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CONVERGENCE OF THE NEWTON METHOD FOR AUBIN CONTINUOUS MAPS

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Abstract. Motivated by optimization considerations we revisit the work by Dontchev in [7] involving the convergence of Newton's method to a solution of a generalized equation in a Banach space setting. Using the same hypotheses and under the same computational cost we provide a finer convergence analysis for Newton's method by using more precise estimates.

1. Introduction

In this study we are concerned with the problem of approximating a solution x of the generalized equation of the form

$$y \in f(x) + F(x), \quad x \in X \tag{1}$$

where y is a given parameter, f is a Fréchet–differentiable operator between Banach spaces X, Y and F is a map, possibly set-valued from X to 2^Y with a closed graph. If $F = \{0\}$, then (1) becomes an equation. Moreover if $F = \mathbb{R}^i_+$, the positive orthant in $\mathbb{R}^i$, then (1) is a system of inequalities. Furthermore, if F is a normal cone to a subset of X, then (1) is a variational inequality.

The most popular method for generating a sequence approximating x is undoubtedly Newton's method

$$y \in f(x_n) + f'(x_n)(x_{n+1} - x_n) + F(x_{n+1}), \tag{2}$$

where $f'(x)$ denotes the Fréchet–derivative of the operator f evaluated at x.

A survey on local as well as semilocal convergence results for Newton's method (2) can be found in [1]–[11] and the references there.

Here motivated by optimization considerations we revisit the work by Dontchev in [7]. Using the same hypotheses but more precise estimates and under the same computational cost we provide a finer convergence analysis for Newton's

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method (2).

2. Local convergence analysis of method (2)

We need to restate some terminology inaugurated in [1]. The distance from a point $x \in X$ to a set $S \subset X$ is given by

$$\operatorname{dist}(x, S) = \inf\{\|x - y\|, y \in S\}.$$

The excess e from the set S to the set W is given by

$$e(W, S) = \sup\{\operatorname{dist}(x, S), x \in W\}.$$

Given $F : X \longrightarrow 2^Y$, the inverse map F^{-1} is defined as $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ and Graph F is the set $\{(x, y) \in X \times Y,\ y \in F(x)\}$.

Aubin in [1] first introduced the concept of Aubin continuity: The map $\Gamma : X \longrightarrow 2^Y$ is said to be pseudo–Lipschitz about $(x_0, y_0) \in$ Graph Γ with modulus M if there exist neighborhoods V of y_0 and U of x_0 such that

$$e\big(\Gamma(y_1) \cap V, \Gamma(y_2)\big) \leq M\|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in V. \tag{3}$$

We need the auxiliary result:

Lemma 1. *Let $(x^*, y^*) \in$ Graph$(f + F)$, let f be a Fréchet–differentiable operator in an open neighborhood of x^*, let f' be continuous at x^* and let F have a closed graph. Moreover assume that the map $(f + F)^{-1}$ is Aubin continuous at (y^*, x^*). Then there exist positive constants α, β and M such that for every*

$$x \in U(x^*, \alpha) = \{x \in X \mid \|x - x^*\| \leq \alpha\},$$

if

$$G_x = \big[f(x) + f'(x)(\cdot - x) + F(\cdot)\big]^{-1},$$

then

$$e\big(G_x(v) \cap U(x^*, \alpha), G_x(w)\big) \leq M\|v - w\| \quad \textit{for all } v, w \in U(y^*, \beta). \tag{4}$$

Proof. The map $T = [f(x^*) + f'(x^*)(\cdot - x^*) + F(\cdot)]^{-1}$ is Aubin continuous at (y^*, x^*) [5]. Let a, b and M' be the corresponding constants. Choose $\varepsilon_0 > 0$ such that

$$M'\varepsilon_0 < 1, \tag{5}$$

$\alpha > 0$ such that

$$\|f'(x) - f'(x^*)\| \leq \varepsilon_0 \quad \text{for all } x \in V((x^*, \alpha), \tag{6}$$

and $\beta > 0$ such that

$$\beta + 4\varepsilon_0\alpha \leq b \quad \text{and} \quad \frac{2M'\beta}{1 - M'\varepsilon_0} \leq \alpha. \tag{7}$$

The rest of the proof follows exactly as in Lemma 1 in [7, p. 388] by simply replacing ε used there by ε_0 used here and setting

$$M = \frac{M'}{1 - M'\varepsilon_0}\,. \tag{8}$$

That completes the proof of Lemma 1. □

We can now show the following local convergence result for Newton's method:

