Some Fixed Point Theorems for Multivalued Maps Satisfying an Implicit Relation on Metrically Convex Spaces

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Abstract. In this paper, we give some fixed point theorems for multivalued maps satisfying an implicit relation on metrically convex spaces. Our results extend and generalize some fixed point theorem in the literature.

1. Introduction

In recent years several fixed point results have been obtained on metrically convex spaces. Assad and Kirk [6] gave a sufficient condition enunciating fixed point of set-valued mappings enjoying specific boundary condition in metrically convex metric spaces. A significant generalization of the fixed point theorem of Assad [5] and the theorem of Assad and Kirk [6] for multivalued contraction non-self mappings is obtained by Itoh [17] in 1977. In the current years the work due to Assad and Kirk [6] has inspired extensive activities which includes Ahmad and Imdad [1], [2], Imdad et al. [13], Imdad and Ali [14], Itoh [17], Khan [19] and some others. Most recently, Dhage et al. [9] and Huang and Cho [12] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Ahmad and Khan [3], Itoh [17], Khan [19] and others. The purpose of this paper is to prove some coincidence and common fixed point theorems for multivalued mappings satisfying an implicit relation on metrically convex spaces. Our results either partially or completely generalize earlier results due to Ahmad and Imdad [1], [2], Ahmad and Khan [3], Ćirić [7], Imdad and Khan [15], Itoh [17], Khan [19], Khan et al. [20], Rhoades [26] and several others. See also the related Theorem 3.1 of [14].

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2. Preliminaries

Let (X, d) be a metric space. Then X said to be metrically convex if for every pair $x, y \in X, x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$.

Lemma 1([6]). *Let K be non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \delta K$ such that $d(x, y) = d(x, z) + d(z, y)$, where δK denotes the boundary of K.*

Let $CB(X)$ denotes the family of all non-empty closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

The function H is a metric on $CB(X)$ and is called Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$.

Lemma 2([21]). *Let $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in B such that $qd(a, b) \leq H(A, B)$.*

Definition 1([15]). Let K be a nonempty subset of a metric space $(X, d), T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be pointwise R-weakly commuting on K if for given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that $d(Ty, FTx) \leq Rd(Tx, Fx)$ for each $y \in K \cap Fx$.

Moreover, the pair (F, T) will be called R-weakly commuting on K if $d(Ty, FTx) \leq Rd(Tx, Fx)$ holds for each $x \in K, Tx \in K$ with some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F, T) on K due to Hadzic and Gajic [11]. For $K = X$ Definition 1 reduces to "pointwise R-weak commutativity and R-weak commutativity" for single valued self mappings due to Pant [22].

Definition 2([10],[11]). Let K be a nonempty subset of a metric space $(X, d), T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be weakly commuting if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have $d(Tx, FTy) \leq d(Ty, Fy)$, whereas the pair (F, T) is said to be compatible if for every sequence $\{x_n\} \subset K$, from the relation $\lim_{n \to \infty} d(Fx_n, Tx_n) = 0$ and $Tx_n \in K$ (for every $n \in N$) it follows that $\lim_{n \to \infty} d(Ty_n, FTx_n) = 0$ for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n, n \in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [18].

Definition 3([13]). Let K be a nonempty subset of a metric space $(X, d), T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be quasi-coincidentally

commuting if for all coincidence points "x" of $(T, F), TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 4([13]). A mapping $T : K \to X$ is said to be coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if T is idempotent at the coincidence points of the pair (F, T).

3. Implicit relation

Implicit relations on metric space have been used in many articles (see [4], [16], [23], [24], [25], [27]).

Let R_+ denote the set of all non-negative real numbers and let $\mathcal{G}$ be the set of all continuous functions $G : R_+^5 \to R$ satisfying the following conditions:

$(G_1) : G(t_1, \cdots, t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, \cdots, t_5$.

(G_2) : there exist three constants $a, b \geq 0$, $2a + 3b < q < 1$ such that the inequality

$$G(u, v, w, v, v + w) \leq 0$$

implies $u \leq \max\{(a + b)v + bw, (a + b)w + bv\}$.

$(G_3) : G(qu, u, 0, 0, 2u) > 0, \forall u > 0$.

Now we give some examples. In the following examples, the condition (G_1) is obvious.

Example 1. Let $G(t_1, \cdots, t_5) = t_1 - \alpha \max\{\dfrac{t_2}{2}, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \geq 0, \alpha > 2\beta$ and $2\alpha + 3\beta < q < 1$.

