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Enumeration of Algebraic Tangles with Applications to Theta-curves and Handcuff Graphs

Dedicated to Professor Akio Kawauchi for his 60th birthday.

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Abstract. We enumerate all algebraic tangles of seven crossings or less up to equivalence. These tangles are mutually distinguished by the corresponding links and their double. The result will be used for enumerating θ-curves and handcuff graphs in a forthcoming paper.

1. Introduction

In [4] J. H. Conway introduced the concept of a *tangle* in order to enumerate knots and links. A tangle is a disjoint union of two arcs and some or no loops properly embedded in a 3-ball B^3. Two tangles T and S are *isotopic* if there is an isotopy of the 3-ball B^3 that takes one tangle to the other while fixing each point of the boundary, and *freely equivalent* if there is a homeomorphism of B^3 which takes T to S without restriction that the endpoints stay fixed. In [14] Y. Nakanishi listed a table of algebraic tangles of five crossings or less up to isotopy by using Conway's method. In [17] H. Yamano gave a table of prime 2-string tangles of seven crossings or less up to free equivalence by using Conway's method. In [7] T. Kanenobu, H. Saito and S. Satoh classified 2-string tangles of seven crossings or less up to free equivalence by using disk-graphs. In this paper, we classify algebraic tangles of seven crossings or less up to *equivalence*, which is weaker than isotopy, but stronger than free equivalence (Definition 2.1). In a forthcoming paper ([11] and [12]), we give an enumeration of prime θ-curves and handcuff graphs with up to seven crossings by using the result of this paper and θ*-polyhedra*. A θ-polyhedron is a connected graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. We can obtain a θ-curve or handcuff graph diagram from a θ-polyhedron by substituting algebraic tangles for their 4-valent vertices. This paper is organized as follows: In Section 2, we give some definitions. In Section 3, we list a table of algebraic tangles. In Section 4, we give some applications to θ-curves and handcuff graphs.

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2. Algebraic tangles

We review Conway's method [4]. We define a *tangle* as a pair (B^3, t), where t is a 1-manifold properly embedded in a unit 3-ball $B^3 = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2 \leq 1\}$ with four boundary components

$$\mathrm{NE} = \left(1/\sqrt{2}, 1/\sqrt{2}, 0\right), \quad \mathrm{SE} = \left(1/\sqrt{2}, -1/\sqrt{2}, 0\right),$$
$$\mathrm{SW} = \left(-1/\sqrt{2}, -1/\sqrt{2}, 0\right), \quad \mathrm{NW} = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right);$$

see Figure 1.

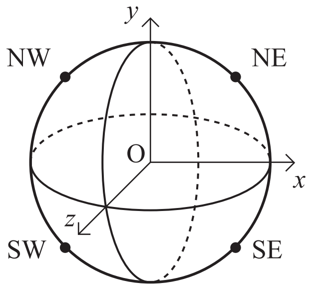


Figure 1: A 3-ball and its 4 boundary components.

Let $T = (B^3, t)$ be a tangle such that t consists of two arcs and n circles. We say T is a V^n*-tangle* (resp. an H^n*-tangle*, an X^n*-tangle*) if T has an arc connecting NE and SE (resp. NW, SW).

We present a tangle by a regular diagram as in Figure 2(a), where we use the projection $(x,y,z) \mapsto (x,y)$. Let R be a tangle. We denote by μR, νR, $\rho_x R$, $\rho_y R$, $\rho_z R$ the tangles obtained from R by reflecting with regard to the xy-plane; $\mu(x,y,z) = (x,y,-z)$, by turning it counter-clockwise by $\pi/2$; $\nu(x,y,z) = (-y,x,z)$, by rotating it through angle π; $\rho_x(x,y,z) = (x,-y,-z)$, $\rho_y(x,y,z) = (-x,y,-z)$, and $\rho_z(x,y,z) = (-x,-y,z)$, respectively. We present these tangles diagrammatically as shown in Figure 2. We call μR the *mirror image* of R.

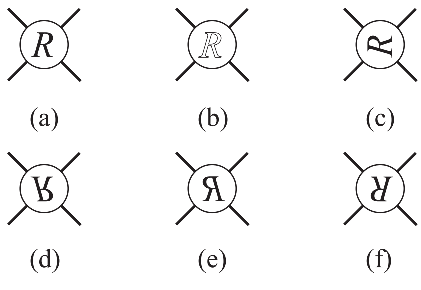


Figure 2: (a) A tangle R. (b) The tangle μR. (c) The tangle νR. (d) The tangle $\rho_x R$. (e) The tangle $\rho_y R$. (f) The tangle $\rho_z R$.

We say that two tangles are *isotopic* if there is an isotopy of the 3-ball B^3 that takes one tangle to the other while fixing each point of the boundary, that is, their diagrams are related by a finite sequence of *Reidemeister moves* as shown in Figure 3 inside the circle defining the tangle while the endpoints of the strings remain fixed.

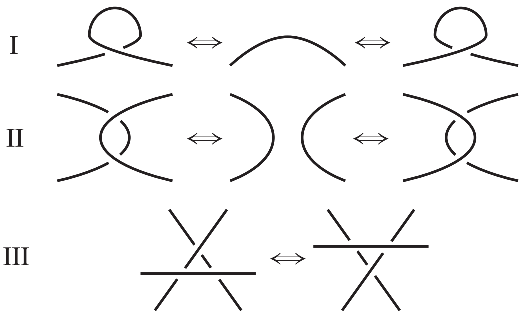


Figure 3: Reidemeister moves.

Definition 2.1. We say that two tangles T and T' are *equivalent* if T is isotopic to one of the following eight tangles:

$$T',\ \rho_x T', \rho_y T', \rho_z T', \nu T', \nu\rho_x T', \nu\rho_y T', \nu\rho_z T'.$$

For a tangle diagram D, we denote by $c(D)$ the number of crossings of D. The *crossing number* of a tangle T, denoted by $c(T)$, is the minimal number of $c(D)$'s for all the diagrams D which present the equivalence class of T.

Given two tangles T and S, we define new tangles $T+S$, TS, $T+$ and $T-$ as shown in Figure 4; $T+S$ and TS are the *sum* and *product* of T and S respectively. Notice that $TS = \rho_x\mu\nu(T) + S$, where $\rho_x\mu\nu(T)$ is the tangle obtained from T by reflecting across the NW and SE diagonal line.

