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COMPUTATION OF THE NIELSEN TYPE NUMBERS FOR MAPS ON THE KLEIN BOTTLE

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Abstract. Let $f : M \to M$ be a self-map on the Klein bottle M. We compute the Lefschetz number and the Nielsen number of f by using the infra-nilmanifold structure of the Klein bottle and the averaging formulas for the Lefschetz numbers and the Nielsen numbers of maps on infra-nilmanifolds. For each positive integer n, we provide an explicit algorithm for a complete computation of the Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$ of f^n.

1. Introduction

Let M be a closed manifold, and let $f : M \to M$ be a self-map. Then we define

$$\mathrm{Fix}(f) = \{x \in M \mid f(x) = x\}$$

the *fixed point set* of f. There are well known invariants in fixed point theory, the Lefschetz number $L(f)$ and the Nielsen number $N(f)$. It is known that the Nielsen number is much more powerful than the Lefschetz number but computing it is very hard.

In [2], Brooks, Brown, Pak, and Taylor show that for a self map $f : M \to M$ on a torus, the Nielsen number $N(f)$ and the Lefschetz number $L(f)$ are equal up to a sign, i.e., $N(f) = |L(f)| = |\det(I - f_*)|$, where $f_* : \pi_1(M) \to \pi_1(M)$ is the homomorphism on $\pi_1(M)$ induced by f. In [1], this result is extended to compact nilmanifolds. Let L be a connected, simply connected nilpotent Lie group, Γ a uniform lattice of L, and $M = \Gamma\backslash L$ a nilmanifold. Any $f : M \to M$ is homotopic to a map obtained from an endomorphism $F : L \to L$ for which $F(\Gamma) \subset \Gamma$. Let F_* be the corresponding endomorphism of the Lie algebra of L. Then $N(f) = |L(f)| = |\det(I - F_*)|$. In [10] and [12], the averaging formula

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for the Nielsen number on infra-nilmanifolds is obtained:

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|,$$

where $f : M \to M$ is a self-map on the infra-nilmanifold M with holonomy group Ψ.

In dynamical systems, it is often the case that topological information can be used to study qualitative and quantitative properties (like the set of periods) of the system. For the periodic points, two Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$ are lower bounds for the number of periodic points of least period exactly n and the set of periodic points of period n, respectively.

In this paper we will give a complete computation (see Theorem 2.3) of the Lefschetz numbers $L(f^n)$ and the Nielsen numbers $N(f^n)$ of all iterates of f when f is a self-map on the Klein bottle. To this end, we will use the fact that the Klein bottle is an infra-nilmanifold, and then use the above averaging formula for Nielsen numbers of continuous self-maps on the infra-nilmanifold.

It is known that on solvmanifolds, the Nielsen numbers of iterates of a map are related with the Nielsen type numbers $N\Phi_n(f)$ and $NP_n(f)$ (cf. [4], [5], [6], [7], [8]). In particular, if f^n is weakly Jiang with $N(f^n) \neq 0$, then two Nielsen type numbers are related to each other:

$$N\Phi_n(f) = \sum_{m|n} NP_m(f), \quad NP_n(f) = \sum_{m|n} \mu(m) N\Phi_{\frac{n}{m}}(f).$$

The second purpose of this paper is to give an explicit algorithm for a complete computation of the Nielsen type numbers $NP_n(f)$ for periodic points of self-maps on the Klein bottle. In a series of papers [5], [6], [7] and [8], Heath and Keppelmann explored calculation of these numbers under some conditions, see Corollary 3.6 of this paper.

2. The Klein bottle maps

Let $\alpha = (a, A)$ and $t_i = (e_i, I_2)$ be elements of $\mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2)$, where

$$a = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then A has period 2, $(a, A)^2 = (a + Aa, I_2) = (e_1, I_2)$, and $t_2\alpha = \alpha t_2^{-1}$. Let Γ be the subgroup generated by t_1 and t_2. Then it forms a lattice in $\mathbb{R}^2$ and $\Gamma\backslash\mathbb{R}^2$ is the 2-torus. It is easy to check that the subgroup

$$\Pi = \langle \Gamma, (a, A) \rangle \subset \mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2)$$

generated by the lattice Γ and the element (a, A) is discrete and torsion free. Furthermore, Γ is a normal subgroup of Π of index 2. Thus Π is an (almost) Bieberbach group, which is the Klein bottle group, and the quotient space $\Pi\backslash\mathbb{R}^2$ is the Klein bottle. Thus $\Gamma\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ is a double covering projection.

Lemma 2.1. *Any homomorphism $\varphi : \Pi \to \Pi$ on the Klein bottle group Π is given as follows:*

$$\varphi(\alpha) = \alpha^r t_2^\ell, \quad \varphi(t_2) = t_2^q,$$

where either r is odd, or r is even and $q = 0$.

Proof. Since Π is generated by α and t_2 subject to $t_2\alpha = \alpha t_2^{-1}$, every element of Π is of the form $\alpha^m t_2^n$. Thus $\varphi(\alpha) = \alpha^r t_2^\ell$ and $\varphi(t_2) = \alpha^s t_2^q$ for some integers r, ℓ, s, q. Since $t_2\alpha = \alpha t_2^{-1}$, we have $\varphi(t_2)\varphi(\alpha) = \varphi(\alpha)\varphi(t_2)^{-1}$. Inspection of this equation induces that $s = 0$, and r is odd or $q = 0$. □

Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$. Fix a lifting $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of f. Then the lifting $\tilde{f}$ induces a homomorphism $\varphi : \Pi \to \Pi$ which is defined by the following rule:

$$\varphi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha \quad \text{for all } \alpha \in \Pi.$$

The homomorphism φ is called a *homomorphism of type* (r, ℓ, q) induced by f. In this case, f is said to be of *of type* (r, ℓ, q).

Recall from [11, Proposition 3.3 and Theorem 3.4] the following homotopy classification of all maps on the Klein bottle.

Lemma 2.2. *Every continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ is homotopic to a map of type (r, ℓ, q) where if r is odd then $\ell = 0, 1$ and $q \geq 0$; and if r is even and $q = 0$, then $\ell \geq 0$. Furthermore, two such maps of type (r, ℓ, q) and (r', ℓ', q') are homotopic if and only if $r = r'$, $q = q'$ and $\ell = \ell'$.*

The following was observed in [6, Example 5.5] using fiber space techniques for solvmanifolds. See also [3, Theorem 5.7], in which another algebraic method is used. Our calculation is obtained using the averaging formula for infra-nilmanifolds, [10, Theorem 3.5] and [12, Theorem 1.4]. This method is much explicit and entirely different from fiber space techniques. In particular, the element $(c, F) \in \mathrm{Aff}(2)$ (see below) will play a crucial role in calculating the Nielsen type numbers, see Sections 5 and 6.

