Probability Density Function of Samples' Amplitude of ASSS OFDM Signal

Lei Wang · Dongweon Yoon · Sang Kyu Park

Abstract

The adaptive symbol selection scheme(ASSS) is popular in reducing peak to average power ratio(PAPR) for orthogonal frequency division multiplexing(OFDM) signals. The probability density function(pdf) of the samples' amplitudes of the adaptively selected OFDM signal without over-sampling has been considered to be approximately equal to the Rayleigh pdf. In this paper, we derive a more precise pdf which shows the relationship between the probability distribution of the samples' amplitudes and the number of the candidate symbols for ASSS. Using the newly derived pdf in the theoretical analysis, more accurate calculation results can be obtained.

Key words : Adaptive Symbol Selection Scheme, Clipping, PAPR Reduction, OFDM.

I. Introduction

Orthogonal frequency division multiplexing(OFDM) is considered to be a promising technique for wireless mobile communications. However, OFDM has a serious peak to average power ratio(PAPR) problem. There are many PAPR reduction techniques which are currently used to solve the PAPR problem[1]. For example, clipping technique[1] is the typical distortion technique, while the selected mapping(SLM) technique and the interleaving technique[1] are the typical distortionless techniques. Both SLM and interleaving technique use an adaptive symbol selection scheme(ASSS) in which one OFDM symbol with the smallest PAPR is selected from a number of statistically independent reference OFDM symbols[2]. In the previous works, for the adaptively selected OFDM signal without over-sampling, the probability distribution of the samples' amplitudes is considered as the Rayleigh distribution[2], and the exact probability density function (pdf) is derived in [3]. However, the computer simulation shows that the exact pdf significantly increases the calculation difficulty. Therefore, in this paper, we derive an approximated pdf for the samples' amplitudes of the adaptively selected OFDM signal. The newly derived pdf shows the effect of ASSS on the probability distribution of the samples' amplitudes. By using the derived pdf in the analysis of bit error rate (BER) performance for the adaptively selected OFDM signal, we can obtain more accurate calculation results than using Raleigh pdf and less calculation difficulty than the exact pdf.

II. Backgrounds

An OFDM symbol can be expressed as the sum of many independent symbols modulated onto subchannels of equal bandwidth. Let $X_k (k=0, 1, \cdots, N-1)$ denote the input data symbols whose period is T. Then the complex representation of an OFDM symbol is given as

$$x(t)=\sum_{k=0}^{N-1} X_k \cdot e^{j2\pi k \Delta f t}, \quad 0 \le t < NT \tag{1}$$

where N is the number of subcarriers, and $\Delta f=1/NT$ is the subcarrier spacing. The samples are denoted by $x_n (n=0, 1, \cdots, N-1)$ for the OFDM symbols with the Nyquist rate. The amplitude of the nth sample of an OFDM symbol is given as $r_n \triangleq |x_n|$. If N is a sufficiently large number, r_n is considered to be approximately equal to a Rayleigh random variable with pdf[2].

$$f_R(r_n)=\frac{2r_n}{P_{in}} e^{-r_n^2/P_{in}}, \quad r_n \ge 0 \tag{2}$$

where $P_{in}=2\sigma^2$ is the input power of the OFDM signal. The PAPR of the OFDM symbol is defined as the ratio of the peak power and the average power.

$$PAPR \triangleq \frac{\max\left[r_n^2\right]}{E\left[r_n^2\right]}=\frac{\max\left[r_n^2\right]}{P_{in}}, \quad n \in [0, N-1] \tag{3}$$

where E[·] denotes the statistical expectation function and max[·] gives the peak amplitude among the samples. For one OFDM symbol, the probability of the peak amplitude being smaller than a given threshold W can

Manuscript received April 3, 2008 ; June 4, 2008. (ID No. 20080403-016J)
Dept. of Electronics & Computer Engineering, Hanyang University, Seoul, Korea.

