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Some properties of a Certain family of Meromorphically Univalent Functions defined by an Integral Operator

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Abstract. Making use of a linear operator, we introduce certain subclass of meromorphically univalent functions in the punctured unit disk and study its properties including some inclusion results, coefficient and distortion problems. Our result generalize many results known in the literature.

1. Introduction

Let $\sum$ denote the class of functions $f(z)$ of the form

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^{k-1}. \tag{1.1}$$

which are analytic and univalent in the punctured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \; and \; 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

The class $\sum$ is closed under the Hadamard product or convolution

$$(f * g)(z) = z^{-1} + \sum_{k=1}^{\infty} a_k b_k z^{k-1} = (g * f)(z),$$

where $f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^{k-1}, g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^{k-1}$.

In terms of the Pochhammer symbol (or the shifted factorial) $(\lambda)_n$ given by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)...(\lambda+n+1) \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = z^{-1} + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} z^{k-1} \tag{1.2}$$

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$$(z \in \mathbb{U}^*; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}).$$

Corresponding to the function $\phi(a, c; z)$, we introduce a linear operator $\text{Ł}(a, c)$ which is defined by means of the following Hadamard product :

$$\text{Ł}(a, c)f(z) := \phi(a, c; z) * f(z), \quad (f \in \sum).$$

The operator $\text{Ł}(a, c)$ has been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see, for details, [4]; see also [11],[12]) and on the space of meromorfically functions by Liu and Srivastava [8].

Furthermore corresponding to the function $\phi(a, c; z)$ defined by (1.2), we introduce a function $\phi^\lambda(a, c; z)$ given by

$$\phi(a, c; z) * \phi^\lambda(a, c; z) = \frac{1}{z(1-z)^{\lambda+1}},$$

$$(a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -1; z \in \mathbb{U}^*; f \in \sum).$$

which leads us to the following family of linear operator $\text{Ł}^\lambda(a, c)$:

$$\text{Ł}^\lambda(a, c)f(z) = \phi^\lambda(a, c; z) * f(z) \tag{1.3}$$

$$(a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -1; z \in \mathbb{U}^*; f \in \sum).$$

It is easily verified from the definition (1.3) that

$$z(\text{Ł}^\lambda(a, c)f(z))' = (\lambda + 1)\text{Ł}^{\lambda+1}(a, c)f(z) - (\lambda + 2)\text{Ł}^\lambda(a, c)f(z), \tag{1.4}$$

$$z(\text{Ł}^\lambda(a + 1, c)f(z))' = a\text{Ł}^\lambda(a, c)f(z) - (a + 1)\text{Ł}^\lambda(a + 1, c)f(z). \tag{1.5}$$

We note that a linear operator $\tau^\lambda(a, c)$, analogous to $\text{Ł}^\lambda(a, c)$ defined here, has been studied widely on the space of analytic functions by [2], [3], [6], [7], [9].

Let M be the class of analytic functions ψ in $\mathbb{U}$ normalized by $\psi(0) = 1$, and let N be the subclass of M consisting of that functions ψ which are univalent in $\mathbb{U}$ and for which $\psi(\mathbb{U})$ is convex and $\Re\{\psi(z)\} > 0, (z \in \mathbb{U})$.

By using the general linear operator $\text{Ł}^\lambda(a, c)$, we define a new subclass of $\sum$ by

$$M^\lambda(a, c; h) := \left\{ f : f \in \sum \text{ and } -\frac{z(\text{Ł}^\lambda(a, c)f)'(z)}{\text{Ł}^\lambda(a, c)f(z)} \prec h \right\}, \ (z \in \mathbb{U}, h \in N). \tag{1.6}$$

If we set $h(z) = \dfrac{1 + Az}{1 + Bz}, (-1 \le B < A \le 1)$, then $h(0) = 1$, h is convex in $\mathbb{U}$ and $\Re\left\{\dfrac{1 + Az}{1 + Bz}\right\} > 0$. Hence for convenience when $h(z) = \dfrac{1 + Az}{1 + Bz}, (-1 \le B < A \le 1)$, set $M^\lambda(a, c; h) = M^\lambda(a, c; A, B)$.