Theorem 1. *Let x^* be a solution of equation (1) for $y = 0$, let f be a Fréchet-differentiable operator in an open neighborhood D of x^*, and let f' be continuous in D. Let F have a closed graph.*

Then the following are equivalent:

(a) *The map $(f + F)^{-1}$ is Aubin continuous at $(0, x^*)$;*
(b) *There exist positive constants σ, b, and c such that for every $y \in U(0, b)$ and for every $x_0 \in U(x^*, \sigma)$ there exists a Newton sequence $\{x_n\}$ starting from x_0 which converges to a solution x of (1) for y.*

Moreover, if x_0 is a solution of (1) for y_0, then

$$\|x - x_0\| \le c\|y - y_0\|. \tag{9}$$

Proof. (1) (b) $\Rightarrow$ (a) follows from the definition of Aubin continuity.
(2) (a) $\Rightarrow$ (b). We use Lemma 1. Let α, β and M be the constants introduced in Lemma 1. Define mapping G_x on $U(x^*, \alpha)$ by

$$G_x = \big[f(x) + f'(x)(\cdot - x) + F(\cdot)\big]^{-1}.$$

Let $\varepsilon > 0$ such that

$$M\varepsilon < 1, \tag{10}$$

and choose $a > 0$, $\sigma > 0$, $b > 0$ such that $U(x^*, a) \subseteq D$ and

$$\|f'(v) - f'(w)\| \le \varepsilon \quad \text{for all } v, w \in U(x^*, a), \tag{11}$$

$$\sigma \le \alpha, \quad \frac{2\sigma}{1 - M\varepsilon} < a, \quad 2\varepsilon\sigma < \beta \tag{12}$$

$$b(1 + M\varepsilon) + 2\varepsilon\sigma \le \beta \ \text{ and } \ \frac{Mb + 2\sigma}{1 - M\varepsilon} \le a. \tag{13}$$

The rest of the proof follows exactly as in Theorem 1 in [7, p. 390] with

$$c = \frac{M}{1 - M\varepsilon}\,. \tag{14}$$

That completes the proof of Theorem 1. □

If f' is Lipschitz continuous about x^*, then the Aubin continuity implies the existence of a Q–quadratically convergent Newton sequence:

Theorem 2. *Let x^* be a solution of (1) for $y = 0$, let f be a Fréchet–differentiable operator in an open neighborhood D of x^*, let f' be L–Lipschitz continuous in D. Let F have closed graph and let $(f+F)^{-1}$ be Aubin continuous at $(0, x^*)$. Then there exist positive constants σ, b and γ such that for every $y \in U(0,b)$ and for every $x_0 \in U(x^*, \sigma)$ there exists a sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton's method (2) and starting at x_0 converging to a solution x of (1) for y so that*

$$\|x_{n+1} - x^*\| \leq \gamma \|x_n - x^*\|^2 \quad (n \geq 0), \tag{15}$$

where,

$$\gamma \geq \frac{ML}{2}\,. \tag{16}$$

Proof. Exactly as the proof of Theorem 2 in [7, p. 393]. □

Remark 1. In view of (6) and (11) we have that

$$\varepsilon_0 \leq \varepsilon \tag{17}$$

holds in general and $\dfrac{\varepsilon}{\varepsilon_0}$ can be arbitrarily large [2], [3]. Note that we can certainly set $\varepsilon \geq 2\varepsilon_0$. If $\varepsilon_0 = \varepsilon$ our results reduce to the corresponding ones in [7]. Otherwise our results constitute an improvement under the same hypotheses and computational cost. Indeed denote by $\overline{M}$, $\overline{c}$, $\overline{\gamma}$ the corresponding to M, c, γ constants used in [7], respectively. That is

$$\overline{M} = \frac{M'}{1 - M'\varepsilon}\,, \tag{18}$$

$$\overline{c} = \frac{\overline{M}}{1 - \overline{M}\varepsilon}\,, \tag{19}$$

and

$$\overline{\gamma} \geq \frac{\overline{M}L}{2}\,. \tag{20}$$

If strict inequality holds in (17) it follows by (8), (13), (15), (18) and (19) that

$$M < \overline{M}, \tag{21}$$

$$c < \overline{c}, \tag{22}$$

and

$$\gamma < \overline{\gamma}. \tag{23}$$

Due to (4), (9), (15) and (21)–(23) the claims made in the introduction are satisfied. Hence the usefulness of our results follows.

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