be obtained by [4]

$$F_l(W)=\Pr\left(\max_{0\le n<N} r_n<W\right)=\Pr(r_0<W)\cdot\Pr(r_1<W)\cdots$$
$$\Pr(r_{N-1}<W)=\left(1-e^{-W^2/P_{in}}\right)^N. \tag{4}$$

Thus, the cumulative distribution and the complementary cumulative distribution of the peak amplitude can be respectively given as

$$F_l(l)=\left(1-e^{-l^2/P_{in}}\right)^N \tag{5}$$

and

$$F_l^c(l)\triangleq 1-F_l(l)=1-\left(1-e^{-l^2/P_{in}}\right)^N. \tag{6}$$

One of the simplest methods to reduce PAPR might be deliberate clipping. Deliberate clipping limits the samples' amplitudes of the input OFDM signal to a predetermined value. The amplitude of the nth output sample of the clipped OFDM signal is given as

$$\tilde{r}_n \triangleq \begin{cases} r_n, & for\ r_n\le W \\ W, & for\ r_n\ge W, \end{cases} \tag{7}$$

and the power of the clipped OFDM signal therefore becomes

$$P_{out}=E\left[\tilde{r}_n^2\right]=\int_0^\infty \tilde{r}_n^2 f_R(r_n)dr_n. \tag{8}$$

Equation (7) shows that deliberate clipping is a memoryless nonlinear transformation. The output of the memoryless nonlinear transformation of OFDM signal x_n can be expressed as [2]

$$\tilde{x}_n=\alpha x_n+d_n \tag{9}$$

where d_n is the distortion term uncorrelated with x_n, and α is an attenuation factor, is expressed as

$$\alpha=\frac{E_{r_n,\phi_n}\left[r_n\cos\phi_n\tilde{r}_n\cos\phi_n\right]}{\sigma^2}=$$
$$\frac{\int_0^\infty r_n\tilde{r}_n f_R(r_n)dr_n\cdot\int_0^{2\pi}\cos^2\phi_n\cdot\frac{1}{2\pi}d\phi_n}{\sigma^2} \tag{10}$$

where ϕ_n is a random variable which has uniformly distribution on $[0,2\pi)$. As derived in [2], the signal-to-noise-plus-distortion-ratio(SNDR) of the clipped OFDM signal $\tilde{x}_n$ can be presented as

$$SNDR=\frac{\alpha^2P_{in}}{P_{out}-\alpha^2P_{in}+N_0} \tag{11}$$

where N_0 is the total variance of the AWGN. Therefore, the BER of QPSK OFDM signal after deliberate clipping can be given as

$$P_b=Q\left(\sqrt{SNDR}\right) \tag{12}$$

where $Q(x)=(1/2)erfc(x/\sqrt{2})=(1/\sqrt{2\pi})\int_x^\infty e^{-t^2/2}dt$.

Ⅲ. pdf of the Amplitudes of the Adaptively Selected OFDM Signal Samples

Deliberate clipping can be used with ASSS to obtain effective PAPR reduction and moderate system complexity[2]. The clipped-adaptively-selected(CAS) OFDM signal is the output of clipping an adaptively selected OFDM signal with ASSS. In ASSS, one OFDM symbol with the smallest PAPR is selected from M statistically independent OFDM symbols. In the previous work[2], the pdf of the peak amplitude of the selected OFDM signal samples was considered to be same as that of the non-selected OFDM signal. However, we found this consideration is not very precise. In the simulation test, we randomly generated 1 million non-selected OFDM symbols and 1 million adaptively selected OFDM symbols, then the probability distributions of their samples amplitudes were observed. Fig. 1 shows that the probability distribution for the non-selected OFDM signal can be approximated as the Rayleigh distribution, and the probability distribution for the adaptively selected OFDM signal has little difference from the Rayleigh distribution.