The main object of this paper is to present a systematic investigation of the various important properties and characteristics of the general class $M^{\lambda}(a,c;h)$.

To prove our main results, we need the following lemma.

Lemma 1.1(Eenigenburg et al. [5]). *Let h be convex univalent in $\mathbb{U}$ with $h(0)=1$ and $\Re(kh(z)+\nu)>0$, $(k,\nu\in\mathbb{C}; z\in\mathbb{U})$.*

If $q(z)$ is analytic in $\mathbb{U}$ with $q(0)=1$, then the subordination

$$q(z)+\frac{zq'(z)}{kq(z)+\nu}\prec h(z)\quad(z\in\mathbb{U}),$$

implies that $q(z)\prec h(z)$ $(z\in\mathbb{U})$.

2. Main results

We begin with the following :

Theorem 2.1. *Let $\lambda\geq 0$ and $a>0$. Then*

(1) $M^{\lambda+1}(a,c;h)\subset M^{\lambda}(a,c;h)$ *for* $(h\in N$ *and* $\Re(h(z))<2+\lambda, z\in\mathbb{U})$;

(2) $M^{\lambda}(a,c;h)\subset M^{\lambda}(a+1,c;h)$ *for* $(h\in N$ *and* $\Re(h(z))<1+a, z\in\mathbb{U})$.

Proof. First of all, we show that

$$M^{\lambda+1}(a,c;h)\subset M^{\lambda}(a,c;h)\ for\ (h\in N\ and\ \Re(h(z))<2+\lambda, z\in\mathbb{U}).$$

Let $f\in M^{\lambda+1}(a,c;h)$ $(h\in N$ *and* $\Re(h(z))<2+\lambda)$ and set

$$p(z)=-\frac{z(\text{Ł}^{\lambda}(a,c)f)'(z)}{\text{Ł}^{\lambda}(a,c)f(z)},\ (z\in\mathbb{U}) \tag{2.1}$$

where p is analytic in $\mathbb{U}$ with $p(0)=1$ and $p(z)\neq 0$ for all $z\in\mathbb{U}$. Applying (1.4) and (2.1), we obtain

$$(\lambda+1)\frac{\text{Ł}^{\lambda+1}(a,c)f(z)}{\text{Ł}^{\lambda}(a,c)f(z)}=-p(z)+(\lambda+2). \tag{2.2}$$

By logarithmically differentiating both sides of (2.2) and multiplying the resulting equation by z, we have

$$-\frac{z(\text{Ł}^{\lambda+1}(a,c)f)'(z)}{\text{Ł}^{\lambda+1}(a,c)f(z)}=p(z)+\frac{zp'(z)}{-p(z)+(\lambda+2)}\quad(z\in\mathbb{U}).$$

Thus, by using Lemma 1.1 and (2.1), we observe that

$$p(z)\prec h(z)\quad(z\in\mathbb{U}),$$

so that $f(z) \in M^{\lambda}(a,c;h)$.

To prove the second part, Let $f(z) \in M^{\lambda}(a,c;h)$, $(h \in N$ and $\Re\{h(z)\} < 1+a)$ and put

$$p(z) = -\frac{z(\text{Ł}^{\lambda}(a+1,c)f)'(z)}{\text{Ł}^{\lambda}(a+1,c)f(z)}, \tag{2.3}$$

where $p(z) = 1+c_1z+c_2z^2+\cdots$ is analytic in $\mathbb{U}$. Then, by using (2.3) and arguments similar to those detailed above, we obtain $p(z) \prec h(z)$ in $\mathbb{U}$, which implies that $f \in M^{\lambda}(a+1,c;h)$. The proof of Theorem 2.1 is thus completed. □

By taking $h(z) = \dfrac{1+Az}{1+Bz}$ we have the following corollary of Theorem 2.1.

