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On Certain Extension of Hilbert's Integral Inequality with Best Constants

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Abstract. In this paper, by introducing a new function with two parameters, we give another generalizations of the Hilbert's integral inequality with a mixed kernel $k(x,y) = \frac{1}{A(x+y)+B|x-y|}$ and a best constant factors. As applications, some particular results with the best constant factors are considered.

1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [4])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}, \tag{1.1}$$

$$\int_0^\infty \int_0^\infty \frac{lnx - lny}{x-y} f(x)g(y)dxdy < \pi^2\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor π and π^2 are the best possible. Inequality (1.1) and (1.2) are the well known Hilbert's inequality. They have been studied and generalized in many directions by a number of mathematicians(see [1]-[3], [6]-[12]).

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In this paper, we give a generalization of Hilbert's inequality as the following.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy < C(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy)^{\frac{1}{2}},$$

where C is a constant.

2. Main results

Lemma 2.1 *Setting*

$$\omega(u) = \int_0^\infty \frac{1}{A(u+v)+B|u-v|}(\frac{u}{v})^{\frac{1}{2}}dv,$$

where $A > 0$ and $B > -A$, then $\omega(u) = C(A,B)$ is a constant.
In particular $C(1,0) = \pi,\ C(1,1) = 2$.

Proof. For fixed u, letting $t = v/u$, we get

$$\begin{aligned}
\omega(u) &= \int_0^\infty \frac{1}{A(u+tu)+B|u-tu|}(\frac{u}{tu})^{\frac{1}{2}}dtu \\
&= \int_0^\infty \frac{1}{A(1+t)+B|1-t|}(\frac{1}{t})^{\frac{1}{2}}dt \\
&= \int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt + \int_1^\infty \frac{1}{(A-B)+(A+B)t}t^{-\frac{1}{2}}dt.
\end{aligned}$$

Setting $t = \frac{1}{x}$ for the second integral, we get

$$\begin{aligned}
\omega(u) &= \int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt + \int_1^\infty \frac{1}{(A-B)+(A+B)\frac{1}{x}}x^{\frac{1}{2}} \cdot \frac{-1}{x^2}dx \\
&= \int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt + \int_0^1 \frac{1}{A+B+(A-B)x}x^{-\frac{1}{2}}dx \\
&= 2\int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt.
\end{aligned}$$

Setting $t^{\frac{1}{2}} = x$, we have

$$\begin{aligned}
\omega(u) &= 2\int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt \\
&= 2\int_0^1 \frac{1}{A+B+(A-B)x^2} \cdot \frac{1}{x} \cdot 2xdx \\
&= \frac{4}{A+B}\int_0^1 \frac{1}{1+\frac{A-B}{A+B}x^2}dx.
\end{aligned}$$

(i) For $-A < B < A$, we get

$$\omega(u) = \frac{4}{A+B}\int_0^1 \frac{1}{1+\frac{A-B}{A+B}x^2}dx = \frac{4}{\sqrt{A^2-B^2}}\arctan\sqrt{\frac{A-B}{A+B}},$$

(ii) For $B > A > 0$, we get

$$\begin{aligned}\omega(u) &= \frac{4}{A+B}\int_0^1 \frac{1}{1+\frac{A-B}{A+B}x^2}dx \\ &= \frac{4}{\sqrt{B^2-A^2}}\int_0^1 \frac{1}{1-(\sqrt{\frac{B-A}{A+B}}x)^2}d\sqrt{\frac{B-A}{A+B}}x \\ &= \frac{4}{\sqrt{B^2-A^2}}\ln\frac{B+\sqrt{B^2-A^2}}{A},\end{aligned}$$

(iii) For $B = A$, it turns into

$$\omega(u) = 2\int_0^1 \frac{1}{A+B+(A-B)t}t^{-\frac{1}{2}}dt = \frac{4}{A}.$$

Thus $\omega(u) = C$. In particular

$$C(1,0) = \int_0^\infty \frac{1}{u+v}(\frac{u}{v})^{\frac{1}{2}}dv = \int_0^\infty \frac{1}{1+t}t^{-\frac{1}{2}}dt = \pi,$$

$$C(1,1) = \int_0^\infty \frac{1}{u+v+|u-v|}(\frac{u}{v})^{\frac{1}{2}}dv = \int_0^\infty \frac{1}{2max\{u,v\}}(\frac{u}{v})^{\frac{1}{2}}dv = 2.$$