As given in [2], the probability of the peak amplitude of the selected OFDM symbol exceeding the given threshold W can be given as

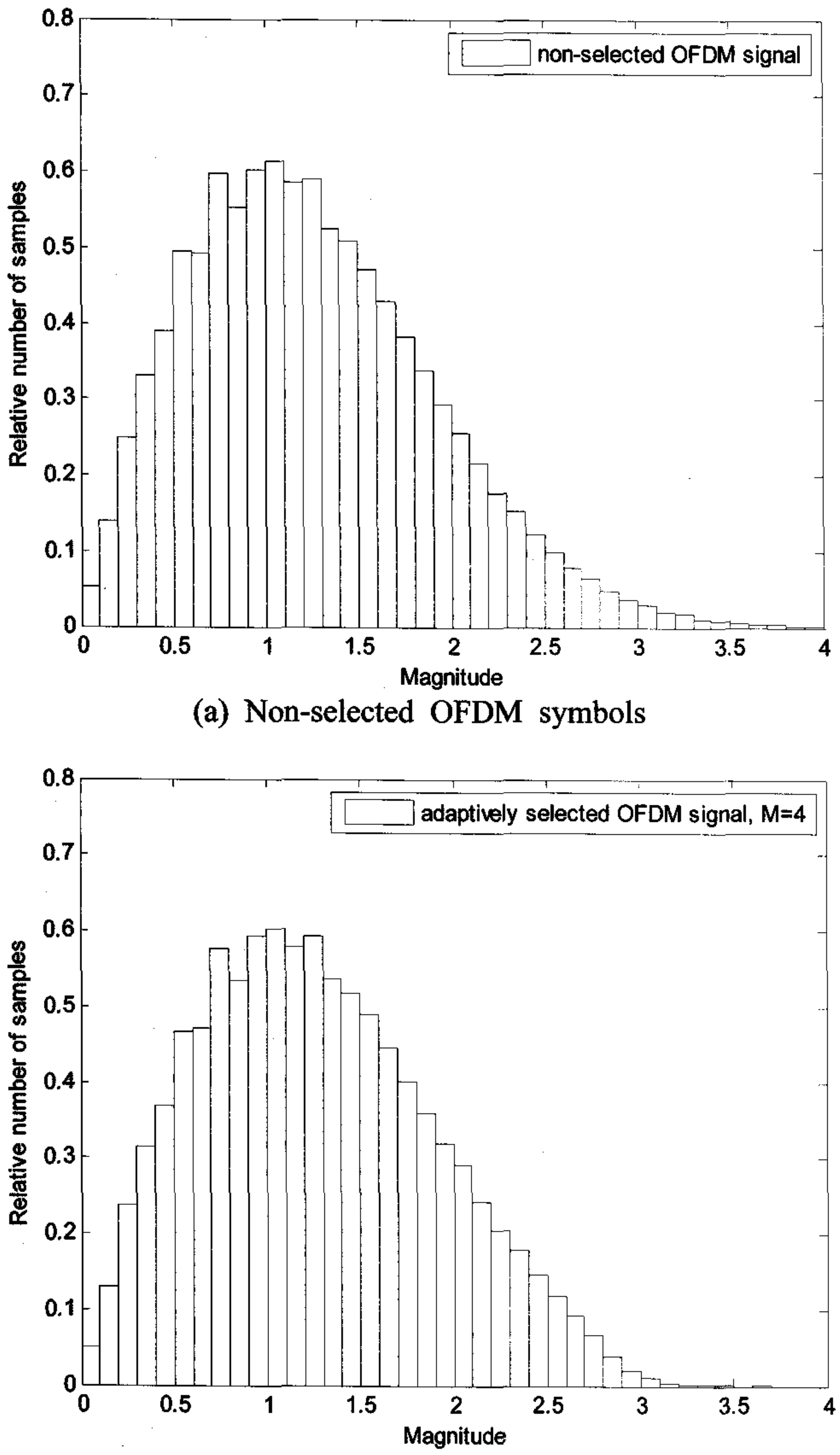
$$F_l^c(W)=\Pr\left(\min_{1\le m\le M} l_m>W\right)=\Pr(l_1>W)\Pr(l_2>W)\cdots$$
$$\Pr(l_m>W)=\left(1-F_{l_m}(W)\right)^M \tag{13}$$