Let $G(u, v, w, v, v + w) = u - \alpha \max\{w, v\} - \beta(w + v) \leq 0$. Thus $u \leq \max\{(\alpha + \beta)v + \beta w, (\alpha + \beta)w + \beta v\}$ and so (G_2) is satisfied with $a = \alpha$ and $b = \beta$.

$G(qu, u, 0, 0, 2u) = u(q - \dfrac{\alpha}{2} - 2\beta) > 0, \forall u > 0$.

Therefore $G \in \mathcal{G}$.

Example 2. Let $G(t_1, \cdots, t_5) = t_1 - \alpha t_2 - \beta(t_3 + t_4) - \gamma t_5$, where $\alpha, \beta, \gamma \geq 0$ and $2\alpha + 3\beta + 3\gamma < q < 1$.

Let $G(u, v, w, v, v + w) = u - \alpha v - \beta(w + v) - \gamma(w + v) \leq 0$. Thus $u \leq (\alpha + \beta + \gamma)v + (\beta + \gamma)w \leq \max\{(\alpha + \beta + \gamma)v + (\beta + \gamma)w, (\alpha + \beta + \gamma)w + (\beta + \gamma)v\}$ and so (G_2) is satisfied with $a = \alpha$, $b = \beta + \gamma$.

$G(qu, u, 0, 0, 2u) = u(q - \alpha - 2\gamma), \forall u > 0$.

Therefore $G \in \mathcal{G}$.

Example 3. Let $G(t_1, \cdots, t_5) = t_1 - \alpha t_2 - \beta \max\{t_3 + t_4, t_5\}$, where $\alpha, \beta, \gamma \geq 0$ and $2\alpha + 3\beta < q < 1$.

Let $G(u, v, w, v, v + w) = u - \alpha v - \beta(v + w) \leq 0$. Thus $u \leq (\alpha + \beta)v + \beta w \leq \max\{(\alpha + \beta)v + \beta w, (\alpha + \beta)w + \beta v\}$ and so (G_2) is satisfied with $a = \alpha, b = \beta$.

$G(qu, u, 0, 0, 2u) = u(q - \alpha - 2\beta), \forall u > 0$.

Therefore $G \in \mathcal{G}$.

Example 4. Let $G(t_1, \cdots, t_5) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma t_5$, where $\alpha, \beta, \gamma \geq 0$ and $2\alpha + 3\beta + 3\gamma < q < 1$.

Let $G(u, v, w, v, v + w) = u - \alpha v - \beta \max\{w, v\} - \gamma(w + v) \leq 0$. Thus $u \leq \max\{(\alpha + \beta + \gamma)v + (\beta + \gamma)w, (\alpha + \beta + \gamma)w + (\beta + \gamma)v\}$ and so (G_2) is satisfied with $a = \alpha, b = \beta + \gamma$

$G(qu, u, 0, 0, 2u) = u(q - \alpha - 2\gamma) > 0, \forall u > 0$.

Therefore $G \in \mathcal{G}$.

Example 5. Let $G(t_1, \cdots, t_5) = t_1 - \max\left\{\dfrac{\alpha t_3 t_5}{t_3 + t_5 + 1}, \dfrac{\beta t_4 t_5}{t_4 + t_5 + 1}\right\} - \gamma(t_3 + t_4)$,

where $\alpha, \beta, \gamma \geq 0$, $2\alpha + 2\beta + 3\gamma < q < 1$.

Let $G(u, v, w, v, v + w) = u - \max\left\{\dfrac{\alpha v(v + w)}{v + 2w + 1}, \dfrac{\beta w(v + w)}{2v + w + 1}\right\} - \gamma(v + w) \leq 0$.

Thus

$$
\begin{aligned}
u &\leq \max\left\{\frac{\alpha v(v + w)}{v + 2w + 1}, \frac{\beta w(v + w)}{2v + w + 1}\right\} + \gamma(v + w) \\
&\leq \max\{\alpha v, \beta w\} + \gamma(v + w) \\
&= \max\{(\alpha + \gamma)v + \gamma w, (\beta + \gamma)w + \gamma v\} \\
&\leq \max\{(\alpha + \beta + \gamma)v + \gamma w, (\alpha + \beta + \gamma)w + \gamma v\}
\end{aligned}
$$

and so (G_2) is satisfied with $a = \alpha + \beta$, $b = \gamma$.

$G(qu, u, 0, 0, 2u) = qu > 0, \forall u > 0$.