Theorem 2.3. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). Then for any $n \in \mathbb{N}$, the Lefschetz number and the Nielsen number of the nth iterate of f are*

$$L(f^n) = 1 - r^n, \quad N(f^n) = \begin{cases} |q^n(1 - r^n)| & \text{if } r \text{ is odd and } q \neq 0; \\ |1 - r^n| & \text{if } q = 0. \end{cases}$$

Proof. Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ induce a homomorphism φ on Π of type (r, ℓ, q). By Lemma 2.1, φ must map Γ into Γ itself. Thus f always has a lifting $\bar{f} : \Gamma\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$ so that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\tilde{f}"] \arrow[d, "p'"] & \mathbb{R}^2 \arrow[d, "p'"] \\
\Gamma\backslash\mathbb{R}^2 \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"] & \Gamma\backslash\mathbb{R}^2 \arrow[d, "\bar{p}"] \\
\Pi\backslash\mathbb{R}^2 \arrow[r, "f"] & \Pi\backslash\mathbb{R}^2
\end{tikzcd}$$

On the other hand, for such a homomorphism φ there exists an affine map $(c, F) \in \mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2)$ such that

$$\varphi(\alpha)(c, F) = (c, F)\alpha, \ \varphi(t_2)(c, F) = (c, F)t_2.$$

This is due to Theorem 1.1 of [13]. These equalities yield that

$$(*) \qquad (c, F) = \begin{cases} \left(-\frac{1}{2}\begin{bmatrix} * \\ \ell \end{bmatrix}, \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix}\right) & \text{if } r \text{ is odd;} \\ \left(\begin{bmatrix} * \\ * \end{bmatrix}, \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix}\right) & \text{if } r \text{ is even and } q = 0. \end{cases}$$

Furthermore, from the above equalities we see that the affine map (c, F) on $\mathbb{R}^2$ induces a map $\Phi_{(c,F)} : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$. Since $\varphi|_\Gamma = F|_\Gamma$, the endomorphism F on $\mathbb{R}^2$ induces a map $\phi_F : \Gamma\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$. Clearly the maps $\Phi_{(c,F)}$ and ϕ_F induce homomorphisms φ and $\varphi|_\Gamma$, respectively. Since $\Pi\backslash\mathbb{R}^2$ and $\Gamma\backslash\mathbb{R}^2$ are $K(\pi, 1)$ manifolds, it follows that $\Phi_{(c,F)} \simeq f$ and $\phi_F \simeq \bar{f}$.

Since the Nielsen numbers are homotopy invariants, we may assume in what follows that $\Phi_{(c,F)} = f$ and $\phi_F = \bar{f}$ so that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "(c,F)"] \arrow[d, "p'"] & \mathbb{R}^2 \arrow[d, "p'"] \\
\Gamma\backslash\mathbb{R}^2 \arrow[r, "\phi_F"] \arrow[d, "\bar{p}"] & \Gamma\backslash\mathbb{R}^2 \arrow[d, "\bar{p}"] \\
\Pi\backslash\mathbb{R}^2 \arrow[r, "\Phi_{(c,F)}"] & \Pi\backslash\mathbb{R}^2
\end{tikzcd}$$

Now we recall the averaging formula for the Nielsen numbers on infra-nilmanifolds from [10, Theorem 3.5] and [12, Theorem 1.4]

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det A_*},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|.$$

Here f is any continuous map on an infra-nilmanifold with holonomy group Ψ. Thus for the case of the Klein bottle, we have for any $n \in \mathbb{N}$

$$\begin{aligned} L(f^n) = L(\Phi^n_{(c,F)}) &= \frac{1}{2}\Big(\frac{\det(I-F^n)}{\det I} + \frac{\det(A-F^n)}{\det A}\Big) \\ &= 1 - r^n, \\ N(f^n) = N(\Phi^n_{(c,F)}) &= \frac{1}{2}(|\det(I-F^n)| + |\det(A-F^n)|) \\ &= \begin{cases} |q^n(1-r^n)| & \text{if } r \text{ is odd and } q \neq 0; \\ |1-r^n| & \text{if } q = 0. \end{cases} \end{aligned}$$

Therefore we have proved our theorem. □

3. Preliminaries on the Nielsen type numbers

For each $n = 1, 2, \ldots$, $\tilde{f}^n$ is a lifting of f^n, and the homomorphism determined by the lifting $\tilde{f}^n$ is $\varphi^n : \Pi \to \Pi$. The homomorphism φ^n defines the *Reidemeister action* of Π on Π as follows:

$$\Pi \times \Pi \longrightarrow \Pi, \quad (\gamma, \alpha) \mapsto \gamma\alpha\varphi^n(\gamma)^{-1}.$$

The Reidemeister class containing α will be denoted by $[\alpha]^n$ and the set of Reidemeister classes of Π determined by φ^n will be denoted by $\mathcal{R}[\varphi^n]$. Write $R[\varphi^n] = |\mathcal{R}[\varphi^n]|$.

For each $\alpha \in \Pi$, $\alpha\tilde{f}$ is a lifting of f and $\alpha\tilde{f}^n$ is a lifting of f^n. They induce homomorphisms $\tau_\alpha\varphi$ and $\tau_\alpha\varphi^n$, where τ_α denotes the conjugation by α, i.e.,

$$\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}.$$

It is known that the periodic point classes of f^n are the subsets $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ ($\alpha \in \Pi$) of the periodic point set $\mathrm{Fix}(f^n)$ of f. Each periodic point class $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ is determined by the Reidemeister class $[\alpha]^n \in \mathcal{R}[\varphi^n]$. The periodic point set $\mathrm{Fix}(f^n)$ splits into a disjoint union of periodic point classes (cf. [10]). That is,

$$\mathrm{Fix}(f^n) = \coprod_{[\alpha]^n \in \mathcal{R}[\varphi^n]} p(\mathrm{Fix}(\alpha\tilde{f}^n)).$$

For $m \mid n$, $\mathrm{Fix}(f^m) \subset \mathrm{Fix}(f^n)$. Let $x \in \mathrm{Fix}(f^m)$ and $\tilde{x} \in p^{-1}(x)$. Then there exist unique $\alpha, \beta \in \Pi$ such that $\alpha\tilde{f}^m(\tilde{x}) = \tilde{x}$ and $\beta\tilde{f}^n(\tilde{x}) = \tilde{x}$. It can be easily derived that

$$\beta = \alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha).$$

This defines a function $\iota_{m,n} : \mathcal{R}[\varphi^m] \to \mathcal{R}[\varphi^n]$ defined by

$$\iota_{m,n}([\alpha]^m) = [\alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha)]^n.$$

Moreover, the following diagram is commuting:

$$\begin{tikzcd} p(\mathrm{Fix}(\alpha\tilde{f}^m)) \arrow[r, mapsto, "\gamma_{m,n}"] \arrow[d, leftrightarrow] & p(\mathrm{Fix}(\alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha)\tilde{f}^n)) \arrow[d, leftrightarrow] \\ {[\alpha]^m} \arrow[r, mapsto, "\iota_{m,n}"] & {[\alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha)]^n} \end{tikzcd}$$

Clearly, as a set $p(\mathrm{Fix}(\alpha\tilde{f}^m)) \subset p(\mathrm{Fix}(\alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha)\tilde{f}^n))$, i.e., if $p(\mathrm{Fix}(\alpha\tilde{f}^m))$ is the periodic point class of f^m determined by x, then

$$p(\mathrm{Fix}(\alpha\varphi^m(\alpha)\varphi^{2m}(\alpha)\cdots\varphi^{n-m}(\alpha)\tilde{f}^n))$$

is the periodic point class of f^n determined by x.