where l_m is the peak amplitude of the mth OFDM symbol and $F_{l_m}(l)$ is the complementary cumulative distribution of l_m. Therefore the cumulative distribution of the peak amplitude of the selected OFDM symbol can be given as

$$F_l(l)=1-F_l^c(l)=1-\left(1-F_{l_m}(l)\right)^M. \tag{14}$$

For the adaptively selected OFDM symbol without oversampling, there are N statistically independent samples for one OFDM symbol period. Just as in (5), equation (14) can be also expressed by multiplying N cumulative density functions. Since these cumulative density functions are considered to be same for all samples of one OFDM symbol, the cumulative density function of the amplitude of the nth sample can be calculated from (5) and (14) as

$$\Pr(r_n<l)=\left(F_l(l)\right)^{1/N}=\left(1-\left(1-F_{l_m}(l)\right)^M\right)^{1/N}=$$
$$\left(1-\left(1-\left(1-e^{-l^2/P_{in}}\right)^N\right)^M\right)^{1/N}. \tag{15}$$



(a) Non-selected OFDM symbols

(b) Adaptively selected OFDM symbols

Fig. 1. Probability distribution of samples' amplitudes(QPSK modulation, 64 subcarriers).

By calculating the derivative of the function (15) and substituting r_n for l, we obtain the pdf of r_n as

$$\begin{aligned}\bar{f}_R(r_n) &= M\cdot\left(1-\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^M\right)^{1/N-1}\cdot\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^{M-1}\cdot\\ &\quad\left(1-e^{-r_n^2/P_{in}}\right)^{N-1}\cdot\frac{2r_n}{P_{in}}e^{-r_n^2/P_{in}}\\ &= M\cdot\left(1-\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^M\right)^{1/N-1}\cdot\left(1-e^{-r_n^2/P_{in}}\right)^{N-1}\cdot\\ &\quad\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^{M-1}\cdot\frac{2r_n}{P_{in}}e^{-r_n^2/P_{in}}\\ &= \mathbb{Z}\cdot\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^{M-1}\cdot\frac{2r_n}{P_{in}}e^{-r_n^2/P_{in}}.\\ \mathbb{Z} &= M\cdot\left(1-\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^M\right)^{1/N-1}\cdot\left(1-e^{-r_n^2/P_{in}}\right)^{N-1}\end{aligned} \tag{16}$$

To verify the validity of the pdf expression in (16), we confirmed that the integral $\int_0^\infty \bar{f}_R(r_n)dr_n$ becomes 1. From (16), we can see the relationship between the probability distribution of the samples' amplitudes and the number of the candidate OFDM symbols for ASSS. Computer simulation shows the calculation of equation (16) with large value of N is difficult by using conventional personal computer. For example, when N=64 and $r_n \ll 1$, we can get

$$A=\left(1-\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^M\right)^{1/N-1} \Rightarrow +\infty \tag{17}$$

and

$$B=\left(1-e^{-r_n^2/P_{in}}\right)^{N-1} \Rightarrow -\infty. \tag{18}$$

Because A is a very large number which is over the memory limit of computer, the computer cannot show the value of A, and approximates A as an infinitely large number. Therefore, $\mathbb{Z}=M\cdot A\cdot B$ is approximated as an infinitely large number. Consequently, $\bar{f}_R(r_n)$ is considered as an infinitely large number for $r_n \ll 1$. Obviously, this result is not correct. To overcome the problem caused by the approximation of computer for A, we try to find a new expression for equation (16). Observing the distribution of $\bar{f}_R(r_n)$ with a smaller N, we find that the shape of $\bar{f}_R(r_n)$ is very slightly affected by $\mathbb{Z}$, and $\mathbb{Z}$ mainly changes the scale of $\bar{f}_R(r_n)$. Therefore, we can presume $\mathbb{Z}$ is a constant number and approximate equation (16) as

$$\bar{f}_R(r_n)\approx\hat{f}_R(r_n)=a\cdot\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^{M-1}\cdot\frac{2r_n}{P_{in}}e^{-r_n^2/P_{in}} \tag{19}$$

where $a=1/\left(\int_0^\infty\left(1-\left(1-e^{-r_n^2/P_{in}}\right)^N\right)^{M-1}\cdot\frac{2r_n}{P_{in}}e^{-r_n^2/P_{in}}dr_n\right)$.