Therefore $G \in \mathcal{G}$.

4. Main result

Now we give our main theorem.

Theorem 1. *Let (X, d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying*

(a) $\delta K \subseteq SK \cap TK, F_i(K) \cap K \subseteq SK, F_j(K) \cap K \subseteq TK$,

(b) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K, Sx \in \delta K \Rightarrow F_j(x) \subseteq K$, and

(4.1)
$$
\begin{aligned}
G\left(H(F_i(x), F_j(y)), d(Tx, Sy), d(Tx, F_i(x)), d(Sy, F_j(y)),\right. \\
\left. d(Tx, F_j(y)) + d(Sy, F_i(x))\right) \leq 0
\end{aligned}
$$

for all $x, y \in K$, where $G \in \mathcal{G}$, $i = 2n - 1, j = 2n, (n \in N)$,

(c) (F_i, T) and (F_j, S) are compatible pairs,

(d) $\{F_n\}, S$ and T are continuous on K.

Then (F_i, T) as well as (F_j, S) has a point of coincidence.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \delta K$. Then (due to $\delta K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. From the implication $Tx \in \delta K$ which implies $F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK$,

let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$ such that

$$qd(y_1, y_2) \leq H(F_1(x_0), F_2(x_1)).$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K) \cap K \subseteq TK$ implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \delta K$ such that
$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

$y_3 \in F_3(x_2)$ be such that $qd(y_2, y_3) \leq H(F_2(x_1), F_3(x_2))$.

Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that
  (e) $y_{2n} \in F_{2n}(x_{2n-1}), y_{2n+1} \in F_{2n+1}(x_{2n})$,
  (f) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}),$$

  (g) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote
$$P_0 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}$$
$$P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}$$
$$Q_0 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}$$
$$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then using (4.1), we have

$$G\left(H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})), d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})),\right.$$
$$\left. d(Sx_{2n-1}, F_{2n}(x_{2n-1})), d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\right) \leq 0$$

or

$$G\left(H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}),\right.$$
$$\left. d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\right) \leq 0.$$

From (G_2), there exist three constants $a, b \geq 0$, $2a + 3b < q < 1$ such that

(4.2) $$H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1}))$$
$$\leq \max \begin{cases} (a+b)d(y_{2n-1}, y_{2n}) + bd(y_{2n}, y_{2n+1}), \\ (a+b)d(y_{2n}, y_{2n+1}) + bd(y_{2n-1}, y_{2n}) \end{cases}.$$

Since

$$
\begin{aligned}
qd(y_{2n}, y_{2n+1}) &= qd(Tx_{2n}, Sx_{2n+1}) \\
&\leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})),
\end{aligned}
$$

from (4.2), we have

$$
d(y_{2n}, y_{2n+1}) \leq \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(y_{2n-1}, y_{2n})
$$

or

$$
d(Tx_{2n}, Sx_{2n+1}) \leq \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(Sx_{2n-1}, Tx_{2n}).
$$

Note that $\max\left\{\dfrac{a+b}{q-b}, \dfrac{b}{q-a-b}\right\} < 1$ since $2a + 3b < q < 1$.

Similarly if $(Sx_{2n-1}, Tx_{2n}) \in P_0 \times Q_1$, then

$$
d(Sx_{2n-1}, Tx_{2n}) \leq \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(Sx_{2n-1}, Tx_{2n-2}).
$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then

$$
d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})
$$

which in turn yields

$$
d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})
$$

and hence

$$
qd(Tx_{2n}, Sx_{2n+1}) \leq qd(y_{2n}, y_{2n+1}) \leq H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})).
$$

Now, proceeding as in Case 1, we have

$$
d(Tx_{2n}, Sx_{2n+1}) \leq \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(Sx_{2n-1}, Tx_{2n}).
$$

In case $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_0$, then as earlier, one also obtains

$$
d(Sx_{2n-1}, Tx_{2n}) \leq \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(Sx_{2n-1}, Tx_{2n-2}).
$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} = y_{2n-1}$ and

$$
\begin{aligned}
(4.3) \quad qd(Tx_{2n}, Sx_{2n+1}) &= qd(Tx_{2n}, y_{2n+1}) \\
&\leq qd(Tx_{2n}, y_{2n}) + qd(y_{2n}, y_{2n+1}) \\
&\leq qd(Tx_{2n}, y_{2n}) + H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})).
\end{aligned}
$$

