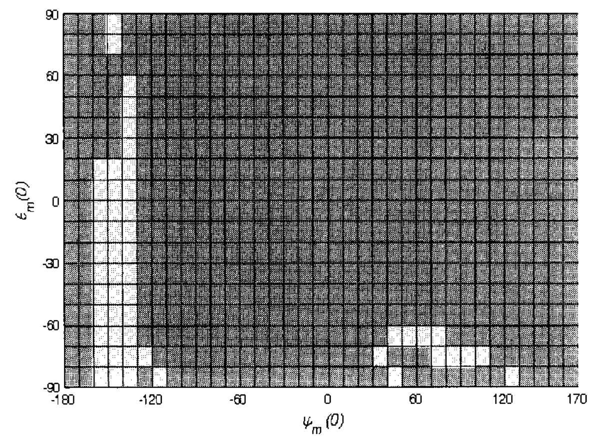


Fig. 7. Capture region (shaded area) of the initial heading angles for NLG ($u_{\lim}$ =50g).

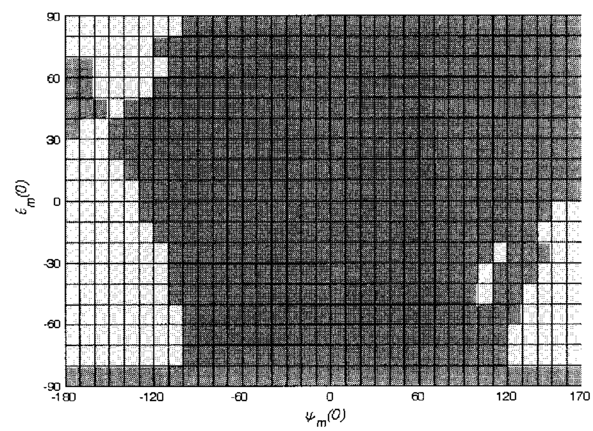


Fig. 8. Capture region (shaded area) of the initial heading angles for APN ($u_{\lim}$ =50g).

this simulation, we considered the same target maneuver as in the previous case. A miss distance of up to 10 meters is regarded as a successful interception of the target. Cases in which the flight time is greater than 60 seconds are treated as failures, even if the miss distance criterion is satisfied. Fig. 7 shows that NLG can capture the target for most of the $\psi_m(0)-\theta_m(0)$ plane, even if the capture region of APN is somewhat restricted. For NLG, if the missile is launched with the initial heading angles of $-120^o \le \psi_m(0) \le -150^o$, the missile cannot intercept the target due to the command limitation. For several initial heading angles in the range of $40^o \le \psi_m(0) \le 120^o$, the missile with NLG does not satisfy the flight time limitation even though the miss distance is within 10 meters. These performances are largely due to the fact that the guidance command from NLG is too small to initially correct the initial heading errors and does not grow as r increases. In contrast to the case of NLG, all of the miss distances of APN are caused by the command limitation.

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6. CONCLUSION

We have proposed a nonlinear guidance law for aerodynamically controlled missiles against maneuvering targets. Because the proposed guidance law is derived from Sontag's formula, it has global capturability as well as optimality. The basic requirements for practical implementation of the proposed law are an onboard IMU and a target-tracking filter combined with a seeker; this equipment is also required in the implementation of a conventional APN. The proposed law has a larger launch envelope than APN against maneuvering targets even if the command limit is introduced. This phenomenon occurs mainly because the magnitude of the command produced by the proposed law is smaller than that of APN.

The proposed law is a special case of Sontag's formula for guidance application. Hence, it can be used to derive an alternative guidance law with the proper selection of a Lyapunov function and the cost function.

APPENDIX: AUGMENTED PROPORTIONAL NAVIGATION LAWS

APN, which is a variant of TPN, is concerned with missile guidance against a constantly accelerated target from the energy optimal perspective. Three-dimensional APN in the L-frame based on the TPN formulation can be obtained as follows:

$$\vec{u}^L = \begin{bmatrix} u_x^{TPN} & u_y^{TPN} & u_z^{TPN} \end{bmatrix}^T + \frac{1}{2} C_T^L \vec{a}_t, \tag{55}$$

where $\vec{a}_t$ is the target acceleration defined in the T-frame and C_T^L denotes the direction cosine matrix from the T-frame to the L-frame, and

$$\begin{bmatrix} u_x^{TPN} & u_z^{TPN} & -u_y^{TPN} \end{bmatrix}^T \triangleq N V_c \vec{\omega}_L. \tag{56}$$

Here, the closing velocity is given by

$$V_c = -v_x, \tag{57}$$

and the LOS rate is given by

$$\vec{\omega}_L = \frac{1}{r} \begin{bmatrix} v_y \tan\theta_L & -v_z & v_y \end{bmatrix}^T. \tag{58}$$

The TPN command vector in the M-frame, $\vec{u}^M$, is given by

$$\begin{aligned}\vec{u}^M &\triangleq \begin{bmatrix} u_{mx}^{ATPN} & u_{my}^{ATPN} & u_{mz}^{ATPN} \end{bmatrix}^T \\ &= C_L^M \begin{bmatrix} u_x^{TPN} & u_y^{TPN} & u_z^{TPN} \end{bmatrix}^T + \frac{1}{2} C_L^M C_T^L \vec{a}_t. \end{aligned} \tag{59}$$

For aerodynamically controlled missiles, only the normal components to the missile velocity, u_{my}^{ATPN} and u_{mz}^{ATPN}, are available.