Computer calculation shows that when N is a large number and $r_n \ll 1$, a is not over the computer's memory limit. Therefore, we can obtain a limited value of $\hat{f}_R(r_n)$ for $r_n \ll 1$.

Ⅳ. Simulation Result

We verify the validity of the presumption and measure the error of derived pdf by a computer simulation. We randomly generated 1 million adaptively selected OFDM symbols with M=2, 4, and 8, respectively. The exact probability distributions of their samples' amplitudes were observed, where the sampling rate is 10. We sampled the Rayleigh pdf and corresponding derived pdf, and calculated the mean square errors(MSE) of the pdf to the exact probability distributions. In Table 1, we can see the MSE of derived pdf to the exact probability distribution is obviously smaller than the MSE of Rayleigh pdf to the exact probability distribution. As the candidate symbols for ASSS increases, the difference between exact probability distribution and the Rayleigh pdf becomes larger. From the table, we can see MSE of Rayleigh pdf increases as M increases. But the difference between the derived pdf and the exact probability distribution always maintains at a smaller level, regardless of changes of M.

Fig. 2 shows the normalized pdf $f_R(r_n)$ and $\hat{f}_R(r_n)$ with N=64 and M=4, and compares them with the probability distribution of samples' amplitudes for adaptively selected OFDM symbols given in Fig.1(a). We can see that the derived pdf is more close to the real probability distribution than the Rayleigh pdf. We applied the derived pdf in the BER analysis for CAS OFDM signal. Substituting (19) into (8), (10)~(12), we can calculate more accurate BER performance for CAS OFDM signal. Fig. 3 shows the theoretical BER performance of the CAS OFDM signal obtained by using the derived pdf and Rayleigh pdf, respectively. The clipping ratio(CR) is given as the ratio of the maximum permissible amplitude and rms power of the OFDM signal for deliberate clipping. Because the selected OFDM signal has smaller PAPR than non-selected OFDM signal, the distortion caused by deliberate clipping is less serious for selected OFDM signal. We can see that the BER performance obtained by derived pdf is better than that obtained by the Rayleigh pdf, and the difference of BER performances becomes larger as CR increases.

Table 1. MSE of pdf to real probability distribution.

	MSE of presumed pdf	MSE of Rayleigh pdf
M=2	1.1536×10^{-4}	2.1401×10^{-4}
M=4	1.1535×10^{-4}	4.3697×10^{-4}
M=8	1.1223×10^{-4}	7.9880×10^{-4}