On the other hand, for $x \in p(\mathrm{Fix}(\alpha\tilde{f}^n))$ we choose $\tilde{x} \in p^{-1}(x)$ so that $\alpha\tilde{f}^n(\tilde{x}) = \tilde{x}$. Then

$$\varphi(\alpha)\tilde{f}^n\tilde{f}(\tilde{x}) = \varphi(\alpha)\tilde{f}\tilde{f}^n(\tilde{x}) = \tilde{f}\alpha\tilde{f}^n(\tilde{x}) = \tilde{f}(\tilde{x})$$

and so $f(x) \in p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n))$. Namely, $p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n))$ is the periodic point class determined by $f(x)$. Therefore f induces a function on the periodic point classes of f^n, which we denote by $[f]$, defined as follows:

$$[f] : p(\mathrm{Fix}(\alpha\tilde{f}^n)) \mapsto p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n)).$$

By [9, Theorem III.1.12], $[f]$ is an index-preserving bijection on the periodic point classes of f^n. We say that $[\alpha]^n$ is *essential* if the corresponding class $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ is essential. Evidently,

$$\begin{tikzcd} \mathrm{Fix}(\alpha\tilde{f}^n) \arrow[r, "\tilde{f}"] \arrow[rr, bend right, "\text{identity}"'] & \mathrm{Fix}(\varphi(\alpha)\tilde{f}^n) \arrow[r, "\alpha\tilde{f}^{n-1}"] & \mathrm{Fix}(\alpha\tilde{f}^n). \end{tikzcd}$$

This implies that for each $\alpha \in \Pi$, the restrictions of f

$$f| : p(\mathrm{Fix}(\alpha\tilde{f}^n)) \longrightarrow p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n))$$

are homeomorphisms such that $[f]^n$ is the identity. In particular,

$$p(\mathrm{Fix}(\alpha\tilde{f}^n)) = \emptyset \quad \Longleftrightarrow \quad p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n)) = \emptyset.$$

Obviously, φ induces a well-defined function on the Reidemeister classes of φ^n, which we will denote by $[\varphi]$, given by $[\varphi] : [\alpha]^n \mapsto [\varphi(\alpha)]^n$. Then the following diagram is commuting:

$$\begin{tikzcd} p(\mathrm{Fix}(\alpha\tilde{f}^n)) \arrow[r, mapsto, "{[f]}"] \arrow[d, leftrightarrow] & p(\mathrm{Fix}(\varphi(\alpha)\tilde{f}^n)) \arrow[d, leftrightarrow] \\ {[\alpha]^n} \arrow[r, mapsto, "{[\varphi]}"] & {[\varphi(\alpha)]^n} \end{tikzcd}$$

Note that since $[f]^n$ is the identity, $[\varphi]^n$ is the identity. Moreover, $\iota_{m,n} \circ [\varphi] = [\varphi] \circ \iota_{m,n}$.

Definition 3.1. The *length* of the element $[\alpha]^n \in \mathcal{R}[\varphi^n]$, denoted by $p([\alpha]^n)$, is the smallest positive integer p such that $[\varphi]^p([\alpha]^n) = [\alpha]^n$. The *$\varphi$-orbit* of $[\alpha]^n$ is the set

$$\langle[\alpha]^n\rangle = \{[\alpha]^n, [\varphi]([\alpha]^n), \ldots, [\varphi]^{p-1}([\alpha]^n)\},$$

where $p = p([\alpha]^n)$. The element $[\alpha]^n \in \mathcal{R}[\varphi^n]$ is *reducible* to m if there exists $[\beta]^m \in \mathcal{R}[\varphi^m]$ such that $\iota_{m,n}([\beta]^m) = [\alpha]^n$. Note that if $[\alpha]^n$ is reducible to m, then $m \mid n$. If $[\alpha]^n$ is not reducible to any $m < n$, we say that $[\alpha]^n$ is *irreducible*. The *depth* of $[\alpha]^n$, denoted by $d([\alpha]^n)$, is the smallest integer m to which $[\alpha]^n$ is reducible. Since clearly $d([\alpha]^n) = d([\varphi]([\alpha]^n))$, we can define the *depth* of the orbit $\langle[\alpha]^n\rangle$: $d(\langle[\alpha]^n\rangle) = d([\alpha]^n)$. If $n = d([\alpha]^n)$, the element $[\alpha]^n$ or the orbit $\langle[\alpha]^n\rangle$ is called *irreducible*.

Let $O_n(\varphi)$ be the number of irreducible, essential periodic point orbits of $\mathcal{R}[\varphi^n]$. If $[\alpha]^n$ is irreducible and essential, then so is $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ and its f-orbit contains at least n periodic points of least period n.

Definition 3.2. The *Nielsen type number of period n* is defined by the formula

$$NP_n(f) = n \times O_n(\varphi).$$

Take the set of all the essential orbits, of any period $m \mid n$, which do not contain any essential orbits of lower period. To each such an orbit, find the lowest period which it can be reduced to. The *Nielsen type number for the n-th iterate*, denoted by $N\Phi_n(f)$, is the sum of these numbers.

Then the Nielsen type numbers $NP_n(f)$ and $N\Phi_n(f)$ are homotopy invariant, non-negative integers [9, Theorem III.4.10].

Definition 3.3. The map $f : M \to M$ is called *weakly Jiang* if either $N(f) = 0$ or $N(f) = R[\varphi]$.

Now let us recall some of the main results in [5], [6], [7].

Theorem 3.4 ([5, Theorem 1], [6, Theorems 1.2]). *Let $f : M \to M$ be a self-map of a nilmanifold or $\mathcal{NR}$ solvmanifold, or if M is an arbitrary solvmanifold suppose that f^n is weakly Jiang. If $N(f^n) \neq 0$, then for all $m \mid n$*

$$N(f^m) = \sum_{k|m} NP_k(f), \quad NP_m(f) = \sum_{k|m} \mu(k)N(f^{\frac{m}{k}}),$$

where μ is the Möbius function.

Theorem 3.5 ([6, Corollary 4.6]). *Let $f : M \to M$ be a self-map. If M is a solvmanifold, then*

$$N\Phi_n(f) = \sum_{k|n} NP_k(f), \quad NP_n(f) = \sum_{k|n} \mu(k)N\Phi_{\frac{n}{k}}(f).$$

The following is observed in [5], [6], [7] using the structure of fibrations on solvmanifolds.

Corollary 3.6. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). Then for odd r,*

(1) *if $q \neq \pm 1$, then for all n, the Nielsen type numbers of periodic points of f are*

$$N\Phi_n(f) = N(f^n) = |q^n(1-r^n)|,$$
$$NP_n(f) = \sum_{k|n} \mu(k)|q^{\frac{n}{k}}(1-r^{\frac{n}{k}})|;$$

(2) *if $q = 1$ and $n = 2^k$, then*

$$NP_n(f) = N(f^n) = |1 - r^{2^k}|,$$
$$N\Phi_n(f) = \sum_{m|n} N(f^m) = \sum_{i=0}^{k} |1 - r^{2^i}|.$$

4. Weakly Jiang maps on the Klein bottle

Suppose that $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ induces a homomorphism φ on Π of type (r, ℓ, q). Then f^n induces a homomorphism φ^n of type

$$\begin{cases} (r^n, \ell + q\ell + \cdots + q^{n-1}\ell, q^n) & \text{if } r \text{ is odd;} \\ (r^n, r^{n-1}\ell, 0) & \text{if } r \text{ is even and } q = 0. \end{cases}$$

By Lemma 2.2, we may assume that if r is odd, then $\ell = 0, 1$ and $q \geq 0$; if r is even, then $\ell \geq 0$ and $q = 0$.

Now we will discuss the case where f^n is weakly Jiang.