Now again using (4.1), we have

$$G\left(H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})), d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})),\right.$$
$$\left. d(Sx_{2n-1}, F_{2n}(x_{2n-1})), d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\right) \le 0$$

or

$$G\left(H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}),\right.$$
$$\left. d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\right) \le 0.$$

From (G_2),

(4.4)
$$H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1}))$$
$$\le \max\left\{\begin{array}{l} (a+b)d(y_{2n-1}, y_{2n}) + bd(y_{2n}, y_{2n+1}), \\ (a+b)d(y_{2n}, y_{2n+1}) + bd(y_{2n-1}, y_{2n}) \end{array}\right\}.$$

From (4.3) and (4.4), we have

$$d(Tx_{2n}, Sx_{2n+1}) \le \max\left\{\frac{q+b}{q-a-b}, \frac{q+a+b}{q-b}\right\} d(Tx_{2n}, Sx_{2n-1}).$$

Now, proceeding as earlier, one also obtains

$$d(Sx_{2n-1}, Tx_{2n}) \le \max\left\{\frac{a+b}{q-b}, \frac{b}{q-a-b}\right\} d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le kd(Sx_{2n-1}, Tx_{2n-2}),$$

where

(4.5)
$$k = \max\left\{\begin{array}{l} \left(\dfrac{a+b}{q-b}\right)\left(\dfrac{q+b}{q-a-b}\right), \left(\dfrac{a+b}{q-b}\right)\left(\dfrac{q+a+b}{q-b}\right), \\ \left(\dfrac{b}{q-a-b}\right)\left(\dfrac{q+b}{q-a-b}\right), \left(\dfrac{b}{q-a-b}\right)\left(\dfrac{q+a+b}{q-b}\right) \end{array}\right\} < 1$$

since $2a + 3b < q < 1$.

To see (4.5), $2a + 3b < q < 1$ yields

$$a + b < q - a - 2b$$
$$\Rightarrow aq + bq < q^2 - aq - 2bq$$
$$\Rightarrow aq + bq + ab + b^2 < q^2 - aq - 2bq + ab + b^2$$
$$\Rightarrow (a+b)(q+b) < (q-b)(q-a-b)$$
$$\Rightarrow \frac{(a+b)(q+b)}{(q-b)(q-a-b)} < 1$$

and

$$
\begin{aligned}
& 3b < q \\
\Rightarrow \quad & \frac{q}{q-b} < \frac{3}{2} \\
\Rightarrow \quad & \frac{q}{2(q-b)} + \frac{1}{4} < 1 \\
\Rightarrow \quad & \frac{1}{2}\left(\frac{q}{(q-b)} + \frac{1}{2}\right) < 1 \\
\Rightarrow \quad & \frac{a+b}{q-b}\left(\frac{q}{q-b} + \frac{a+b}{q-b}\right) < 1 \\
\Rightarrow \quad & \frac{(a+b)(q+a+b)}{(q-b)^2} < 1
\end{aligned}
$$

and

$$
\begin{aligned}
& b < q - 2a - 2b \\
\Rightarrow \quad & bq < q^2 - 2aq - 2bq \\
\Rightarrow \quad & bq + b^2 < q^2 - 2aq - 2bq + b^2 + a^2 + 2ab \\
\Rightarrow \quad & b(q+b) < (q-a-b)^2 \\
\Rightarrow \quad & \frac{b(q+b)}{(q-a-b)^2} < 1
\end{aligned}
$$

and

$$
\begin{aligned}
& a + 3b < q \\
\Rightarrow \quad & aq + 3bq < q^2 \\
\Rightarrow \quad & bq < q^2 - aq - 2bq \\
\Rightarrow \quad & bq + ab + b^2 < q^2 - aq - 2bq + ab + b^2 \\
\Rightarrow \quad & b(q+a+b) < (q-a-b)(q-b) \\
\Rightarrow \quad & \frac{b(q+a+b)}{(q-a-b)(q-b)} < 1.
\end{aligned}
$$

Thus in all the cases, we have

$$
d(Tx_{2n}, Sx_{2n+1}) \le k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}
$$

whereas

$$
d(Sx_{2n+1}, Tx_{2n+2}) \le k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}.
$$

Now on the lines of Assad and Kirk [6], it can be shown by induction that for $n \ge 1$, we have

$$
d(Tx_{2n}, Sx_{2n+1}) < k^n \delta, \quad d(Sx_{2n+1}, Tx_{2n+2}) < k^{n+\frac{1}{2}} \delta
$$

whereas

$$
\delta = k^{-\frac{1}{2}} \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.
$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \cdots, Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, \cdots\}$ is Cauchy and hence converges to the point z in X. Then as noted in [10] there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 or Q_0 respectively.