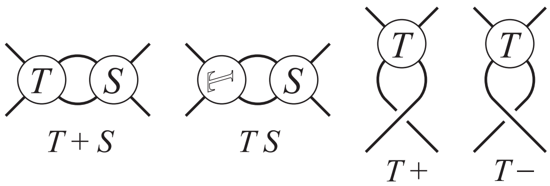


Figure 4: The operations.

The simplest tangles are the 0 and ∞ tangles as shown in Figures 5 (a) and (b). Further, for a positive integer n we define the n tangle and the $-n$ tangle as shown in Figures 5 (c) and (d), which are called *integral tangles*.

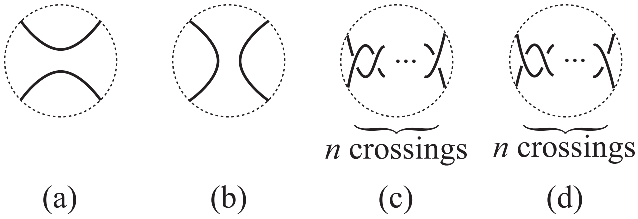


Figure 5: (a) The 0 tangle. (b) The ∞ tangle. (c) The n tangle. (d) The $-n$ tangle.

A tangle T is said to be *algebraic* if T is obtained from the 0 and ∞ tangles by a finite sequence of the operations given in Figure 4. Thus, an algebraic tangle is obtained from the 0, ∞, and integral tangles by the operations of addition and multiplication. We denote the n tangle simply by n, and the $-n$ tangle by $\bar{n}$. For integral tangles a_1, a_2, $a_3, \cdots, a_{i-1}$, a_i, the tangle $a_1 a_2 a_3 \cdots a_{i-1} a_i$, abbreviating $((\cdots(a_1\ a_2)a_3 \cdots a_{i-1})a_i)$, is called a *rational tangle*. Two rational tangles $a_1 a_2 \cdots a_{i-1} a_i$ and $b_1 b_2 \cdots b_{j-1} b_j$ are isotopic if and only if the corresponding rational numbers (including $1/0 = \infty$)

$$a_i + \cfrac{1}{a_{i-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}} \quad \text{and} \quad b_j + \cfrac{1}{b_{j-1} + \cfrac{1}{\ddots + \cfrac{1}{b_2 + \cfrac{1}{b_1}}}}$$

are the same.

Remark 2.2. For the above continued fraction, we can assume that each a_m $(1 \leq m \leq i)$ has the same sign.

The comma notation $(a_1, a_2, \cdots, a_i) = (a_1\, 0) + (a_2\, 0) + \cdots + (a_i\, 0)$ is preferred to the sum notation, but is only used with two or more terms in the bracket. Figure 6 shows the step-by-step formation of two algebraic tangles 2 1 1 1 and 2 1, 2 as examples.

A tangle $T = (B^3, t)$ is said to be *splittable* if there exists a disk Δ such that Δ does not meet t, but splits two arcs of t in B^3.

Let T be a tangle. We define the *numerator*, $N(T)$, and *denominator*, $D(T)$, the links as shown in Figure 7. We call the set of links $\{N(T), D(T)\}$ the *corresponding links* for T. Clearly, we have

Proposition 2.3. *Suppose that T and S are equivalent tangles. Then their corresponding links $\{N(T), D(T)\}$ and $\{N(S), D(S)\}$ present the same set of isotopic links.*

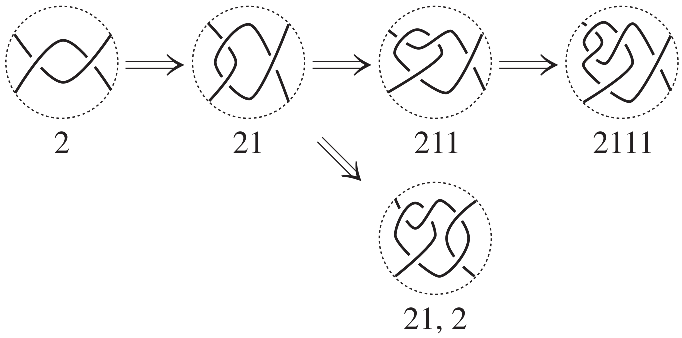


Figure 6: Algebraic tangles.

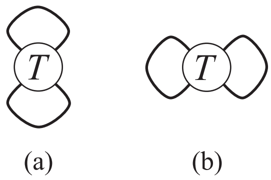


Figure 7: (a) The numerator $N(T)$. (b) The denominator $D(T)$.

Let R be a tangle. We define the *double*, $W(R)$ by the link as shown in Figure 8;

$$W(R) = N(R + \rho_y \mu R).$$

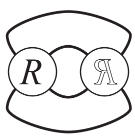


Figure 8: The double $W(R)$.

Clearly, we have

Proposition 2.4. *Suppose that T and S are equivalent tangles. Then, their doubles $W(T)$ and $W(S)$ are isotopic.*

3. Table of algebraic tangles

We list a table of unsplittable algebraic tangles with seven crossings or less up to equivalence in Definition 2.1. However, we list either T or μT for each tangle T, even if they are not equivalent.

Theorem 3.1. *Table* 1 *exhibit diagrams of unsplittable algebraic tangles with up to seven crossings.*

Links in the second column correspond to Rolfsen's knot table [15]. Specifically, 0 is the trivial knot, $L_1 \# L_2$ is a connected sum of links L_1 and L_2, and $\overline{L}$ is the mirror image of L. The last column gives the type of the algebraic tangle, where $X = X^0$, $H = H^0$, $V = V^0$.

Table 1: Algebraic tangles with up to 7 crossings.

T	$\big(N(T), D(T)\big)$	type
1	$(0,\ 0)$	X
2	$(2^2_1,\ 0)$	H
3	$(3_1,\ 0)$	X
2 1	$(\overline{3_1},\ 2^2_1)$	V
4	$(4^2_1,\ 0)$	H
3 1	$(\overline{4^2_1},\ \overline{3_1})$	H
2 2	$(4_1,\ 2^2_1)$	V
2 1 1	$(4_1,\ 3_1)$	X
$2, 2$	$(\overline{4^2_1},\ 2^2_1 \# 2^2_1)$	V^1
$2, \bar{2}$	$(0^2_1,\ 2^2_1 \# 2^2_1)$	V^1
5	$(5_1,\ 0)$	X
4 1	$(\overline{5_1},\ \overline{4^2_1})$	V
3 2	$(5_2,\ \overline{3_1})$	X
3 1 1	$(\overline{5_2},\ 4^2_1)$	V
2 3	$(\overline{5_2},\ 2^2_1)$	V
2 2 1	$(5_2, 4_1)$	X
2 1 2	$(5^2_1,\ \overline{3_1})$	H
2 1 1 1	$(\overline{5^2_1},\ 4_1)$	H
$2, 2+$	$(\overline{5^2_1},\ 2^2_1 \# 2^2_1)$	V^1
$(2, 2)1$	$(5^2_1,\ 4^2_1)$	X^1
$(2, 2)\bar{1}$	$(0^2_1,\ 4^2_1)$	X^1
$3, 2$	$(\overline{5_1},\ \overline{3_1} \# 2^2_1)$	V
$3, \bar{2}$	$(0,\ \overline{3_1} \# 2^2_1)$	V
$2\ 1, 2$	$(5_2,\ 3_1 \# 2^2_1)$	V

Table 1: Algebraic tangles with up to 7 crossings (continued).