Theorem 4.1. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). Then f^n is weakly Jiang if and only if one of the following holds*

(1) *Case: $N(f^n) = 0$*
- *r odd and $q = 0$*
- *$r = 1$*
- *n even and $r = -1$.*

(2) *Case: $N(f^n) = R[\varphi^n]$*
- *r even and $q = 0$*
- *n odd, $r = -1$ and $q \neq 0, 1$*
- *$r \neq \pm 1$ odd and $q \neq 0, 1$.*

Proof. Note that $N(f^n) = 0$ if and only if either $q = 0$ or $r = \pm 1$ when n is even and $r = 1$ when n is odd. Thus we need to know when f^n is weakly Jiang in the case where $N(f^n) \neq 0$, i.e., the case where r is even and $q = 0$, the case where $q \neq 0$ and $r \neq \pm 1$, and the case where $q \neq 0$ and $r \neq 1$ if n is odd.

Case A: $q = 0$ and r even.
In this case $N(f^n) = |1 - r^n| \neq 0$, and φ^n is of type $(r^n, r^{n-1}\ell, 0)$. Noting that

$$\varphi^n(\alpha^a t_2^b) = \alpha^{ar^n} t_2^{ar^{n-1}\ell},$$
$$\varphi^n(\alpha^a t_2^b)^{-1} = \alpha^{-ar^n} t_2^{-ar^{n-1}\ell},$$

we have, for each $i, j \in \mathbb{Z}$,

$$[\alpha^i t_2^j]^n = \left\{\alpha^{(1-r^n)k+i} t_2^{-r^{n-1}\ell k+j\pm h} \mid h, k \in \mathbb{Z}\right\}.$$

It follows that $\mathcal{R}[\varphi^n] = \left\{[\alpha]^n, [\alpha^2]^n, \ldots, [\alpha^{|1-r^n|}]^n\right\}$. Thus $N(f^n) = |1 - r^n| = R[\varphi^n]$. Hence in this case f^n is always weakly Jiang.

Case B: n odd, $q \neq 0$ and $r = -1$.
In this case $N(f^n) = 2q^n \neq 0$, and φ^n is of type $(-1, M, q^n)$, where $M = \ell + q\ell + \cdots + q^{n-1}\ell$. Noting that

$$\varphi^n(\alpha^a t_2^b) = \begin{cases} \alpha^{-a} t_2^{bq^n+M}, & \text{if } a \text{ is odd;} \\ \alpha^{-a} t_2^{bq^n}, & \text{if } a \text{ is even,} \end{cases}$$
$$\varphi^n(\alpha^a t_2^b)^{-1} = \begin{cases} \alpha^{a} t_2^{bq^n+M}, & \text{if } a \text{ is odd;} \\ \alpha^{a} t_2^{-bq^n}, & \text{if } a \text{ is even,} \end{cases}$$

we have, for each $i, j \in \mathbb{Z}$,

$$[\alpha^{2i+1} t_2^j]^n = \left\{\alpha^{4k+2i+1} t_2^{(q^n+1)h+j},\ \alpha^{4k+2i+3} t_2^{(q^n+1)h+M-j} \mid h, k \in \mathbb{Z}\right\},$$
$$[\alpha^{2i} t_2^j]^n = \left\{\alpha^{4k+2i} t_2^{(q^n-1)h+j},\ \alpha^{4k+2i+2} t_2^{(q^n-1)h+M-j} \mid h, k \in \mathbb{Z}\right\}.$$

It follows that

(1) if $q \neq 1$, then

$$\mathcal{R}[\varphi^n] = \left\{[\alpha]^n, [\alpha t_2]^n, \ldots, [\alpha t_2^{q^n}]^n, [\alpha^2]^n, [\alpha^2 t_2]^n, \ldots, [\alpha^2 t_2^{q^n-2}]^n\right\};$$

(2) if $q = 1$, then

$$\mathcal{R}[\varphi^n] = \left\{[\alpha]^n, [\alpha t_2]^n\right\} \bigcup \left\{[\alpha^2 t_2^k]^n \mid k \in \mathbb{Z}\right\}.$$

Note here that $[\alpha]^n = [\alpha^{4k\pm1}]^n$ for even ℓ, and $[\alpha]^n = [\alpha^{4k+1}]^n = [\alpha^{4k-1} t_2]^n$ for odd ℓ.

In particular, if $q \neq 1$, then $N(f^n) = 2q^n = (q^n + 1) + (q^n - 1) = R[\varphi^n]$. Hence in the case when n is odd, $q \neq 0$ and $r = -1$, we see that f^n is weakly Jiang if and only if $q \neq 1$.

Case C: $q \neq 0$ and $r \neq \pm 1$ odd.
Let $M = \ell + q\ell + \cdots + q^{n-1}\ell$. Then φ^n is of type (r^n, M, q^n). Noting that

$$\varphi^n(\alpha^a t_2^b) = \begin{cases} \alpha^{ar^n} t_2^{bq^n+M}, & \text{if } a \text{ is odd;} \\ \alpha^{ar^n} t_2^{bq^n}, & \text{if } a \text{ is even,} \end{cases}$$

$$\varphi^n(\alpha^a t_2^b)^{-1} = \begin{cases} \alpha^{-ar^n} t_2^{bq^n+M}, & \text{if } a \text{ is odd;} \\ \alpha^{-ar^n} t_2^{-bq^n}, & \text{if } a \text{ is even,} \end{cases}$$

we have, for each $i, j \in \mathbb{Z}$,

$$\begin{aligned} [\alpha^{2i+1} t_2^j]^n = \Big\{ &\alpha^{2(1-r^n)k+2i+1} t_2^{(q^n+1)h+j}, \\ &\alpha^{2(1-r^n)k+(r^n+2i)} t_2^{(q^n+1)h+M-j} \mid k, h \in \mathbb{Z} \Big\}, \\ [\alpha^{2i} t_2^j]^n = \Big\{ &\alpha^{2(1-r^n)k+2i} t_2^{(q^n-1)h+j}, \\ &\alpha^{2(1-r^n)k+(r^n+2i-1)} t_2^{(q^n-1)h+M-j} \mid k, h \in \mathbb{Z} \Big\}. \end{aligned}$$

It follows that

(1) if $q \neq 1$, then

$$\begin{aligned} \mathcal{R}[\varphi^n] = \Big\{ [\alpha^i t_2^j]^n \mid 1 \leq i \leq |1 - r^n|, & \\ 1 \leq j \leq q^n + 1 \text{ if } i \text{ is odd;} & \\ 1 \leq j \leq q^n - 1 \text{ if } i \text{ is even} & \Big\}; \end{aligned}$$

(2) if $q = 1$, then

$$\begin{aligned} \mathcal{R}[\varphi^n] = &\Big\{ [\alpha^i]^n, [\alpha^i t_2]^n \mid 1 \leq i \leq |1 - r^n|, i \text{ odd} \Big\} \\ &\bigcup \Big\{ [\alpha^i t_2^m]^n \mid 1 \leq i \leq |1 - r^n|, i \text{ even}; m \in \mathbb{Z} \Big\}. \end{aligned}$$

In this case $M = n\ell$. Note further that if $n\ell$ is even, then for all $s \in \mathbb{Z}$

$$[\alpha^i]^n = [\alpha^{s|1-r^n|+i}]^n, \quad [\alpha^i t_2]^n = [\alpha^{s|1-r^n|+i} t_2]^n;$$

if $n\ell$ is odd, then $[\alpha^{|1-r^n|+i}]^n = [\alpha^i t_2]^n$.