□

Lemma 2.2 *Suppose $\varepsilon > 0$, $A > 0$ and $B > -A$, then*

(2.1) $$\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} \frac{1}{A(1+t)+B|1-t|}t^{-\frac{1+\varepsilon}{2}}dt = O(1)(\varepsilon \to 0^+).$$

Proof. For $x \geq 1$, There exist $\varepsilon > 0$, which is small enough, such that $1+\frac{-1-\varepsilon}{2} > 0$, we have

$$\int_0^{\frac{1}{x}} \frac{1}{A(1+t)+B|1-t|}t^{-\frac{1+\varepsilon}{2}}dt < k\int_0^{\frac{1}{x}} t^{-\frac{1+\varepsilon}{2}}dt = \frac{kx^{-\beta}}{\beta},$$

where $k = \frac{1}{A}$ or $\frac{1}{A+B}$, $\beta = 1-\frac{1+\varepsilon}{2}$, and we can take $a = \frac{1}{4}$, if $\varepsilon < 1/2$, we get $\frac{x^{-\beta}}{\beta} < \frac{x^{-a}}{a}$, so

$$\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} \frac{1}{A(1+t)+B|1-t|}t^{-\frac{1+\varepsilon}{2}}dtdx < \frac{k}{a}\int_1^\infty x^{-1-a-\varepsilon}dx < \frac{k}{a^2} = 16k.$$

The lemma is proved. □

Now we study the following inequality:

Theorem 2.3 *Suppose* $f(x),\ g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, $0 < \int_0^\infty g^2(x)dx < \infty$, $A > 0$ *and* $B > -A$. *Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy < C\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}, \tag{2.2}$$

where the constant factor C *is the best possible. In particular*
(i) *for* $A = 1, B = 0$, *it reduces to Hilbert's inequality*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}.$$

(ii) *for* $A = 1, B = 1$, *it reduces to Hilbert's type inequality*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{max\{x,y\}}dxdy < 4\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\}^{\frac{1}{2}}.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \\
= & \int_0^\infty \int_0^\infty [\frac{f(x)}{[A(x+y)+B|x-y|]^{\frac{1}{2}}}(\frac{x}{y})^{\frac{1}{4}}] \times [\frac{g(y)}{[A(x+y)+B|x-y|)]^{\frac{1}{2}}}(\frac{y}{x})^{\frac{1}{4}}]dxdy \\
\leq & \int_0^\infty \int_0^\infty \frac{f^2(x)}{A(x+y)+B|x-y|}(\frac{x}{y})^{\frac{1}{2}}dxdy \times \int_0^\infty \int_0^\infty \frac{g^2(y)}{A(x+y)+B|x-y|}(\frac{y}{x})^{\frac{1}{2}}dxdy.
\end{aligned}$$

Define the weight function $\omega(u)$ as

$$\omega(u) := \int_0^\infty \frac{1}{A(u+v)+B|u-v|}(\frac{u}{v})^{\frac{1}{2}}dv,$$

then the above inequality yields

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \\
\leq & [\int_0^\infty \varpi(x)f^2(x)dx]^{\frac{1}{2}}[\int_0^\infty \varpi(y)g^2(y)dy]^{\frac{1}{2}}.
\end{aligned}$$

By lemma 2.1, we have $\varpi(u) = C$, thus

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \leq C\{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}\{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}}. \tag{2.3}$$

If (2.3) takes the form of the equality, then there exist constants a and b, such that they are not all zero and (see [5])

$$a\frac{f^2(x)}{A(x+y)+B|x-y|}(\frac{x}{y})^{\frac{1}{2}} = b\frac{g^2(y)}{A(x+y)+B|x-y|}(\frac{y}{x})^{\frac{1}{2}}$$

a.e. in $(0,\infty)\times(0,\infty)$.
Then we have

$$axf^2(x) = byg^2(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

Hence we have

$$axf^2(x) = byg^2(y) = constant \qquad \text{a.e. in } (0,\infty)\times(0,\infty),$$

which contradicts the facts that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Hence (2.3) takes the form of strict inequality. So we have (2.2).