T	$\big(N(T), D(T)\big)$	type
6	$(6^2_1,\ 0)$	H
$5\ 1$	$(\overline{6^2_1},\ \overline{5_1})$	H
$4\ 2$	$(6_1,\ \overline{4^2_1})$	V
$4\ 1\ 1$	$(\overline{6_1},\ 5_1)$	X
$3\ 3$	$(6^2_2,\ \overline{3_1})$	H
$3\ 2\ 1$	$(\overline{6^2_2},\ \overline{5_2})$	H
$3\ 1\ 2$	$(6_2,\ 4^2_1)$	V
$3\ 1\ 1\ 1$	$(\overline{6_2},\ 5_2)$	X
$2\ 4$	$(\overline{6_1}, 2^2_1)$	V
$2\ 3\ 1$	$(6_1,\ 5_2)$	X
$2\ 2\ 2$	$(6^2_3, 4_1)$	H
$2\ 2\ 1\ 1$	$(\overline{6^2_3},\ \overline{5_2})$	H
$2\ 1\ 3$	$(\overline{6_2},\ 3_1)$	X
$2\ 1\ 2\ 1$	$(6_2,\ \overline{5^2_1})$	V
$2\ 1\ 1\ 2$	$(6_3,\ 4_1)$	X
$2\ 1\ 1\ 1\ 1$	$(6_3,\ 5^2_1)$	V
$2, 2 + +$	$(\overline{6^2_3},\ 2^2_1\#2^2_1)$	V^1
$(2,2)2$	$(\overline{6^3_1},\ 4^2_1)$	H^1
$(2,2)\bar{2}$	$(\overline{6^3_3},\ 4^2_1)$	H^1
$(2,\bar{2})2$	$(\overline{6^3_3},\ 0^2_1)$	H^1
$3, 2+$	$(6_2,\ \overline{3_1}\#2^2_1)$	V
$2\ 1, 2+$	$(6_3,\ 3_1\#2^2_1)$	V
$(3,2)1$	$(6_2,\ 5_1)$	X
$(3,2)\bar{1}$	$(0,\ 5_1)$	X
$(3,\bar{2})\bar{1}$	$(0,\ 5_2)$	X
$(2\ 1,2)1$	$(6_3,\ \overline{5_2})$	X
$(2,2+)1$	$(\overline{6^2_3},\ 5^2_1)$	X^1
$(2,2)1\ 1$	$(6^3_1,\ \overline{5^2_1})$	H^1
$4, 2$	$(\overline{6^2_1},\ \overline{4^2_1}\#2^2_1)$	V^1
$4, \bar{2}$	$(2^2_1,\ \overline{4^2_1}\#2^2_1)$	V^1
$3\ 1, 2$	$(6^2_2,\ 4^2_1\#2^2_1)$	V^1
$2\ 2, 2$	$(6_1,\ 4_1\#2^2_1)$	V
$2\ 2, \bar{2}$	$(0,\ 4_1\#2^2_1)$	V

Table 1: Algebraic tangles with up to 7 crossings (continued).

T	$\big(N(T), D(T)\big)$	type
$2\ 1\ 1, 2$	$(\overline{6_2},\ 4_1\#2_1^2)$	V
$(2,2), 2$	$(\overline{6_3^2},\ 4_1^2\#2_1^2)$	V^1
$(2,2), \bar{2}$	$(\overline{4_1^2},\ 4_1^2\#2_1^2)$	V^1
$(2,\bar{2}), 2$	$(5_1^2,\ 0_1^2\#2_1^2)$	V^1
$3, 3$	$(\overline{6_1^2},\ \overline{3_1}\#\overline{3_1})$	H
$3, \bar{3}$	$(0_1^2,\ 3_1\#\overline{3_1})$	H
$3, 2\ 1$	$(6_1,\ 3_1\#\overline{3_1})$	X
$3, \bar{2}\ \bar{1}$	$(3_1,\ \overline{3_1}\#\overline{3_1})$	X
$2\ 1, 2\ 1$	$(\overline{6_3^2},\ 3_1\#3_1)$	H
$2, 2, 2$	$(6_1^3,\ 2_1^2\#2_1^2\#2_1^2)$	V^2
$2, 2, \bar{2}$	$(6_3^3,\ 2_1^2\#2_1^2\#2_1^2)$	V^2
7	$(7_1,\ 0)$	X
$6\ 1$	$(\overline{7_1},\ \overline{6_1^2})$	V
$5\ 2$	$(7_2,\ \overline{5_1})$	X
$5\ 1\ 1$	$(\overline{7_2},\ 6_1^2)$	V
$4\ 3$	$(7_3,\ \overline{4_1^2})$	V
$4\ 2\ 1$	$(\overline{7_3},\ \overline{6_1})$	X
$4\ 1\ 2$	$(7_1^2,\ 5_1)$	H
$4\ 1\ 1\ 1$	$(\overline{7_1^2},\ 6_1)$	H
$3\ 4$	$(\overline{7_3},\ \overline{3_1})$	X
$3\ 3\ 1$	$(7_3,\ 6_2^2)$	V
$3\ 2\ 2$	$(7_5,\ \overline{5_2})$	X
$3\ 2\ 1\ 1$	$(\overline{7_5},\ 6_2^2)$	V
$3\ 1\ 3$	$(7_4,\ 4_1^2)$	V
$3\ 1\ 2\ 1$	$(\overline{7_4},\ \overline{6_2})$	X
$3\ 1\ 1\ 2$	$(7_2^2,\ 5_2)$	H
$3\ 1\ 1\ 1\ 1$	$(\overline{7_2^2},\ 6_2)$	H
$2\ 5$	$(\overline{7_2},\ 2_1^2)$	V
$2\ 4\ 1$	$(7_2,\ 6_1)$	X
$2\ 3\ 2$	$(7_3^2,\ 5_2)$	H
$2\ 3\ 1\ 1$	$(\overline{7_3^2},\ \overline{6_1})$	H
$2\ 2\ 3$	$(\overline{7_5},\ 4_1)$	X
$2\ 2\ 2\ 1$	$(7_5,\ \overline{6_3^2})$	V

Table 1: Algebraic tangles with up to 7 crossings (continued).