Corollary 2.1. *Let $\lambda \geq 0, a > 0$ and $-1 < B < A \leq 1$. Then*

(1) $M^{\lambda+1}(a,c;A,B) \subset M^{\lambda}(a,c;A,B)$, *if* $\dfrac{A-B}{1+B} < 1+\lambda$;

(2) $M^{\lambda}(a,c;A,B) \subset M^{\lambda}(a+1,c;A,B)$, *if* $\dfrac{A-B}{1+B} < a$.

Theorem 2.2. *Let $0 < \alpha < \mu$ and $h \in N$ be so that $\Re\{h(z)\} < \dfrac{\mu}{\alpha}$. If $f \in M^{\lambda}(a,c;h)$, then the function $g(z)$ defined by*

$$L^{\lambda}(a,c)g(z) = \left(\frac{\mu-\alpha}{z^{\mu}}\int_0^z t^{\mu-1}[L^{\lambda}(a,c)f(z)]^{\alpha}dt\right)^{\frac{1}{\alpha}}, \tag{2.4}$$

is also in the same class $M^{\lambda}(a,c;h)$.

Proof. Suppose that $f \in M^{\lambda}(a,c;h)$ and set

$$p(z) = -\frac{z(\text{Ł}^{\lambda}(a,c)g)'(z)}{\text{Ł}^{\lambda}(a,c)g(z)}, \qquad (z \in \mathbb{U}). \tag{2.5}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By logarithmically differentiating both sides of (2.4) and multiplying the resulting equation by z, we have

$$(\mu-\alpha)\left(\frac{\text{Ł}^{\lambda}(a,c)f(z)}{\text{Ł}^{\lambda}(a,c)g(z)}\right)^{\alpha} = -\alpha p(z) + \mu. \tag{2.6}$$

Then by using (2.5) and (2.6), we find after some computations that

$$-\frac{z(\text{Ł}(a,c)f)'(z)}{\text{Ł}(a,c)f(z)} = p(z) + \frac{zp'(z)}{-\alpha p(z)+\mu}.$$

Thus, by using Lemma 1.1, and (2.5), we observe that $p(z) \prec h(z)$ $(z \in \mathbb{U})$, so that $g(z) \in M^{\lambda}(a,c;h)$, and the proof is completed. □

Corollary 2.2. *Let $0 < \alpha < \mu$ and $-1 < B < A \leq 1$ be so that $\frac{1+A}{1+B} < \frac{\mu}{\alpha}$. If $f \in M^{\lambda}(a,c;A,B)$, then the function $g(z)$ defined by*

$$L^{\lambda}(a,c)g(z) = \left(\frac{\mu-\alpha}{z^{\mu}}\int_0^z t^{\mu-1}[L^{\lambda}(a,c)f(z)]^{\alpha}\right)^{\frac{1}{\alpha}}, \tag{2.7}$$

is also in the same class $M^{\lambda}(a,c;A,B)$.

Theorem 2.3. *Let $\lambda > -1, -1 \leq B < A \leq 1$, and $a > 0, c > 0$. If $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^{n-1} \in M^{\lambda}(a,c;A,B)$, then*

$$|a_n| \leq \frac{(A-B)_n (a)_n}{(c)_n(\lambda+1)_n} \quad (n = 1,2,3,\cdots). \tag{2.8}$$

When $B = -1$, and $A = 1-2\alpha$ $(0 \leq \alpha < 1)$, the result is sharp for the function given by $f(z) = Ł(1,\lambda+1)Ł(a,c)\frac{1}{z(1-z)^{2(1-\alpha)}}$.