For $0<\varepsilon<1$, setting $f_\varepsilon(x) = x^{\frac{-\varepsilon-1}{2}}$, for $x\in[1,\infty)$; $f_\varepsilon(x)=0$, for $x\in(0,1)$; $g_\varepsilon(y) = y^{\frac{-\varepsilon-1}{2}}$, for $y\in[1,\infty)$; $g_\varepsilon(y)=0$, for $y\in(0,1)$. Assume that the constant factor C is not the best possible, then there exists a positive number K with $K<C$, such that (2.2) is valid by changing C to K. We have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy < K\{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}\{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}} = K/\varepsilon.$$

Since

$$\int_0^\infty \frac{1}{A(1+t)+B|1-t|}t^{\frac{-1-\varepsilon}{2}}dt = C+o(1) \ (\varepsilon\to 0^+),$$

setting $y=xt$, by (2.1), we find

$$\begin{aligned} & \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \\ = & \int_1^\infty\int_1^\infty \frac{x^{\frac{-\varepsilon-1}{2}}y^{\frac{-\varepsilon-1}{2}}}{A(x+y)+B|x-y|}dxdy = \int_1^\infty\int_{x^{-1}}^\infty \frac{x^{\frac{-\varepsilon-1}{2}}(tx)^{\frac{-\varepsilon-1}{2}}}{A(1+t)+B|1-t|}dxdt \\ = & \int_1^\infty x^{-\varepsilon-1}[\int_0^\infty \frac{1}{A(1+t)+B|1-t|}t^{\frac{-1-\varepsilon}{2}}dt - \int_0^{x^{-1}} \frac{1}{A(1+t)+B|1-t|}t^{\frac{-1-\varepsilon}{2}}dt]dx \\ = & \frac{1}{\varepsilon}[C+o(1)]. \end{aligned}$$

Since for $\varepsilon>0$ small enough, we have$C+o(1)<K$. Thus we get $C\le K$, which contradicts the hypothesis. Hence the constant factor C in (2.2) is the best possible.
□

Theorem 2.4 *Suppose* $A>0, B>-A,\ f\ge 0$ *and* $0<\int_0^\infty f^2(x)dx<\infty$. *Then*

$$\int_0^\infty[\int_0^\infty \frac{f(x)}{A(x+y)+B|x-y|}dx]^2dy < C^2\int_0^\infty f^2(x)dx, \tag{2.4}$$

where the constant factor C^2 *is the best possible. Inequality* (2.4) *is equivalent to* (2.2).

Proof. Setting $g(y)$ as

$$\int_0^\infty \frac{f(x)}{A(x+y)+B|x-y|}dx,\ y \in (0,\infty),$$

then by (2.2), we find

$$\begin{aligned} 0 &< \int_0^\infty g^2(y)dy = \int_0^\infty [\int_0^\infty \frac{f(x)}{A(x+y)+B|x-y|}dx]^2 dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \le C\{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}\{\int_0^\infty g^2(y)dy\}^{\frac{1}{2}}. \end{aligned}$$

Hence we obtain

$$0 < \int_0^\infty g^2(y)dy \le C^2 \int_0^\infty f^2(x)dx < \infty. \tag{2.5}$$

By (2.2), both (2.5) and (2.6) take the form of strict inequality, so we have (2.4). On the other hand, suppose that (2.4) is valid. By Hölder's inequality, we find

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A(x+y)+B|x-y|}dxdy \\ = &\int_0^\infty [\int_0^\infty \frac{f(x)}{A(x+y)+B|x-y|}dx]g(y)dy \\ \le &\{\int_0^\infty [\int_0^\infty \frac{f(x)}{A(x+y)+B|x-y|}dx]^2 dy\}^{1/2}\{\int_0^\infty g^2(y)dy\}^{1/2}. \end{aligned} \tag{2.6}$$

Then by (2.4), we have (2.2). Thus (2.2) and (2.4) are equivalent.

If the constant C^2 in (2.4) is not the best possible, by (2.7), we may get a contradiction that the constant factor C in (2.2) is not the best possible. Thus we complete the proof of the theorem. □

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