T	$\big(N(T), D(T)\big)$	type
2 2 1 2	$(7_6,\ \overline{5_2})$	X
2 2 1 1 1	$(\overline{7_6},\ \overline{6^2_3})$	V
2 1 4	$(\overline{7^2_1},\ 3_1)$	H
2 1 3 1	$(7^2_1,\ 6_2)$	H
2 1 2 2	$(\overline{7_6},\ \overline{5^2_1})$	V
2 1 2 1 1	$(7_6,\ \overline{6_2})$	X
2 1 1 3	$(\overline{7^2_2},\ 4_1)$	H
2 1 1 2 1	$(7^2_2,\ 6_3)$	H
2 1 1 1 2	$(7_7,\ 5^2_1)$	V
2 1 1 1 1 1	$(\overline{7_7},\ 6_3)$	X
$2, 2 + ++$	$(\overline{7^2_3},\ 2^2_1\#2^2_1)$	V^1
$(2, 2)3$	$(\overline{7^2_4},\ 4^2_1)$	X^1
$(2, 2)\bar{3}$	$(\overline{7^2_8},\ 4^2_1)$	X^1
$(2, \bar{2})3$	$(\overline{7^2_7},\ 0^2_1)$	X^1
$3, 2 + +$	$(\overline{7_5},\ \overline{3_1}\#2^2_1)$	V
$2\ 1, 2 + +$	$(7_6,\ 3_1\#2^2_1)$	V
$(3, 2)2$	$(\overline{7^2_4},\ 5_1)$	H
$(3, 2)\bar{2}$	$(\overline{7^2_7},\ 5_1)$	H
$(3, \bar{2})2$	$(\overline{7^2_7},\ 0)$	H
$(3, \bar{2})\bar{2}$	$(7^2_8,\ 0)$	H
$(2\ 1, 2)2$	$(\overline{7^2_5},\ \overline{5_2})$	H
$(2\ 1, 2)\bar{2}$	$(\overline{7^2_8},\ \overline{5_2})$	H
$(2, 2+)2$	$(\overline{7^3_1},\ \overline{5^2_1})$	H^1
$(2, 2)1\ 2$	$(7^2_5,\ \overline{5^2_1})$	X^1
$4, 2+$	$(\overline{7^2_1},\ \overline{4^2_1}\#2^2_1)$	V^1
$3\ 1, 2+$	$(\overline{7^2_2},\ 4^2_1\#2^2_1)$	V^1
$2\ 2, 2+$	$(\overline{7_6},\ 4_1\#2^2_1)$	V
$2\ 1\ 1, 2+$	$(\overline{7_7},\ 4_1\#2^2_1)$	V
$(2, 2), 2+$	$(\overline{7^2_5},\ 4^2_1\#2^2_1)$	V^1
$3, 3+$	$(\overline{7_4},\ \overline{3_1}\#\overline{3_1})$	X
$3, 2\ 1+$	$(7^2_2,\ 3_1\#\overline{3_1})$	H
$2\ 1, 2\ 1+$	$(7_7,\ 3_1\#3_1)$	X

Table 1: Algebraic tangles with up to 7 crossings (continued).

T	$\big(N(T), D(T)\big)$	type
$2, 2, 2+$	$(7^3_1,\ 2^2_1\#2^2_1\#2^2_1)$	V^2
$(2, 2++)1$	$(7^2_3,\ 6^2_3)$	X^1
$(2, 2)2\ 1$	$(7^2_4,\ 6^3_1)$	V^1
$(2, 2)\bar{2}\ \bar{1}$	$(7^2_8,\ 6^3_3)$	V^1
$(2, \bar{2})2\ 1$	$(7^2_7,\ 6^3_3)$	V^1
$(3, 2+)1$	$(7_5,\ 6_2)$	X
$(2\ 1, 2+)1$	$(\overline{7_6},\ 6_3)$	X
$(3, 2)1\ 1$	$(7^2_4,\ \overline{6_2})$	H
$(2\ 1, 2)1\ 1$	$(7^2_5,\ 6_3)$	H
$(2, 2+)1\ 1$	$(7^3_1,\ 6^2_3)$	H^1
$(2, 2)1\ 1\ 1$	$(\overline{7^2_5},\ \overline{6^3_1})$	V^1
$(4, 2)1$	$(7^2_1,\ 6^2_1)$	X^1
$(4, 2)\bar{1}$	$(2^2_1,\ 6^2_1)$	X^1
$(4, \bar{2})\bar{1}$	$(6^2_2,\ 2^2_1)$	X^1
$(3\ 1, 2)1$	$(7^2_2,\ 6^2_2)$	X^1
$(2\ 2, 2)1$	$(7_6,\ \overline{6_1})$	X
$(2\ 2, 2)\bar{1}$	$(0,\ \overline{6_1})$	X
$(2\ 2, \bar{2})\bar{1}$	$(\overline{6_2},\ 0)$	X
$(2\ 1\ 1, 2)1$	$(7_7,\ 6_2)$	X
$((2, 2), 2)1$	$(7^2_5,\ 6^2_3)$	X^1
$((2, 2), 2)\bar{1}$	$(4^2_1,\ 6^2_3)$	X^1
$((2, 2), \bar{2})\bar{1}$	$(7^2_7,\ 4^2_1)$	X^1
$((2, \bar{2}), 2)1$	$(7^2_8,\ \overline{5^2_1})$	X^1
$((2, \bar{2}), 2)\bar{1}$	$(5^2_1,\ \overline{5^2_1})$	X^1
$(3, 3)1$	$(7_4,\ 6^2_1)$	V
$(3, 3)\bar{1}$	$(\overline{3_1},\ 6^2_1)$	V
$(3, \bar{3})1$	$(\overline{6_1},\ 0^2_1)$	V
$(3, 2\ 1)1$	$(7^2_2,\ \overline{6_1})$	H
$(3, \bar{2}\ \bar{1})\bar{1}$	$(6^2_3,\ \overline{3_1})$	H
$(2\ 1, 2\ 1)1$	$(\overline{7_7},\ 6^2_3)$	V
$(2, 2, 2)1$	$(\overline{7^3_1},\ \overline{6^3_1})$	X^2
$(2, 2, 2)\bar{1}$	$(\overline{6^3_3},\ \overline{6^3_1})$	X^2
$(2, 2, \bar{2})\bar{1}$	$(6^3_3,\ \overline{6^3_3})$	X^2

Table 1: Algebraic tangles with up to 7 crossings (continued).