Although TPN-based APN is energy optimal, its capturability is not known. In fact, the capture region of TPN is highly limited even for a target with a constant speed [8]. PPN always guarantees target interception when the ratio of the missile speed to target speed is greater than $\sqrt{2}$ and the navigation constant is greater than 1 [13]. We can therefore deduce that PPN-based APN has global capturability. The PPN-based APN in the M-frame can be formulated as

$$\begin{aligned}\vec{u}_{APN} &\triangleq \begin{bmatrix} u_{mx}^{APPN} & u_{my}^{APPN} & u_{mz}^{APPN} \end{bmatrix}^T \\ &= N\left(\vec{\omega}_L \times \vec{V}_m + \frac{1}{2} C_L^M C_T^L \vec{a}_t \right). \end{aligned} \tag{60}$$

As in the case of TPN-based APN, u_{my}^{APPN} and u_{mz}^{APPN} are only available for aerodynamically controlled missiles. In this case, u_{mx}^{APPN}, which is introduced by the augmented target acceleration, is also neglected and the command components normal to the missile velocity are calculated as

$$\begin{aligned} a_{my}^{APPN} &= -NV_m\left(\dot{\lambda}_x s\theta_m c\psi_m + \dot{\lambda}_y s\theta_m s\psi_m - \dot{\lambda}_z c\theta_m\right) \\ &\quad + 0.5 N a_{ty}^M, \\ a_{mz}^{APPN} &= -NV_m\left(-\dot{\lambda}_x s\psi_m + \dot{\lambda}_y c\psi_m\right) + 0.5 N a_{tz}^M, \end{aligned} \tag{61}$$

where

$$\begin{aligned} a_{ty}^M &= \left(s\psi_m s\psi_t + c\psi_m c\psi_t\right) a_{ty} \\ &\quad + \left(s\psi_m s\theta_t c\psi_t - c\psi_m s\theta_t s\psi_t\right) a_{tz}, \\ a_{tz}^M &= \left(s\theta_m c\psi_m s\psi_t - s\theta_m s\psi_m c\psi_t\right) a_{ty} \\ &\quad + \begin{pmatrix} s\theta_m c\psi_m s\theta_t c\psi_t + s\theta_m s\psi_m s\theta_t s\psi_t \\ + c\theta_m c\theta_t \end{pmatrix} a_{tz}. \end{aligned} \tag{62}$$

In practice, $\dot{\lambda}_x$ can be ignored because it cannot be measured by an on-board seeker.

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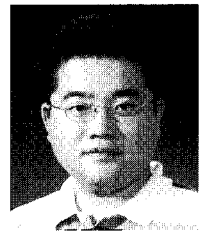
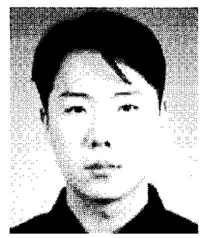
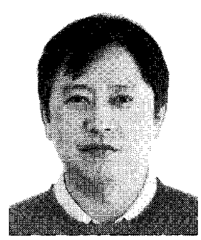


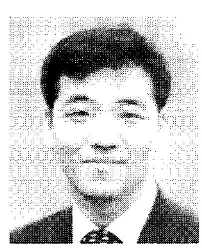
Chang-Kyung Ryoo received the B.S. degree in Aerospace Engineering from Inha University, Korea, in 1989. He received the M.S. and Ph.D. degrees in Aerospace Engineering from KAIST, Korea, in 1991 and 2006, respectively. From 1991 to 2006, he worked for Agency for Defense Development in Korea. Now, he is an Assistant Professor of the Department of Aerospace Engineering, Inha University, Korea. His research interests include nonlinear guidance and control of missiles and UAVs.



Yoon-Hwan Kim received the B.S. degree in Mechanical & Aerospace Engineering from Seoul National University, Korea, in 2002. He received the M.S. degree in Aerospace Engineering from KAIST in 2004. He is currently pursuing a Ph.D. degree in Aerospace Engineering, KAIST. His research areas include missile guidance, UAV formation flight, and attitude control of spacecraft.



Min-Jea Tahk received the B.S. degree in Aerospace Engineering from Seoul National University, Korea, in 1976. He received the M.S. and Ph.D. degrees in Aerospace Engineering from the University of Texas at Austin, in 1983 and 1986, respectively. He is currently a Professor of the Department of Aerospace Engineering, KAIST. His research areas include missile guidance, UAV formation flight, autopilot design, and evolutionary optimization techniques.



Keeyoung Choi received the B.S. and M.S. degrees in Aeronautical Engineering from Seoul National University, Korea, in 1987 and 1989, respectively. He received the Ph.D. degree in Aeronautics and Astronautics from Stanford University in 1994. From 1993 to 1999 he was a Senior Engineer at Advaned Rotorcraft Technology, Inc. He joined the Department of Aerospace Engineering in Inha University in 2000, and is an Associate Professor. His research interests include optimal control theories, navigation and control of UAV, and rotorcraft simulation.