Suppose that the subsequence $\{Tx_{2n_k}\}$ contained in P_0 for each $k \in N$ converges to z. Using compatibility of (F_j, S), we have

$$\lim_{k \to \infty} d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) = 0 \text{ for any even integer } j \in N,$$

which implies that $\lim_{k \to \infty} d(STx_{2n_k}, F_j(Sx_{2n_k-1})) = 0$.

Using the continuity of S and F_j, one obtains $Sz \in F_j(z)$, for any even integer $j \in N$. Similarly continuity of T and F_i implies $Tz \in F_i(z)$, for any odd integer $i \in N$. Now using (4.1), we have

$$G\left(H(F_i(z), F_j(z)), d(Tz, Sz), d(Tz, F_i(z)), d(Sz, F_j(z)), \atop d(Tz, F_j(z)) + d(Sz, F_i(z))\right) \leq 0$$

or, since $qd(Tz, Sz) \leq H(F_i(z), F_j(z))$,

$$G(qd(Tz, Sz), d(Tz, Sz), 0, 0, 2d(Tz, Sz)) \leq 0$$

which is a contradiction with (G_3) if $d(Tz, Sz) > 0$. Thus we obtain $d(Tz, Sz) = 0$ and so $Tz = Sz$ which shows that z is a common coincidence point of the maps $\{F_n\}, S$ and T. $\qquad \square$

Remark 1. By Theorem 1, we get an improved version of main theorem of [4].

Remark 2. Theorem 3.1 of [15], which is a generalization of results of [1], [2], follows from Example 1 and Theorem 1.

Remark 3. Theorem 1 can prove for pointwise R-weakly commuting maps as Theorem 3.4 of [15].

Theorem 2. *Let (X, d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying (4.1), (a) and (b). Suppose that*

(h) TK and SK are closed subspaces of X. Then (F_i, T) has a point of coincidence and (F_j, S) has a point of coincidence.

Moreover, (F_i, T) has a common fixed point if T is quasi-coincidentally commuting and coincidentally idempotent w.r.t. F_i whereas (F_j, S) has a common fixed point provided S is quasi-coincidentally commuting and coincidentally idempotent w.r.t. F_j.

Remark 4. Theorem 3.5 of [15], which is a generalization of results of Khan [19] and Khan et al. [20], follows from Example 1 and Theorem 2.

Theorem 3. *Let (X, d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ satisfying*
(i) $x \in \delta K \Rightarrow F_n(x) \subseteq K$ and

$$G\left(H(F_i(x), F_j(y)), d(x, y), d(x, F_i(x)), d(y, F_j(y)), \ d(x, F_j(y)) + d(y, F_i(x))\right) \leq 0$$

for all $x, y \in K$, *where* $G \in \mathcal{G}$, $i \neq j$,

    Then $\{F_n\}$ *has a common fixed point.*

Remark 5. Corollary 3.6 of [15], which is a generalization of results of [8], follows from Example 1 and Theorem 3.

Remark 6. If we combined Example 5 with Theorem 3, we have the following interesting result.

Corollary 1. *Let* (X, d) *be a metrically convex complete metric space and* K *a non-empty closed subset of* X. *Let* $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ *satisfying*

    (j) $x \in \delta K \Rightarrow F_n(x) \subseteq K$ *and*

$$
H(F_i(x), F_j(y)) \leq \max \left\{
\begin{array}{c}
\dfrac{\alpha d(x, F_i(x)) \left[d(x, F_j(y)) + d(y, F_i(x)) \right]}{d(x, F_i(x)) + d(x, F_j(y)) + d(y, F_i(x)) + 1}, \\
\dfrac{\beta d(x, F_i(x)) \left[d(x, F_j(y)) + d(y, F_i(x)) \right]}{d(y, F_j(y)) + d(x, F_j(y)) + d(y, F_i(x)) + 1}
\end{array}
\right\}
$$
$$
+ \gamma \left[d(x, F_i(x)) + d(y, F_j(y)) \right]
$$

for all $x, y \in K$, *where* $\alpha, \beta, \gamma \geq 0$, $2\alpha + 2\beta + 3\gamma < q < 1$, $i \neq j$.

    Then $\{F_n\}$ *has a common fixed point.*

Remark 7. We can have some new results, if we combined Theorem 1, Theorem 2 or Theorem 3 with some examples of G.

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