T	$\big(N(T), D(T)\big)$	type
$5, 2$	$(\overline{7_1},\ \overline{5_1}\#2^2_1)$	V
$5, \bar{2}$	$(\overline{3_1},\ \overline{5_1}\#2^2_1)$	V
$4\ 1, 2$	$(\overline{7_3},\ 5_1\#2^2_1)$	V
$3\ 2, 2$	$(7_3,\ \overline{5_2}\#2^2_1)$	V
$3\ 2, \bar{2}$	$(0,\ \overline{5_2}\#2^2_1)$	V
$3\ 1\ 1, 2$	$(\overline{7_4},\ 5_2\#2^2_1)$	V
$2\ 3, 2$	$(7_2,\ 5_2\#2^2_1)$	V
$2\ 3, \bar{2}$	$(3_1,\ 5_2\#2^2_1)$	V
$2\ 2\ 1, 2$	$(\overline{7_5},\ \overline{5_2}\#2^2_1)$	V
$2\ 1\ 2, 2$	$(\overline{7^2_1},\ \overline{5^2_1}\#2^2_1)$	V^1
$2\ 1\ 2, \bar{2}$	$(2^2_1,\ \overline{5^2_1}\#2^2_1)$	V^1
$2\ 1\ 1\ 1, 2$	$(7^2_2,\ 5^2_1\#2^2_1)$	V^1
$(3, 2), 2$	$(7_5,\ 5_1\#2^2_1)$	V
$(3, 2), \bar{2}$	$(5_2,\ 5_1\#2^2_1)$	V
$(3, \bar{2}), 2$	$(6_2,\ 2^2_1)$	V
$(3, \bar{2}), \bar{2}$	$(6_3,\ 2^2_1)$	V
$(2\ 1, 2), 2$	$(\overline{7_6},\ \overline{5_2}\#2^2_1)$	V
$(2\ 1, 2), \bar{2}$	$(\overline{5_1},\ \overline{5_2}\#2^2_1)$	V
$(2, 2+), 2$	$(7^2_3,\ 5^2_1\#2^2_1)$	V^1
$(2, 2+), \bar{2}$	$(0^2_1,\ 5^2_1\#2^2_1)$	V^1
$((2, 2)1), 2$	$(7^2_4,\ \overline{5^2_1}\#2^2_1)$	V^1
$((2, 2)\bar{1}), \bar{2}$	$(7^2_8,\ 0^2_1\#2^2_1)$	V^1
$((2, \bar{2})1), 2$	$(7^2_7,\ \overline{4^2_1}\#2^2_1)$	V^1
$4, 3$	$(\overline{7_1},\ \overline{4^2_1}\#3_1)$	V
$4, \bar{3}$	$(0,\ \overline{4^2_1}\#3_1)$	V
$3\ 1, 3$	$(7_3,\ 4^2_1\#\overline{3_1})$	V
$3\ 1, \bar{3}$	$(4_1,\ \overline{4^2_1}\#3_1)$	V
$2\ 2, 3$	$(7_2,\ 4_1\#\overline{3_1})$	X
$2\ 2, \bar{3}$	$(0,\ 4_1\#3_1)$	X
$2\ 1\ 1, 3$	$(\overline{7^2_1},\ 4_1\#\overline{3_1})$	H
$2\ 1\ 1, \bar{3}$	$(4^2_1,\ 4_1\#3_1)$	H

Table 1: Algebraic tangles with up to 7 crossings (continued).

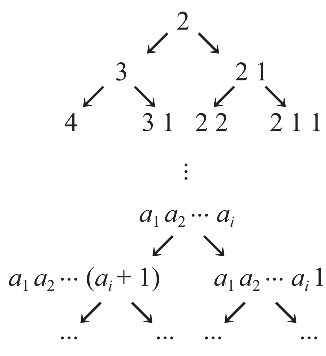
T	$\big(N(T), D(T)\big)$	type
$(2,2),3$	$(7^2_3,\ 4^2_1\#\overline{3_1})$	X^1
$(2,2),\bar{3}$	$(\overline{5^2_1},\ 4^2_1\#3_1)$	X^1
$(2,\bar{2}),3$	$(6^2_3,\ 0^2_1\#\overline{3_1})$	X^1
$4,2\ 1$	$(7_2,\ \overline{4^2_1}\#3_1)$	V
$3\ 1,2\ 1$	$(7_5,\ 4^2_1\#3_1)$	V
$2\ 2,2\ 1$	$(\overline{7^2_3},\ 4_1\#3_1)$	H
$2\ 1\ 1,2\ 1$	$(\overline{7_6},\ 4_1\#3_1)$	X
$(2,2),2\ 1$	$(\overline{7^3_1},\ 4^2_1\#3_1)$	H^1
$(2,2),\bar{2}\ \bar{1}$	$(6^3_3,\ 4^2_1\#\overline{3_1})$	H^1
$(2,\bar{2}),2\ 1$	$(\overline{6^3_1},\ 0^2_1\#3_1)$	H^1
$3,2,2$	$(7^2_4,\ \overline{3_1}\#2^2_1\#2^2_1)$	V^1
$3,2,\bar{2}$	$(7^2_7,\ \overline{3_1}\#2^2_1\#2^2_1)$	V^1
$3,\bar{2},\bar{2}$	$(\overline{7^2_8},\ \overline{3_1}\#2^2_1\#2^2_1)$	V^1
$2\ 1,2,2$	$(7^2_5,\ 3_1\#2^2_1\#2^2_1)$	V^1

Sketch of proof of Theorem 3.1. First, we enumerate the rational tangles with up to seven crossings. Second, we give algebraic (but not rational) tangle diagrams with up to seven crossings by using operations in Figure 4. Third, we classify the tangles up to equivalence.