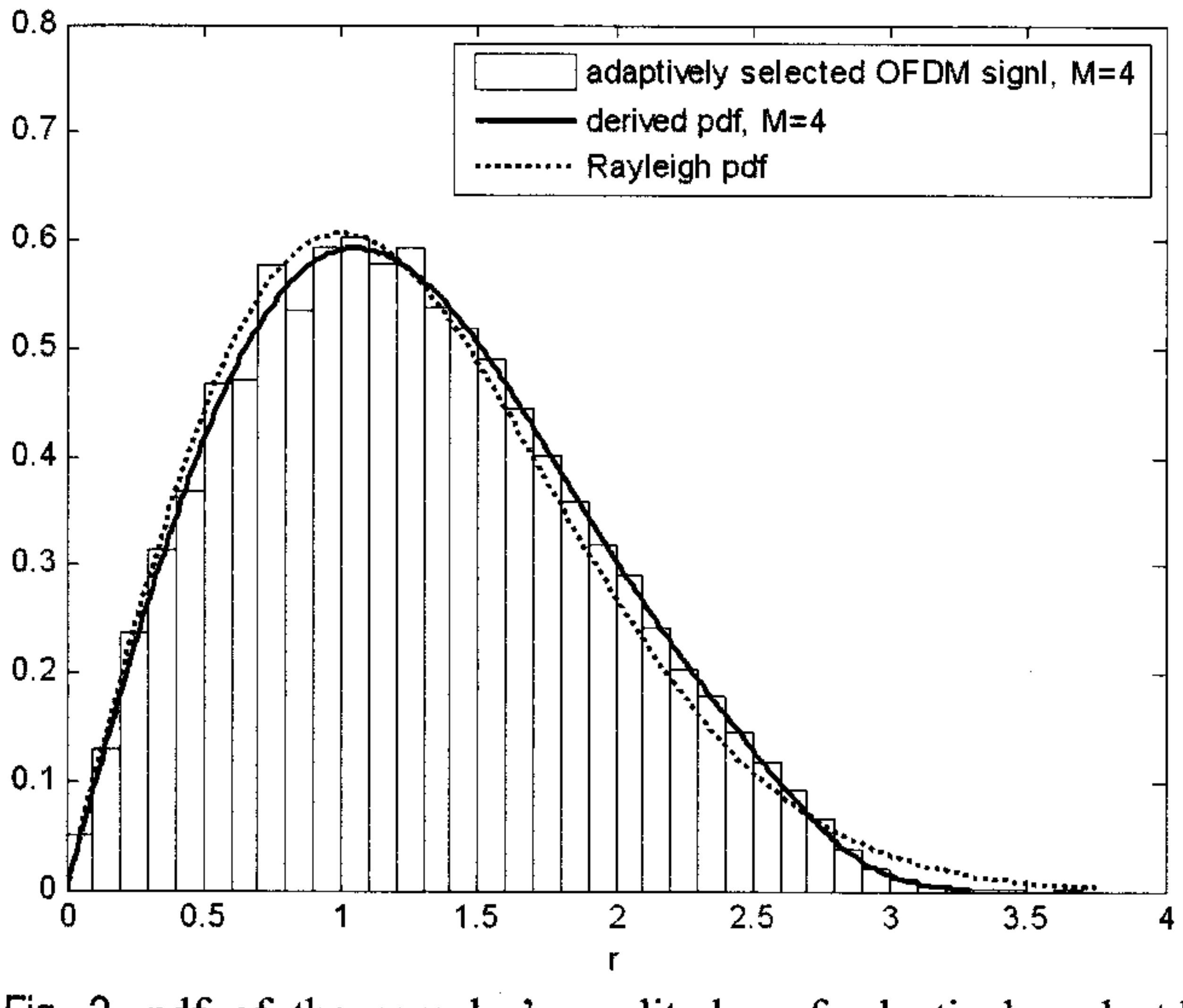


Fig. 2. pdf of the samples' amplitudes of adaptively selectd OFDM signal N=64.