In particular, if $q \neq 1$, then $N(f^n) = |q^n(1-r^n)| = \frac{|1-r^n|}{2}((q^n+1)+(q^n-1)) = R[\varphi^n]$. Hence in the case when $q \neq 0$ and $r \neq \pm 1$ and is odd, we see that f^n is weakly Jiang if and only if $q \neq 1$. □

Corollary 4.2. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type $(r, \ell, 0)$ with r even. Then for all n,*

$$NP_n(f) = \sum_{k|n} \mu(k)|1 - r^{\frac{n}{k}}|, \quad N\Phi_n(f) = |1 - r^n|.$$

Proof. When r is even and $q = 0$, by Theorem 4.1, f^n is weakly Jiang and $N(f^n) = |1 - r^n| \neq 0$. Now the assertion follows directly from Theorems 3.4 and 3.5. $\square$

Therefore, we are left to consider the case where r is odd. Moreover, the case where $q \neq 1$ was treated completely in Corollary 3.6. Hence we may assume in what follows that r is odd and $q = 1$.

5. The Nielsen type numbers: non-weakly Jiang case I

Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). The case where $N(f^n) = R[\varphi^n]$ is equivalent, by Theorem 4.1, to one of the following cases:

- r even and $q = 0$; this is solved in Corollary 4.2.
- n odd, $r = -1$ and $q \neq 0, 1$; this is solved in Corollary 3.6.
- $r \neq \pm 1$ odd and $q \neq 0, 1$; this is solved in Corollary 3.6.

If $N(f^n) = 0$, then by Theorem 4.1, we have either

- r odd and $q = 0$,
- $r = 1$, or
- n even and $r = -1$.

Remark 5.1. In the first two cases, $N(f^k) = 0$ and thus there are no essential Reidemeister classes determined by φ^k for all k. It follows that the Nielsen type numbers of periodic points of f are $NP_n(f) = 0 = N\Phi_n(f)$.

Remark 5.2. Consider the last case, that is, the case where n is even and $r = -1$ (and $q = 1$). By definition, $NP_n(f) = 0$ and $NP_k(f) = 0$ for all even k. By Theorem 3.5 and Theorem 5.3 below,

$$N\Phi_n(f) = \sum_{k|n} NP_k(f) = \sum_{k \text{ odd},\ k|n} NP_k(f) = NP_1(f) = 2.$$

Now we are left to evaluate the Nielsen type number of periodic points of f in the case where $0 \neq N(f^n) \neq R[\varphi^n]$, i.e., f^n is non-weakly Jiang. Explicitly, we should consider the following cases in this section and the next section:

- n is odd, $r = -1$ and $q = 1$
- $r \neq \pm 1$ is odd and $q = 1$.

Theorem 5.3. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). If n is odd, $r = -1$ and $q = 1$, then the Nielsen type numbers of periodic points of f are*

$$N(f^n) = 2;\ NP_1(f) = 2,\ NP_n(f) = 0\,(n > 1);\ N\Phi_n(f) = 2.$$

Proof. Suppose first that $q = 1$. This is Case B in the proof of Theorem 4.1. Thus

$$\mathcal{R}[\varphi^n] = \Big\{[\alpha]^n,\ [\alpha t_2]^n\Big\} \bigcup \Big\{[\alpha^2 t_2^k]^n \mid k \in \mathbb{Z}\Big\}.$$

Recall here that $[\alpha]^n = [\alpha^{4k\pm1}]^n$ for even ℓ, and $[\alpha]^n = [\alpha^{4k+1}]^n = [\alpha^{4k-1}t_2]^n$ for odd ℓ. Now if $m \mid n$, then since n is odd, both m and $\frac{n}{m}$ are odd. For any k, φ^k is of type $((-1)^k, k\ell, 1)$. Observing that

$$\iota_{m,n}([\alpha]^m) = [\alpha t_2^{\frac{n-m}{2}\ell}]^n, \quad \iota_{m,n}([\alpha t_2]^m) = [\alpha t_2^{1+\frac{n-m}{2}\ell}]^n,$$
$$\iota_{m,n}([\alpha^2 t_2^k]^m) = [\alpha^2 t_2^{\frac{n}{m}k}]^n,$$

we see that $\{\langle[\alpha^2 t_2^k]^n\rangle \mid k \in \mathbb{Z} - \bigcup_{m|n} \frac{n}{m}\mathbb{Z}\}$ are the only irreducible periodic orbits of $\mathcal{R}[\varphi^n]$ for $n > 1$, and obviously everything is irreducible for $n = 1$. The Reidemeister class $[\alpha]^n \in \mathcal{R}[\varphi^n]$ corresponds to the fixed point class $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ of f^n. Recalling from the proof of Theorem 2.3 that

$$\tilde{f} = (c, F) = \left(-\frac{1}{2}\begin{bmatrix}*\\ \ell\end{bmatrix}, \begin{bmatrix}r & 0\\ 0 & q\end{bmatrix}\right) = \left(-\frac{1}{2}\begin{bmatrix}*\\ \ell\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\right),$$

since $\alpha\tilde{f}^n = (a, A)(c, F)^n = \left(\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}*\\ -n\ell\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right)$, we can see easily that $\mathrm{Fix}(\alpha\tilde{f}^n)$ and hence $p(\mathrm{Fix}(\alpha\tilde{f}^n))$ consist of single element. This tells that the corresponding Reidemeister class $[\alpha]^n$ is essential. Similarly, we can show that the Reidemeister class $[\alpha t_2]^n$ is also essential. Since $N(f^n) = 2$, it follows that all other classes are inessential.

Next we find the length of each φ-orbit of the essential Reidemeister classes $[\alpha]^n$ and $[\alpha t_2]^n$. Recalling here that $[\alpha]^n = [\alpha^{4k\pm1}]^n$ for even ℓ, and $[\alpha]^n = [\alpha^{4k+1}]^n = [\alpha^{4k-1}t_2]^n$ for odd ℓ, we see that

$$\varphi([\alpha]^n) = [\alpha^{-1}t_2^{\ell}]^n = [\alpha]^n,$$

and hence we have $\langle[\alpha]^n\rangle = \{[\alpha]^n\}$ and $\langle[\alpha t_2]^n\rangle = \{[\alpha t_2]^n\}$. In all, we obtain

$$O_1(\varphi) = 2, \ O_n(\varphi) = 0\,(n > 1).$$

Therefore, $NP_1(f) = 2$ and $NP_n(f) = n \times O_n(\varphi) = 0$ for all odd $n > 1$, and by Theorem 3.5, $N\Phi_n(f) = 2$. □

Corollary 5.4. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). If $r = -1$ and $q = 1$, then for all n, the Nielsen type numbers of periodic points of f are*

$$N\Phi_n(f) = 2.$$

Proof. Follows from Remark 5.2 and Theorem 5.3. □

6. The Nielsen type numbers: non-weakly Jiang case II

In this section, we will consider the remaining cases when $r \neq \pm1$ is odd and $q = 1$. Our computation problem of the Nielsen type numbers can be done by the following general four steps.

Step 1: Find the Reidemeister classes $\mathcal{R}[\varphi^n]$.

Step 2: Find the essential Reidemeister classes.

Step 3: Find the irreducible essential Reidemeister classes.