In order to enumerate all the rational tangles of n crossings, by Definition 2.1 and Remark 2.2, we produce the sequences of positive integers $a_1a_2\cdots a_i$ satisfying

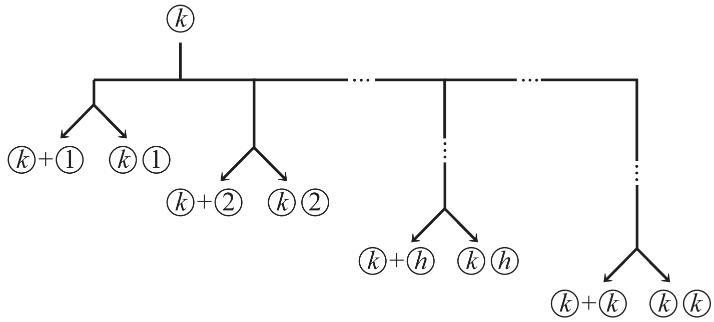
$$a_1 + a_2 + \cdots + a_i = n.$$

Specifically, we construct a binary tree as follows.



The sequence corresponds to the rational tangle $a_1a_2\cdots a_i$ (cf. [6]).

In order to enumerate all the algebraic (but not rational) tangle diagrams of n crossings, we construct the following tree.



Here, ⓚ denotes an algebraic tangle with k crossings, ⓚ+ⓗ and ⓚⓗ are new algebraic tangles with $(k+h)$ crossings (see Figure 9). By Definition 2.1, we may assume $1 \le h \le k$. If ⓚ is the rational tangle $a_1a_2\cdots a_i$ $(a_m > 0)$, ⓚ+① corresponds to $a_1a_2\cdots(a_i+1)$ and ⓚ① corresponds to $a_1a_2\cdots a_i\ 1$.

Example. We give the case where $k = h = 2$ in Figure 9. Note that the algebraic tangles with two crossings are 2, $\bar{2}$, 2 0, $\bar{2}$ 0.

Then, for each tangle T, we investigate the corresponding links $\{N(T), D(T)\}$ and compare them. Except for the tangles 5 and $(3,2)\bar{1}$, we show these tangles are mutually distinct by the corresponding links. In fact, $\{N(5), D(5)\} = \{N((3,2)\bar{1}), D((3,2)\bar{1})\} = \{0, 5_1\}$. However, their doubles $W(5)$ and $W((3,2)\bar{1})$ are not isotopic. So, the tangles 5 and $(3,2)\bar{1}$ are not equivalent. □

4. Applications to spatial graphs

A *spatial graph* is a graph in S^3. Specifically, a *θ-curve* Θ is a spatial graph which consists of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), such that each edge joins the vertices. A *handcuff graph* Φ is also a spatial graph which consists of two vertices (v_1, v_2) and three edges (e_1, e_2, e_3), where e_3 has distinct endpoints v_1 and v_2, and e_1 and e_2 are loops based at v_1 and v_2, respectively. A *constituent knot* is a subgraph of Θ that consists of two vertices (v_1, v_2) and two edges (e_i, e_j). The set of constituent knots is an invariant of θ-curves. A *constituent link* is a subgraph of Φ that consists of two vertices (v_1, v_2) and two edges (e_i, e_j). The constituent link is an invariant of handcuff graphs.

From Theorem 3.1, we can enumerate special θ-curves and handcuff graphs. For a tangle T, we define a spatial graph diagram $G(T)$ as shown in Figure 10 (cf. [10]). If T is a V^0-tangle or an X^0-tangle, then $G(T)$ is a θ-curve. And if T is an H^0-tangle, then $G(T)$ is a handcuff graph.

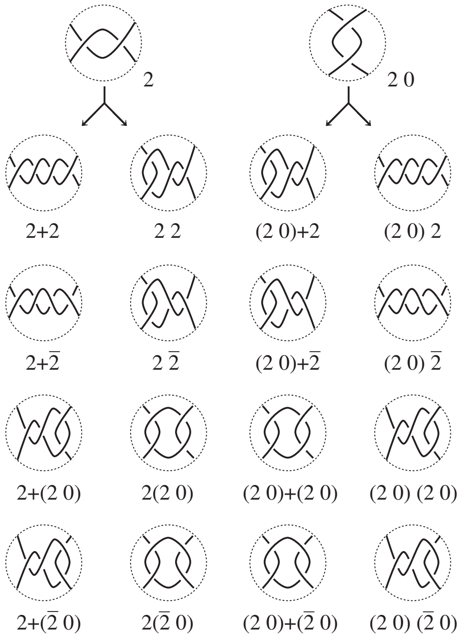


Figure 9: The case $k = h = 2$.

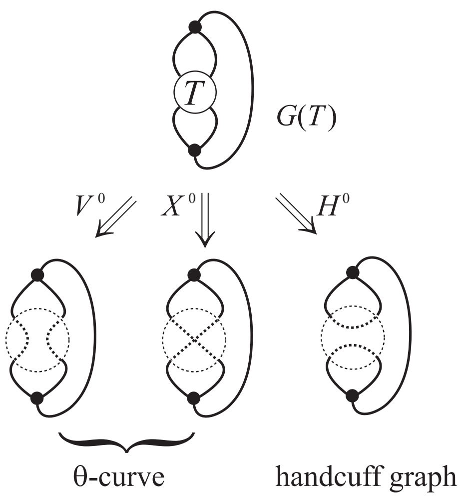


Figure 10: $G(T)$ produces a θ-curve or a handcuff graph.

We give an enumeration of special θ-curves with up to seven crossings as in Table 2. Knots in the second column correspond to Rolfsen's knot table [15], and

θ-curves in the last column correspond to Litherland's table [9]. The θ-curves are ordered so that their constituent knots are in lexicographic order. A knot $\overline{K}$ and a θ-curve $\overline{\Theta}$ denote the mirror images of K and Θ, respectively.

Example. For the tangle $3, \bar{2}$, we obtain the θ-curve by $G(T)$. Its constituent knots are $\overline{3_1}$ and two trivial knot. By deformation as in Figure 11, we conclude the θ-curve is 3_1.