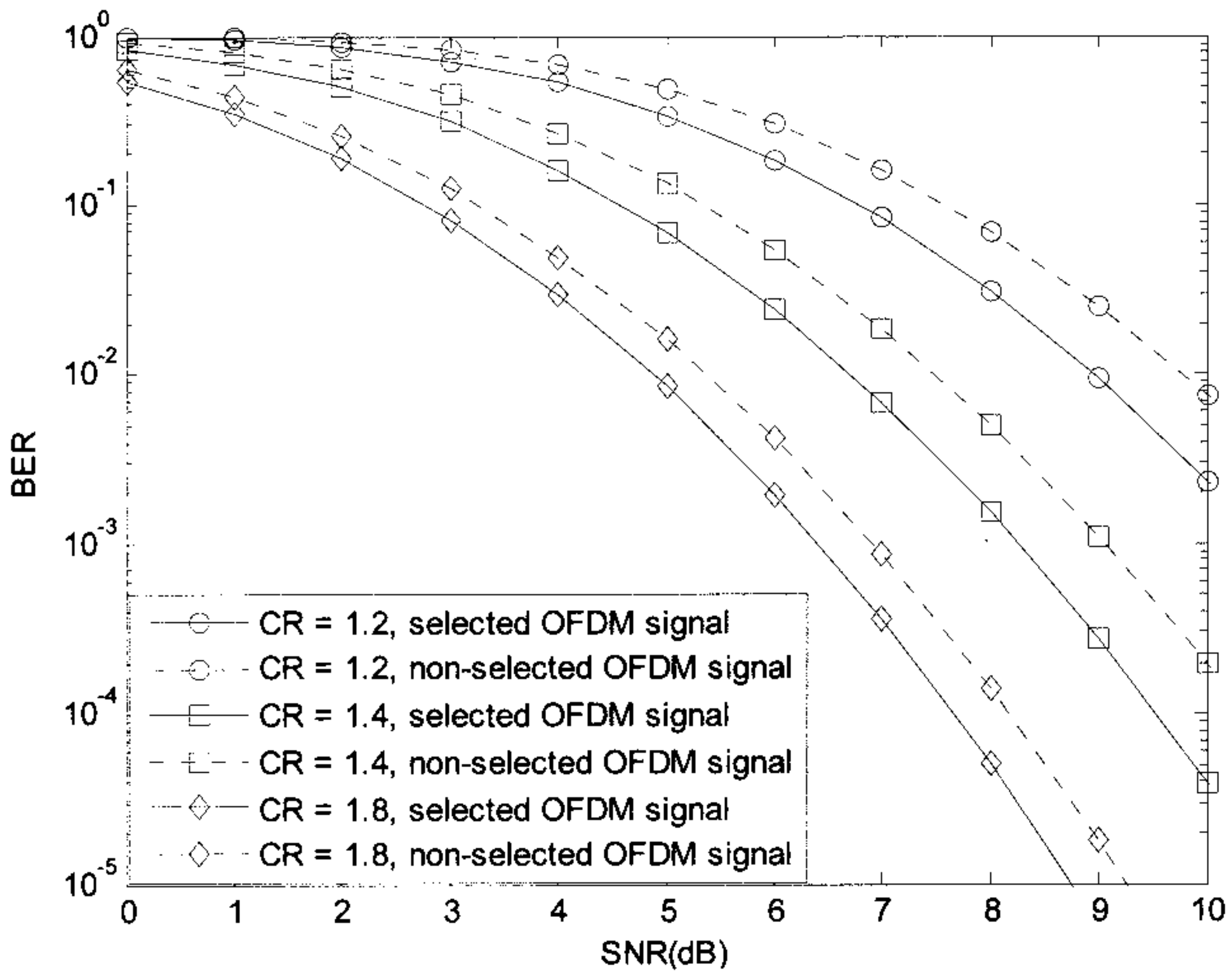


Fig. 3. Theoretical BER performance calculated by using derived pdf and Rayleigh pdf, N=64 and M=4.

V. Conclusion

In this paper, the probability distribution of the amplitudes of the adaptively selected OFDM signal samples has been approximately derived. The newly derived pdf shows the relationship between the probability distribution of the samples' amplitudes and the number of candidate OFDM symbols for ASSS. By using the derived pdf in the analysis of the error performance of the adaptively selected OFDM signal, more accurate calculation can be achieved than using Rayleigh pdf. In addition, the calculation difficulty is significantly reduced comparing with using the exact pdf.

References

[1] S. H. Han, J. H. Lee, "An overview of peak-to- average power ratio reduction techniques for multicarrier transmission", *IEEE Wireless Communications*, vol. 12, no. 2, pp. 56-65, Apr. 2005.

[2] H. Ochiai, H. Imai, "Performance of the deliberate clipping with adaptive symbol selection for strictly band-limited OFDM systems", *IEEE Journal on Selected Areas in Communications*, vol. 18, no. 11, pp. 2270-2277, Nov. 2000.