Step 4: Find the length of the irreducible essential Reidemeister classes. In fact, we will show that all the irreducible essential Reidemeister classes have the same length n.

Step 1: Find the Reidemeister classes $\mathcal{R}[\varphi^n]$.

This is Case C in the proof of Theorem 4.1. Thus

$$\mathcal{R}[\varphi^n] = \Big\{[\alpha^i]^n, [\alpha^i t_2]^n \mid 1 \le i \le |1-r^n|,\ i \text{ odd}\Big\}$$
$$\bigcup \Big\{[\alpha^i t_2^k]^n \mid 1 \le i \le |1-r^n|,\ i \text{ even};\ k \in \mathbb{Z}\Big\}.$$

Note further that if $n\ell$ is even, then for all $s \in \mathbb{Z}$

$$[\alpha^i]^n = [\alpha^{s|1-r^n|+i}]^n, \quad [\alpha^i t_2]^n = [\alpha^{s|1-r^n|+i} t_2]^n;$$

if $n\ell$ is odd, then $[\alpha^{|1-r^n|+i}]^n = [\alpha^i t_2]^n$.

Step 2: Find the essential Reidemeister classes.

Recalling that

$$\tilde{f} = (c, F) = \left(-\frac{1}{2}\begin{bmatrix} * \\ \ell \end{bmatrix}, \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix}\right) = \left(-\frac{1}{2}\begin{bmatrix} * \\ \ell \end{bmatrix}, \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}\right),$$

we can see easily that $\mathrm{Fix}(\alpha^i \tilde{f}^n)$ (i odd) and hence $p(\mathrm{Fix}(\alpha^i \tilde{f}^n))$ consist of single element. This tells that the corresponding Reidemeister classes $[\alpha^i]^n$ are essential. Similarly, we can show that the Reidemeister classes $[\alpha^i t_2]^n$ are also essential. Since $N(f^n) = |1-r^n|$, it follows that

$$\Big\{[\alpha^i]^n,\ [\alpha^i t_2]^n \mid 1 \le i \le |1-r^n|,\ i \text{ odd}\Big\}$$

are the essential Reidemeister classes. Hence if $n\ell$ is even, then $O_n \coprod O_n$, where $O_n = \{i \mid 1 \le i \le |1-r^n|,\ i \text{ odd}\}$, corresponds to the essential classes, and if $n\ell$ is odd, then $R_n = \{i \mid 1 \le i \le 2|1-r^n|,\ i \text{ odd}\}$ corresponds to the essential classes.

Step 3: Find the irreducible essential Reidemeister classes.

Then for odd i, we have

$$\varphi^n(\alpha^i) = \alpha^{ir^n} t_2^{n\ell}, \quad \varphi^n(\alpha^i t_2) = \alpha^{ir^n} t_2^{n\ell+1},$$

$$\iota_{m,n}(\alpha^i) = \begin{cases} \alpha^{i\frac{1-r^n}{1-r^m}} t_2^{\frac{n-m}{2}\ell}, & \frac{n}{m} \text{ is odd;} \\ \alpha^{i\frac{1-r^n}{1-r^m}} t_2^{\frac{n}{2}\ell}, & \frac{n}{m} \text{ is even,} \end{cases}$$

$$\iota_{m,n}(\alpha^i t_2) = \begin{cases} \alpha^{i\frac{1-r^n}{1-r^m}} t_2^{\frac{n-m}{2}\ell+1}, & \frac{n}{m} \text{ is odd;} \\ \alpha^{i\frac{1-r^n}{1-r^m}} t_2^{\frac{n}{2}\ell}, & \frac{n}{m} \text{ is even.} \end{cases}$$

Therefore:

Case: n is of the form $4k$.

Then $\frac{n}{2}\ell$ is even, and if $\frac{n}{m}$ is odd, then m must be even and hence $\frac{n-m}{2}\ell$ is even. Thus,

$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = \begin{cases} [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, & \frac{n}{m} \text{ is even;} \\ [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n, & \frac{n}{m} \text{ is odd.} \end{cases}$$

Case: <u>n is of the form $4k+1$.</u>
Both m and $\frac{n}{m}$ are either of the form $4a+1$ or of the form $4a+3$. If $\frac{n}{m}$ is of the form $4a+1$, then

$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n.$$

If $\frac{n}{m}$ is of the form $4a+3$, then:

- if ℓ is even, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n;$$
- if ℓ is odd, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n.$$

Case: <u>n is of the form $4k+2$.</u>
Either m is of the form $2a+1$ and $\frac{n}{m}$ is of the form $4b+2$, or m is of the form $4a+2$ and $\frac{n}{m}$ is of the form $2b+1$. If m is of the form $4a+2$ and $\frac{n}{m}$ is of the form $2b+1$, then

$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n.$$

If m is of the form $2a+1$ and $\frac{n}{m}$ is of the form $4b+2$, then:

- if ℓ is even, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n;$$
- if ℓ is odd, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n.$$

Case: <u>n is of the form $4k+3$.</u>
Either m is of the form $4a+3$ and $\frac{n}{m}$ is of the form $4b+1$, or m is of the form $4a+1$ and $\frac{n}{m}$ is of the form $4b+3$. If m is of the form $4a+3$ and $\frac{n}{m}$ is of the form $4b+1$, then

$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n.$$

If m is of the form $4a+1$ and $\frac{n}{m}$ is of the form $4b+3$, then:

- if ℓ is even, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n;$$
- if ℓ is odd, then
$$\iota_{m,n}([\alpha^i]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}} t_2]^n, \quad \iota_{m,n}([\alpha^i t_2]^m) = [\alpha^{i\frac{1-r^n}{1-r^m}}]^n.$$

Step 4: Find the length of the essential Reidemeister classes.
We can show that the essential φ-orbits are as follows:

- if ℓ is even, then
$$\langle[\alpha^i]^n\rangle = \{[\alpha^i]^n, [\alpha^{ir}]^n, \dots, [\alpha^{ir^{n-1}}]^n\},$$
$$\langle[\alpha^i t_2]^n\rangle = \{[\alpha^i t_2]^n, [\alpha^{ir} t_2]^n, \dots, [\alpha^{ir^{n-1}} t_2]^n\};$$
- if ℓ is odd, then
$$\langle[\alpha^i]^n\rangle = \{[\alpha^i]^n, [\alpha^{ir} t_2]^n, \dots, [\alpha^{ir^{n-1}} t_2^{\epsilon}]^n\},$$
$$\langle[\alpha^i t_2]^n\rangle = \{[\alpha^i t_2]^n, [\alpha^{ir}]^n, \dots, [\alpha^{ir^{n-1}} t_2^{\varepsilon}]^n\}.$$

Based on the above four steps, we give an explication of how to derive the formulas for the Nielsen type numbers.