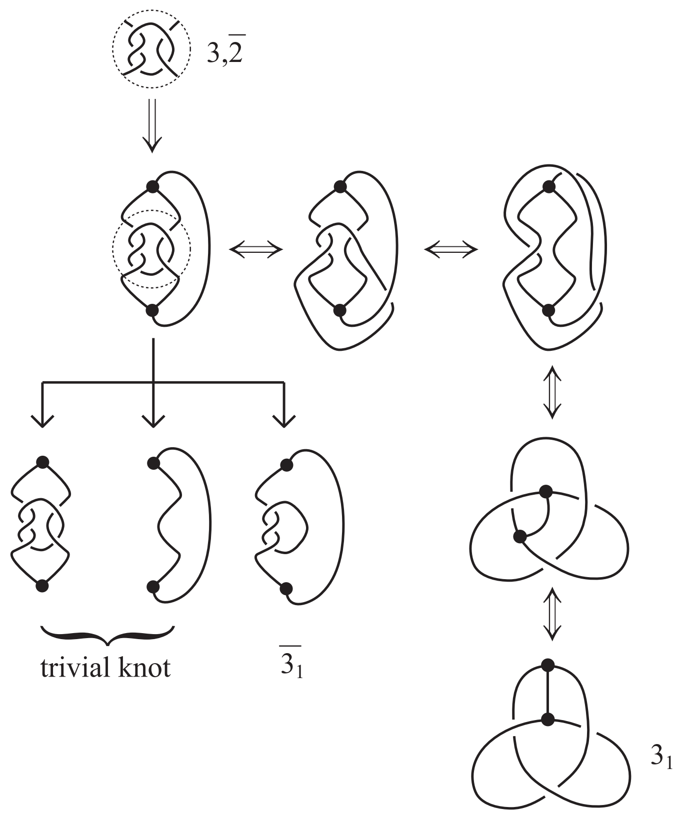


Figure 11: The tangle $3, \bar{2}$ produces the θ-curve 3_1.

Table 2: θ-curves with up to 7 crossings.

Tangle	constituent knot	Θ
1	0, 0, 0	trivial
3	3_1, 0, 0	$\overline{3_1}$
2 1	$\overline{3_1}$, 0, 0	3_1
2 2	4_1, 0, 0	4_1
2 1 1	4_1, 0, 0	$\overline{4_1}$
5	5_1, 0, 0	5_3
4 1	$\overline{5_1}$, 0, 0	$\overline{5_3}$
3 2	5_2, 0, 0	5_6
3 2 1	$\overline{5_2}$, 0, 0	$\overline{5_6}$
2 3	$\overline{5_2}$, 0, 0	$\overline{5_5}$
2 2 1	5_2, 0, 0	5_5
$3, 2$	$\overline{5_1}$, $\overline{3_1}$, 0	$\overline{5_4}$
$3, \bar{2}$	$\overline{3_1}$, 0, 0	3_1
$2\ 1, 2$	5_2, 3_1, 0	5_7
4 2	6_1, 0, 0	6_5
4 1 1	$\overline{6_1}$, 0, 0	$\overline{6_5}$
3 1 2	6_2, 0, 0	6_9
3 1 1 1	$\overline{6_2}$, 0, 0	$\overline{6_9}$
2 4	$\overline{6_1}$, 0, 0	$\overline{6_6}$
2 3 1	6_1, 0, 0	6_6
2 1 3	6_2, 0, 0	6_{10}
2 1 2 1	$\overline{6_2}$, 0, 0	$\overline{6_{10}}$
2 1 1 2	6_3, 0, 0	$\overline{6_{14}}$
2 1 1 1 1	6_3, 0, 0	6_{14}
$3, 2+$	$\overline{6_2}$, $\overline{3_1}$, 0	$\overline{6_{12}}$
$2\ 1, 2+$	6_3, 3_1, 0	$\overline{6_{16}}$
$(3, 2)1$	6_2, 3_1, 0	6_{12}
$(3, 2)\bar{1}$	3_1, 0, 0	$\overline{3_1}$
$(3, \bar{2})\bar{1}$	5_2, 3_1, 0	5_7
$(2\ 1, 2)1$	6_3, $\overline{3_1}$, 0	6_{16}
$2\ 2, 2$	6_1, 4_1, 0	6_8
$2\ 2, \bar{2}$	4_1, 0, 0	4_1
$2\ 1\ 1, 2$	$\overline{6_2}$, 4_1, 0	$\overline{6_{13}}$
$3, 2\ 1$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
$3, \bar{2}\ \bar{1}$	3_1, 0, 0	$\overline{5_2}$

Table 2: θ-curves with up to 7 crossings (continued).

Tangle	constituent knot	Θ
7	7_1, 0, 0	7_{25}
6 1	$\overline{7_1}$, 0, 0	$\overline{7_{25}}$
5 2	7_2, 0, 0	$\overline{7_{29}}$
5 1 1	$\overline{7_2}$, 0, 0	7_{29}
4 3	7_3, 0, 0	7_{33}
4 2 1	$\overline{7_3}$, 0, 0	$\overline{7_{33}}$
3 4	$\overline{7_3}$, 0, 0	$\overline{7_{34}}$
3 3 1	7_3, 0, 0	7_{34}
3 2 2	7_5, 0, 0	$\overline{7_{43}}$
3 2 1 1	$\overline{7_5}$, 0, 0	7_{43}
3 1 3	7_4, 0, 0	7_{38}
3 1 2 1	$\overline{7_4}$, 0, 0	$\overline{7_{38}}$
2 5	$\overline{7_2}$, 0, 0	7_{28}
2 4 1	7_2, 0, 0	$\overline{7_{28}}$
2 2 3	7_5, 0, 0	$\overline{7_{44}}$
2 2 2 1	$\overline{7_5}$, 0, 0	7_{44}
2 2 1 2	7_6, 0, 0	$\overline{7_{53}}$
2 2 1 1 1	$\overline{7_6}$, 0, 0	7_{53}
2 1 2 2	$\overline{7_6}$, 0, 0	7_{50}
2 1 2 1 1	7_6, 0, 0	$\overline{7_{50}}$
2 1 1 1 2	7_7, 0, 0	7_{59}
2 1 1 1 1 1	$\overline{7_7}$, 0, 0	$\overline{7_{59}}$
$3, 2 + +$	$\overline{7_5}$, $\overline{3_1}$, 0	7_{46}
$2\ 1, 2 + +$	7_6, 3_1, 0	$\overline{7_{56}}$
$2\ 2, 2+$	$\overline{7_6}$, 4_1, 0	7_{57}
$2\ 1\ 1, 2+$	$\overline{7_7}$, 4_1, 0	$\overline{7_{65}}$
$3, 3+$	$\overline{7_4}$, 0, 0	$\overline{7_{39}}$
$2\ 1, 2\ 1+$	7_7, 0, 0	7_{62}
$(3, 2+)1$	7_5, 3_1, 0	$\overline{7_{46}}$
$(2\ 1, 2+)1$	$\overline{7_6}$, $\overline{3_1}$, 0	7_{56}
$(2\ 2, 2)1$	7_6, 4_1, 0	$\overline{7_{57}}$
$(2\ 2, 2)\bar{1}$	4_1, 0, 0	$\overline{4_1}$
$(2\ 2, \bar{2})\bar{1}$	$\overline{6_2}$, 4_1, 0	$\overline{6_{13}}$
$(2\ 1\ 1, 2)1$	7_7, 4_1, 0	7_{65}

Table 2: θ-curves with 7 crossings (continued).