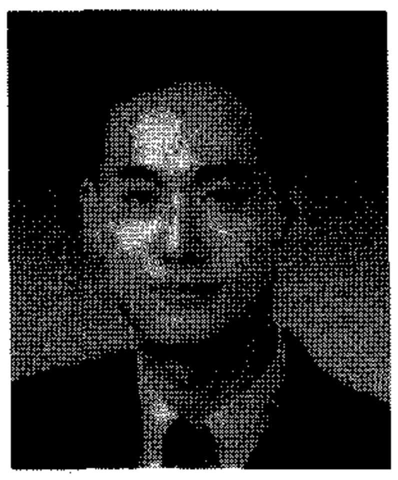
[3] L. Wang, D. Yoon, and S. K. Park, "Exact distribution of the amplitude of adaptively selected OFDM

signal samples", *IEICE Trannsaction on Communications*, vol. E91-B, no. 1, pp. 355-358, Jan. 2008.
[4] H. Ochiai, H. Imai, "On the distribution of the peak-to-average power ratio in OFDM symbols", *IEEE Trannsaction on Communications*, vol. 49, no. 2, pp. 282-289, Feb. 2001.

Lei Wang

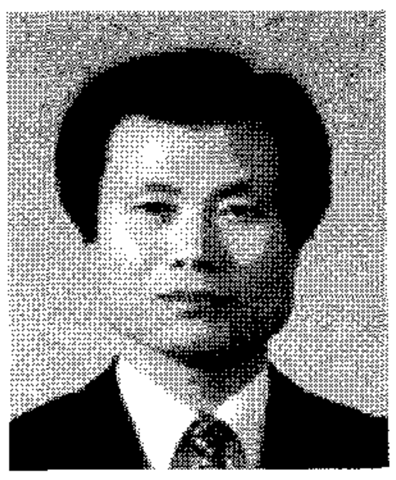
received the B.S. degree in school of electronics and information engineering of Dalian University of Technology, Dalian, China, in 2001, and M.S. degree in school of electrical, electronics and computer engineering of University of Newcastle upon Tyne, Newcastle, U.K., in 2003. He is currently working toward a Ph.D. degree in the department of electronics and computer engineering of Hanyang University, Seoul, Korea. He is a student member of the IEEE. His research interests lie in the field of new modulation techniques and digital communications theory, especially OFDM system.

Dongweon Yoon



received the B.S.(summa cum laude), M.S. and Ph.D. degrees in electronic communications engineering from Hanyang University, Seoul, Korea, in 1989, 1992 and 1995. From March 1995 to August 1997, he was an assistant professor in the division of electronic and information engineering of Dongseo University, Pusan, Korea. From September 1997 to February 2004, he was an Associate Professor in the division of Information and Communications Engineering of Daejeon University, Daejeon, Korea. Since March 2004, he has been on the faculty of Hanyang University, Seoul, Korea, where he is now a Professor in the department of electronics and computer engineering. He has twice been an Invited Researcher at the Electronics and Telecommunications Research Institute(ETRI), Daejeon, Korea (February-December 1997, November 2002- December 2005). He was a visiting Professor at the Pennsylvania State University, University Park, Pennsylvania, for the academic year 2001-2002. He has served as a consultant for a number of companies and given many lectures on the topics of digital communications and wireless communications. His research interests include new modulation techniques, accurate performance evaluations, digital communications theory and system, spread spectrum communications, and wireless communications.

Sang Kyu Park



received the B.S. degree from Seoul National University, Korea, in 1974, and the M.S. degree from Duke University, U.S.A. in 1980, and the Ph.D. degree from the University of Michigan, U.S.A. in 1987, all in electrical engineering. From July 1976 to October 1978, he was a Research Engineer at Agency for Defense Development, Korea. From August 1990 to August 1991, he was a Visiting Scholar at the University of Southern California, U.S.A. Since March 1987, he has been with the department of electronics and computer engineering at Hanyang University, Seoul, Korea, where he is currently a Professor. His research interests are in the areas of communications theory, wireless communications, mobile communications, spread spectrum communications, and secure communications.