Recall that
$$R_m = \{i \mid 1 \le i \le 2|1 - r^m|,\ i \text{ odd}\},$$
$$O_m = \{i \mid 1 \le i \le |1 - r^m|,\ i \text{ odd}\}.$$
Suppose first that $n\ell$ is odd. Then n and ℓ are odd, and if $m \mid n$, then m and $\frac{n}{m}$ are odd. By Steps 1 and 2, R_n corresponds to the essential Reidemeister classes. By Step 3, we can see that if $\frac{n}{m} \equiv 1 \pmod 4$, then $\iota_{m,n}$ sends each i to $i\frac{1-r^n}{1-r^m}$ and each $i + |1 - r^n|$ to $i\frac{1-r^n}{1-r^m} + |1 - r^n|$, and if $\frac{n}{m} \equiv 3 \pmod 4$, then $\iota_{m,n}$ sends each i to $i\frac{1-r^n}{1-r^m} + |1 - r^n|$ and each $i + |1 - r^n|$ to $i\frac{1-r^n}{1-r^m}$ where $1 \le i \le |1 - r^m|$ is odd. Thus
$$U_n := R_n - \bigcup_{m|n,\ m<n} \left\{\frac{1-r^n}{1-r^m} O_m \bigcup \frac{1-r^n}{1-r^m} O_m + |1 - r^n|\right\}$$
corresponds to the irreducible essential Reidemeister classes. According to the equalities $[\alpha^{(1-r^n)+i}]^n = [\alpha^i t_2]^n$ of Step 1, for each $i \in U_n$ we define a subset $\langle i\rangle$ of U_n for which
$$\langle i\rangle \equiv \{i, ir^2, ir^4, \dots, ir^{n-1}, ir + |1 - r^n|, ir^3 + |1 - r^n|, \dots, ir^{n-2} + |1 - r^n|\} \pmod{2(1 - r^n)}.$$
For each odd integer i with $1 \le i \le 2|1 - r^n|$, the smallest positive integer $k(i)$ for which $[\alpha^i]^n = [\varphi^{k(i)}(\alpha^i)]^n$ is the length of the orbit $\langle[\alpha^i]^n\rangle$. Equivalently, we consider the smallest even positive integer k_1 such that $i(1 - r^{k_1}) \equiv 0 \pmod{2(1 - r^n)}$, and consider the smallest odd positive integer k_2 such that $i(1 - r^{k_2}) \equiv 1 - r^n \pmod{2(1 - r^n)}$. But since the even integer k_1 cannot divide the odd integer n, there is no solution k_1. Thus $k(i) = k_2$ is the length of $\langle i\rangle$, and $|\langle i\rangle| = k(i)$. Since $\langle i\rangle$ corresponds to the φ-orbit $\langle[\alpha^i]^n\rangle$, the number of subsets $\langle i\rangle$ of U_n is the number of irreducible essential φ-orbits, $O_n(\varphi)$.

For each $i \in U_n$, we claim that $k(i) = n$. For simplicity write k instead of $k(i)$. Consider $i(1-r^k) \equiv 1-r^n \pmod{2(1-r^n)}$, or $i(1-r^k) = (2a+1)(1-r^n)$

for some integer a. Then $2a+1 \leq i$. If $2a+1 < i$, then $k < n$ and $1 \leq 2a+1 \leq 2|1-r^k|$. This implies that $i = \frac{1-r^n}{1-r^k}(2a+1) \notin U_n$. This is a contradiction, which yields that $2a+1 = i$, and hence $k = n$. Therefore all the orbits $\langle i \rangle$ have the same length n. This implies that $O_n(\varphi) = |U_n|/n$.

In all, the Nielsen type numbers are

$$\begin{aligned} NP_n(f) &= n \times O_n(\varphi) = n \times \frac{|U_n|}{n} = |U_n|, \\ N\Phi_n(f) &= \sum_{m|n} NP_m(f) = \sum_{m|n} |U_m|. \end{aligned}$$

Finally, note that since $R_m = O_m \bigcup (O_m + |1-r^m|)$, we have $\frac{1-r^n}{1-r^m} R_m = \frac{1-r^n}{1-r^m} O_m \bigcup \frac{1-r^n}{1-r^m} (O_m + |1-r^n|)$, and hence

$$U_n = R_n - \bigcup_{m|n,\ m<n} \frac{1-r^n}{1-r^m} R_m.$$

Suppose next that $n\ell$ is even. By Steps 1 and 2, $O_n \coprod O_n$ corresponds to the essential Reidemeister classes. Note that one of the following cases holds:

(i) ℓ is even and n is odd.
(ii) both ℓ and n are even.
(iii) ℓ is odd and n is even.

Consider the first case, that is, assume that ℓ is even and n is odd. Since n is odd, both m and $\frac{n}{m}$ are odd. By Step 3, we can see that if

$$V_n := O_n - \bigcup_{m|n,\ m<n} \frac{1-r^n}{1-r^m} O_m,$$

then $V_n \coprod V_n$ corresponds to the irreducible essential Reidemeister classes. Next, for each $i \in V_n$ we define a subset $\langle i \rangle$ of V_n for which

$$\langle i \rangle \equiv \{i, ir, \ldots, ir^{n-1}\} \pmod{1-r^n}.$$

For each $i \in V_n$, consider the smallest positive integer $k = k(i)$ such that $i(1-r^k) \equiv 0 \pmod{1-r^n}$, which is the length of $\langle i \rangle$, and $|\langle i \rangle| = k(i)$. Since $\langle i \rangle$ corresponds to the φ-orbits $\langle [\alpha^i]^n \rangle$ and $\langle [\alpha^i t_2]^n \rangle$, twice the number of subsets $\langle i \rangle$ of V_n is the number of irreducible essential φ-orbits, $O_n(\varphi)$. Similarly as before, we can show that all the orbits $\langle i \rangle$ have the same length $k(i) = n$. Therefore, the Nielsen type numbers are

$$\begin{aligned} NP_n(f) &= n \times O_n(\varphi) = n \times 2\frac{|V_n|}{n} = 2|V_n|, \\ N\Phi_n(f) &= \sum_{m|n} NP_m(f) = 2\sum_{m|n} |V_m|. \end{aligned}$$

Consider the second case, that is, assume that both ℓ and n are even. By Step 3, we can see that if

$$V_n := O_n - \bigcup_{m|n,\ m<n} \frac{1-r^n}{1-r^m} O_m, \quad W_n := O_n - \bigcup_{\substack{m|n,\ m<n \\ \frac{n}{m}\ \text{odd}}} \frac{1-r^n}{1-r^m} O_m,$$

then $V_n \coprod W_n$ corresponds to the irreducible essential Reidemeister classes. If $\frac{n}{m}$ is even, $\frac{1-r^n}{1-r^m}$ is also even. Since O_n contains only odd integers, it follows that $V_n = W_n$. Therefore, $V_n \coprod V_n$ corresponds to the irreducible essential Reidemeister classes. Next, for each $i \in V_n$ we define a subset $\langle i \rangle$ of V_n for which

$$\langle i \rangle \equiv \{i, ir, \ldots, ir^{n-1}\} \pmod{1 - r^n}.$$

For each $i \in V_n$, consider the smallest positive integer $k = k(i)$ such that $i(1-r^k) \equiv 0 \pmod{1-r^n}$, which is the length of $\langle i \rangle$, and $|\langle i \rangle| = k(i)$. Since $\langle i \rangle$ corresponds to the φ-orbit $\langle [\alpha^i]^n \rangle$ or the φ-orbit $\langle [\alpha^j t_2]^n \rangle$, twice the number of subsets $\langle i \rangle$ of V_n is the number of irreducible essential φ-orbits, $O_n(\varphi)$. On the other hand, similarly as before, we can show that all the orbits $\langle i \rangle$ have the same length $k(i) = n$.