Proof. Suppose that $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^{n-1} \in M^{\lambda}(a,c;A,B)$, and put

$$-\frac{z(Ł^{\lambda}(a,c)f)'(z)}{Ł^{\lambda}(a,c)f(z)} := p(z), \tag{2.9}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $p(z) \prec \frac{1+Az}{1+Bz}$. Substituting the series expansion of $f(z)$ and $p(z)$ in (2.9) and equating the coefficients of z^n on both sides of the resulting equation, we obtain

$$-nd_n = d_{n-1}p_1 + d_{n-2}p_2 + \cdots + d_1 p_{n-1} + p_n \quad (n = 1,2,3,\cdots), \tag{2.10}$$

where $p_0 := d_0 := 1$ and $d_n := \frac{(c)_n(\lambda+1)_n}{(a)_n(1)_n} a_n, (n := 1,2,3,\cdots)$. Using the well known coefficient estimates (see for details [1])

$$|p_n| \leq A - B, \qquad (n = 1,2,3,\cdots)$$

in (2.10), we get the required result (2.8). It is easily verified the result is sharp for the function f defined by $f(z) = Ł(1,\lambda+1)Ł(a,c)\frac{1}{z(1-z)^{2(1-\alpha)}}$. □

Finally by making use of the well known hypergeometric function $F(a,b;c;z)$ defined by $F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n$ we prove.

Theorem 2.4. *Let $a > 0, c > 0, -1 \leq B < A \leq 1$ and $\lambda > -1$. If $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^{n-1} \in M^{\lambda}(a,c;A,B)$, then*

$$|f(z)| \leq Ł(a,c)Ł(1,\lambda+1)\left[\frac{1}{r}F(A-B,1;1;r)\right] \quad (|z| < r), \tag{2.11}$$

and

$$(2.12)\qquad |f'(z)| \le \frac{2}{r^2} + Ł(a,c)Ł(1,\lambda+1)\left(\frac{1}{z}F(A-B,1;1;z)\right)'_{z=r}, \quad (|z|<r).$$

By considering the function $f(z) = Ł(a,c)Ł(1,\lambda+1)\left(\frac{1}{z(1-z)^{2(1-\alpha)}}\right)'_{z=r}$, *one can show that the first inequality is sharp when* $A = 1-2\alpha, B = -1$.

Proof. Let $f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^{n-1} \in M^{\lambda}(a,c;A;B)$. Then, using Theorem 2.3, we obtain that

$$\begin{aligned} |f(z)| &\le \frac{1}{r} + \sum_{n=1}^{\infty} |a_n||r|^{n-1} \\ &= \frac{1}{r}\left(1 + \sum_{n=1}^{\infty} |a_n| r^n\right) \\ &\le \frac{1}{r}\left(1 + \sum_{n=1}^{\infty} \frac{(A-B)_n (a)_n}{(c)_n(\lambda+1)_n} r^n\right) \\ &= Ł(a,c)Ł(1,\lambda+1)\left[\frac{1}{r}F(A-B,1;1;r)\right]. \end{aligned}$$

and that

$$\begin{aligned} |zf'(z)| &\le \frac{1}{r} + \sum_{n=1}^{\infty} (n-1)|a_n||r|^{n-1} \\ &= \frac{1}{r}\left(1 + \sum_{n=1}^{\infty} (n-1)|a_n| r^n\right) \\ &\le \frac{1}{r}\left(1 + \sum_{n=1}^{\infty} (n-1)\frac{(A-B)_n (a)_n}{(c)_n(\lambda+1)_n} r^n\right) \\ &\le \frac{1}{r}\left(1 + \sum_{n=1}^{\infty} (n+1)\frac{(A-B)_n (a)_n}{(c)_n(\lambda+1)_n} r^n - 2\sum_{n=1}^{\infty} \frac{(A-B)_n (a)_n}{(c)_n(\lambda+1)_n} r^n\right) \\ &= \frac{2}{r} + rŁ(a,c)\left(Ł(1,\lambda+1)\frac{1}{z}F(A-B,1;1;z)\right)'_{z=r}. \end{aligned}$$

By considering the function $f(z) = Ł(a,c)Ł(1,\lambda+1)\frac{1}{z(1-z)^{2(1-2\alpha)}}$, one can show that the first inequality is sharp when $B = -1$ and $A = 1-2\alpha, (0 \le \alpha < 1)$. □

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