Tangle	constituent knot	Θ
$(3,3)1$	7_4, 0, 0	7_{39}
$(3,3)\bar{1}$	$\overline{3_1}$, 0, 0	5_2
$(3,\bar{3})1$	$\overline{6_1}$, 0, 0	$\overline{6_7}$
$(2\ 1,2\ 1)1$	$\overline{7_7}$, 0, 0	$\overline{7_{62}}$
$5,2$	$\overline{7_1}$, $\overline{5_1}$, 0	$\overline{7_{27}}$
$5,\bar{2}$	$\overline{5_1}$, $\overline{3_1}$, 0	$\overline{5_4}$
$4\ 1,2$	$\overline{7_3}$, 5_1, 0	$\overline{7_{36}}$
$3\ 2,2$	7_3, $\overline{5_2}$, 0	7_{37}
$3\ 2,\bar{2}$	$\overline{5_2}$, 0, 0	$\overline{5_5}$
$3\ 1\ 1,2$	$\overline{7_4}$, 5_2, 0	$\overline{7_{42}}$
$2\ 3,2$	7_2, 5_2, 0	$\overline{7_{32}}$
$2\ 3,\bar{2}$	5_2, 3_1, 0	5_7
$2\ 2\ 1,2$	$\overline{7_5}$, $\overline{5_2}$,0	7_{49}
$(3,2),2$	7_5, 5_1, 0	$\overline{7_{48}}$
$(3,2),\bar{2}$	5_1, 5_2, 0	7_{18}
$(3,\bar{2}),2$	6_2, 0, 0	6_{11}
$(3,\bar{2}),\bar{2}$	6_3, 0, 0	6_{15}
$(2\ 1,2),2$	$\overline{7_6}$, $\overline{5_2}$, 0	7_{58}
$(2\ 1,2),\bar{2}$	$\overline{5_1}$, $\overline{5_2}$, 0	$\overline{7_{18}}$
$4,3$	$\overline{7_1}$, $\overline{3_1}$, 0	$\overline{7_{26}}$
$4,\bar{3}$	3_1, 0, 0	$\overline{5_2}$
$3\ 1,3$	7_3, $\overline{3_1}$, 0	7_{35}
$3\ 1,\bar{3}$	4_1, 3_1, 0	6_4
$2\ 2,3$	7_2, 0, 0	$\overline{7_{30}}$
$2\ 2,\bar{3}$	0, 0, 0	$\overline{5_1}$
$4,2\ 1$	7_2, 3_1, 0	$\overline{7_{31}}$
$3\ 1,2\ 1$	7_5, 3_1, 0	$\overline{7_{45}}$
$2\ 1\ 1,2\ 1$	$\overline{7_6}$, 0, 0	7_{54}

We also give an enumeration of special handcuff graphs with up to seven crossings as in Table 3. Links in the second column correspond to Rolfsen's knot table [15], and handcuff graphs in the last column correspond to our table [12]. The handcuff graphs are ordered so that their constituent links are in lexicographic order. A link $\overline{L}$ and a handcuff graph $\overline{\Phi}$ denote the mirror images of L and Φ, respectively. Moreover, $\#_3$ denotes *an order* 3 *vertex connected sum* (see [16]).

Table 3: Handcuff graphs with up to 7 crossings.

Tangle	constituent link	Φ
2	2_1^2	2_1
4	4_1^2	4_1
3 1	$\overline{4_1^2}$	$\overline{4_1}$
2 1 2	5_1^2	5_1
2 1 1 1	$\overline{5_1^2}$	$\overline{5_1}$
6	6_1^2	6_5
5 1	$\overline{6_1^2}$	$\overline{6_5}$
3 3	6_2^2	6_7
3 2 1	$\overline{6_2^2}$	$\overline{6_7}$
2 2 2	6_3^2	6_8
2 2 1 1	$\overline{6_3^2}$	$\overline{6_8}$
$3, 3$	$\overline{6_1^2}$	6_6
$3, \bar{3}$	0_1^2	$2_1 \#_3 2_1$
$2\ 1, 2\ 1$	$\overline{6_3^2}$	6_9
4 1 2	7_1^2	7_{18}
4 1 1 1	$\overline{7_1^2}$	$\overline{7_{18}}$
3 1 1 2	7_2^2	7_{22}
3 1 1 1 1	$\overline{7_2^2}$	$\overline{7_{22}}$
2 3 2	7_3^2	7_{26}
2 3 1 1	$\overline{7_3^2}$	$\overline{7_{26}}$
2 1 4	$\overline{7_1^2}$	$\overline{7_{19}}$
2 1 3 1	7_1^2	7_{19}
2 1 1 3	$\overline{7_2^2}$	7_{23}
2 1 1 2 1	7_2^2	$\overline{7_{23}}$
$(3, 2)2$	$\overline{7_4^2}$	7_{28}
$(3, 2)\bar{2}$	$\overline{7_7^2}$	7_{35}
$(3, \bar{2})2$	$\overline{7_7^2}$	7_{35}
$(3, \bar{2})\bar{2}$	$\overline{7_8^2}$	$\overline{7_{36}}$
$(2\ 1, 2)2$	$\overline{7_5^2}$	7_{30}
$(2\ 1, 2)\bar{2}$	$\overline{7_8^2}$	7_{36}
$3, 2\ 1+$	7_2^2	7_{24}
$(3, 2)1\ 1$	7_4^2	$\overline{7_{28}}$

Table 3: Handcuff graphs with up to 7 crossings (continued).

Tangle	constituent link	Φ
$(2\ 1, 2)1\ 1$	7^2_5	$\overline{7_{30}}$
$(3, 2\ 1)1$	$\overline{7^2_2}$	$\overline{7_{24}}$
$(3, \bar{2}\ \bar{1})\bar{1}$	6^2_3	6_9
$2\ 1\ 1, 3$	$\overline{7^2_1}$	7_{20}
$2\ 1\ 1, \bar{3}$	4^2_1	$\overline{6_4}$
$2\ 2, 2\ 1$	$\overline{7^2_3}$	7_{21}

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