Consider the last case, that is, assume that ℓ is odd and n is even. By Step 3, we can see that the irreducible essential Reidemeister classes corresponds to the set $V_n \coprod W_n$ or $W_n \coprod V_n$ according as $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. Since, as above, $W_n = V_n$, $V_n \coprod V_n$ is the set of all irreducible essential Reidemeister classes. Similarly as before, we can show that all the orbits $\langle i \rangle$ have the same length $= n$.

Therefore, in the second and the third cases, i.e., in the case where n is even, the Nielsen type numbers are

$$\begin{aligned} NP_n(f) &= n \times O_n(\varphi) = n \times 2\frac{|V_n|}{n} = 2|V_n|, \\ N\Phi_n(f) &= \sum_{m|n} NP_m(f) = \sum_{\substack{m|n, \\ m\ \text{even}}} NP_m(f) + \sum_{\substack{m|n, \\ m\ \text{odd}}} NP_m(f) \\ &= \sum_{\substack{m|n, \\ m\ \text{even}}} 2|V_m| + \sum_{\substack{m|n, \\ m\ \text{odd}}} 2|V_m| = 2\sum_{m|n} |V_m|. \end{aligned}$$

Now observe easily that $|U_m| = 2|V_m|$ for all m, we will express $|V_n|$ in terms of n, r, ℓ and $q = 1$.

Notation. Let $n = 2^e p_1^{e_1} \cdots p_t^{e_t}$ be the prime decomposition of a positive integer n (so that the p_j's are distinct odd primes). Write for each $j = 1, 2, \ldots, t$,

$$n_j = \frac{n}{p_j}.$$

Lemma 6.1. *We have*

$$|V_n| = \frac{1}{2}\left(|1-r^n| + (-1)^s \sum |1 - r^{(n_{k_1},\dots,n_{k_s})}|\right),$$

where $\{k_1, \dots, k_s\} \subset \{1, 2, \dots, t\}$.

Proof. Assume that n is even. First we observe that if m is a divisor of n with n/m is even, then $\frac{1-r^n}{1-r^m} O_m$ consists of only certain even integers. By the definition of the set V_n, we only need to consider the divisors m of n which are of the form $2^e n'$.

Next we observe that if $k \mid \ell \mid n$ where both k and ℓ are of the form $2^e n'$, then

$$\frac{1-r^n}{1-r^k} O_k \subset \frac{1-r^n}{1-r^\ell} O_\ell.$$

For, if i is odd with $1 \le i \le |1-r^k|$, then

$$\frac{1-r^n}{1-r^k} \times i = \frac{1-r^n}{1-r^\ell} \times \frac{1-r^\ell}{1-r^k} \times i$$

and $\frac{1-r^\ell}{1-r^k} \times i$ is odd between 1 and $|1-r^\ell|$.

Finally we observe that if $n_1, n_2, \dots, n_t$ are the distinct maximal proper divisors of n both of which are of the form $2^e n'$, then

$$\bigcap_{i=1}^{s} \frac{1-r^n}{1-r^{n_{k_i}}} O_{n_{k_i}} = \frac{1-r^n}{1-r^{(n_{k_1},\dots,n_{k_s})}} O_{(n_{k_1},\dots,n_{k_s})},$$

where $(n_{k_1}, \dots, n_{k_s})$ is the least common multiple of $n_{k_1}, \dots, n_{k_s}$.

The above observations yield that the size of the set V_n is

$$|V_n| = |O_n| + (-1)^s \sum \left|\frac{1-r^n}{1-r^{(n_{k_1},\dots,n_{k_s})}} O_{(n_{k_1},\dots,n_{k_s})}\right|,$$

where $\{k_1, \dots, k_s\} \subset \{1, 2, \dots, t\}$. This proves the assertion for the case where n is even. The assertion for the case where n is odd can be proved in a similar way. $\square$

Immediately we have

Theorem 6.2. *Let* $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ *be any continuous map on the Klein bottle* $\Pi\backslash\mathbb{R}^2$ *of type* (r, ℓ, q) *where* $r \neq \pm 1$ *is odd and* $q = 1$. *Then for any positive integer* n,

$$NP_n(f) = |1-r^n| + (-1)^s \sum |1 - r^{(n_{k_1},\dots,n_{k_s})}|,$$

$$N\Phi_n(f) = \sum_{m|n} \left(|1-r^m| + (-1)^s \sum |1 - r^{(m_{k_1},\dots,m_{k_s})}|\right).$$

Corollary 6.3. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q) where $r \neq \pm 1$ is odd and $q = 1$. Then for any odd integer n,*

$$NP_n(f) = \sum_{k|n} \mu(k)|1 - r^{\frac{n}{k}}|, \quad N\Phi_n(f) = |1 - r^n| = N(f^n).$$

As another application, we consider the very special case where all the non-trivial divisors of n are even, namely, $n = 2^k$. This case was already considered in Corollary 3.6.(2) (cf. [5], [6], [7]). Note that our proof is much simpler and very direct.

Corollary 6.4. *Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). If $r \neq \pm 1$ is odd, $q = 1$ and $n = 2^k$, then*

$$NP_n(f) = N(f^n) = |1 - r^{2^k}|,$$
$$N\Phi_n(f) = \sum_{m|n} N(f^m) = \sum_{i=0}^{k} |1 - r^{2^i}|.$$

7. Summary

Let $f : \Pi\backslash\mathbb{R}^2 \to \Pi\backslash\mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi\backslash\mathbb{R}^2$ of type (r, ℓ, q). The Lefschetz number and the Nielsen number of the n-th iterate f^n of f are given by

$$L(f^n) = 1 - r^n, \quad N(f^n) = \begin{cases} |q^n(1 - r^n)| & \text{if } r \text{ is odd;} \\ |1 - r^n| & \text{if } r \text{ is even and } q = 0. \end{cases}$$

and the Nielsen type number $N\Phi_n(f)$ is given by the formula $N\Phi_n(f) = \sum_{k|n} NP_k(f)$ and vice versa, $NP_n(f) = \sum_{k|n} \mu(k) N\Phi_{\frac{n}{k}}(f)$.

In all, we can tabulate a complete description of the formulas for the Nielsen type numbers $NP_n(f)$ of all maps f on the Klein bottle as follows:

Theorem 7.1. *For each positive integer n and a triple (r, ℓ, q) of integers with $\ell, q \geq 0$, we have*

r		ℓ	q	$NP_n(f)$ or $N\Phi_n(f)$	Proof
even			$\neq 0$	Not a right choice for f	Lemma 2.1
			0	$N\Phi_n(f) = \|1 - r^n\|$	Corollary 4.2
odd			$\neq 0, 1$	$N\Phi_n(f) = \|q^n(1 - r^n)\|$	Corollary 3.6
			0	$N\Phi_n(f) = 0 = NP_n(f)$	Remark 5.1
	1		1	$N\Phi_n(f) = 0 = NP_n(f)$	Remark 5.2
	-1			$N\Phi_n(f) = 2$	Corollary 5.4
	$\neq \pm 1$			$NP_n(f) = \|1 - r^n\| + (-1)^s \sum \|1 - r^{(n_{k_1}, \ldots, n_{k_s})}\|$	Theorem 6.2

Remark 7.2. Suppose that a self-map f on the Klein bottle is of type (r, ℓ, q). All the formulas of $L(f^n)$, $N(f^n)$, $NP_n(f)$ and $N\Phi_n(f)$ are independent of the variable ℓ. This is due to the fact that the "linearization F of f" (see the proof of Theorem 2.3) loses the information on ℓ completely.

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