NONDIFFERENTIABLE SECOND-ORDER MINIMAX MIXED INTEGER SYMMETRIC DUALITY

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Abstract. In this paper, a pair of Wolfe type nondifferentiable second order symmetric minimax mixed integer dual problems is formulated. Symmetric and self-duality theorems are established under η_1-bonvexity/η_2-boncavity assumptions. Several known results are obtained as special cases. Examples of such primal and dual problems are also given.

1. Introduction

Minimax mathematical programming has applications in several optimization problems, e.g., game theory, the design of electronic circuits, approximation theory and various situations relating to decision making under uncertainty.

Dorn [6] introduced symmetric dual for quadratic programming problems. Subsequently, Dantzig et al. [5] and Bazaraa and Goode [3] studied symmetric duality in nonlinear programming assuming the kernel function $f(x, y)$ to be convex in the first variable and concave in the second variable. Balas [2] generalized symmetric duality of Dantzig et al. [5] by constraining some of the primal and dual variables to belong to arbitrary sets and thus introduced minimax symmetric dual programs.

Mangasarian [10] considered a nonlinear programs and discussed second order duality under certain inequalities. Mond [12] established Mangasarian's duality relations assuming $f(x, y)$ to be bonvex/boncave. Mangasarian [10, p. 609], Mond [12, p. 93] and Hanson [8, p. 316] have indicated usefulness of second order dual over the first order dual.

Hou and Yang [9] considered Mond-Weir type second order symmetric duality involving nondifferentiable functions. Gulati and Gupta [7] discussed a pair of Wolfe type nondifferentiable second order symmetric programs involving

support functions and relaxing the nonnegativity conditions in the problems studied by Yang et al. [13].

In this paper, we consider a pair of Wolfe type nondifferentiable second order symmetric minimax mixed integer programs and use the results of Gulati and Gupta [7] to establish symmetric and self-duality theorems under η_1-bonvexity/η_2-boncavity assumptions. The duality results obtained in this paper extend some of the known results in the literature. Examples have also been given in the end.

2. Notations and preliminaries

Let $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Let U and V be two arbitrary sets of integers in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{m_1}$, respectively. Throughout this paper, we constrain some of the components of x and y to belong to arbitrary sets of integers as in Balas [2]. Suppose that the first n_1 $(0 \leq n_1 \leq n)$ components of x belong to U and the first m_1 $(0 \leq m_1 \leq m)$ components of y belong to V. Then we write $(x, y) = (x^1, x^2, y^1, y^2)$ where $x^1 = (x_1, x_2, \ldots, x_{n_1})$ and $y^1 = (y_1, y_2, \ldots, y_{m_1})$, x^2 and y^2 belong to $\mathbb{R}^{n-n_1}$ and $\mathbb{R}^{m-m_1}$, respectively. Let $k(x, y)$ be a real valued twice differentiable function defined on an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\nabla_{x^2} k(\bar{x}, \bar{y})$ denotes the gradient vector of k with respect to x^2 at $(\bar{x}, \bar{y})$. Also let $\nabla_{x^2 x^2} k(\bar{x}, \bar{y})$ denotes the Hessian matrix with respect to x^2 evaluated at $(\bar{x}, \bar{y})$. $\nabla_{y^2} k(\bar{x}, \bar{y})$ and $\nabla_{y^2 y^2} k(\bar{x}, \bar{y})$ are defined similarly.

Definition 1. A real twice differentiable function f defined on a set $X \times Y$, where X and Y are open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, is said to be η_1-bonvex in the first variable at $u \in X$, if there exists a function $\eta_1 : X \times X \to \mathbb{R}^n$ such that for $v \in Y$, $r \in \mathbb{R}^n$, $x \in X$,

$$f(x, v) - f(u, v) \geq \eta_1^T(x, u)[\nabla_x f(u, v) + \nabla_{xx} f(u, v)r] - \frac{1}{2} r^T \nabla_{xx} f(u, v)r,$$

and $f(x, y)$ is said to be η_2-bonvex in the second variable at $v \in Y$, if there exists a function $\eta_2 : Y \times Y \to \mathbb{R}^m$ such that for $u \in X$, $p \in \mathbb{R}^m$, $y \in Y$,

$$f(u, y) - f(u, v) \geq \eta_2^T(y, v)[\nabla_y f(u, v) + \nabla_{yy} f(u, v)p] - \frac{1}{2} p^T \nabla_{yy} f(u, v)p.$$

A twice differentiable function f is η-boncave if $-f$ is η-bonvex.

Definition 2. Let C be a compact convex set in $\mathbb{R}^n$. The support function of C is defined by

$$S(x|C) = \max \{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in \mathbb{R}^n$ such that

$$S(y|C) \geq S(x|C) + z^T(y - x) \text{ for all } y \in C.$$

The subdifferential of $S(x|C)$ is given by

$$\partial S(x|C) = \{z \in C : z^T x = S(x|C)\}.$$

For any set $S \subset \mathbb{R}^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in \mathbb{R}^n : y^T(z - x) \le 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set C, y is in $N_C(x)$ if and only if $S(y|C) = x^T y$, or equivalently, x is in $\partial S(y|C)$.

Definition 3. Let $s^1, s^2, \ldots, s^p$ be elements of an arbitrary vector space. A vector function $G(s^1, s^2, \ldots, s^p)$ will be called additively separable with respect to s^1 if there exist vector functions $H(s^1)$ (independent of $s^2, \ldots, s^p$) and $K(s^2, \ldots, s^p)$ (independent of s^1), such that

$$G(s^1, s^2, \ldots, s^p) = H(s^1) + K(s^2, \ldots, s^p).$$

3. Wolfe type minimax mixed integer programming

We now consider the following pair of Wolfe type nondifferentiable second order minimax mixed integer symmetric dual programs:

Primal Problem (SP):

$$\text{Max}_{x^1} \text{Min}_{x^2, y, z} \; M(x, y, p) = f(x, y) + S(x^2|C) - (y^2)^T \nabla_{y^2} f(x, y)$$
$$- (y^2)^T \nabla_{y^2 y^2} f(x, y) p - \frac{1}{2} p^T \nabla_{y^2 y^2} f(x, y) p$$

subject to

(1) $$\nabla_{y^2} f(x, y) - z + \nabla_{y^2 y^2} f(x, y) p \le 0,$$

(2) $$z \in D,$$

(3) $$x^1 \in U, \; y^1 \in V.$$

Dual Problem (SD):

$$\text{Min}_{v^1} \text{Max}_{u, v^2, w} \; N(u, v, r) = f(u, v) - S(v^2|D) - (u^2)^T \nabla_{x^2} f(u, v)$$
$$- (u^2)^T \nabla_{x^2 x^2} f(u, v) r - \frac{1}{2} r^T \nabla_{x^2 x^2} f(u, v) r$$

subject to

(4) $$\nabla_{x^2} f(u, v) + w + \nabla_{x^2 x^2} f(u, v) r \ge 0,$$

(5) $$w \in C,$$

(6) $$u^1 \in U, \; v^1 \in V,$$

where
  1. f is a differentiable function from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,
  2. r, w are vectors in $\mathbb{R}^{n-n_1}$ and p, z are vectors in $\mathbb{R}^{m-m_1}$, and
  3. C and D are compact convex sets in $\mathbb{R}^{n-n_1}$ and $\mathbb{R}^{m-m_1}$, respectively.

Theorem 1 (Symmetric duality). *Let $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ be an optimal solution for* (SP). *Also, let*
  (i) $f(x, y)$ *be additively separable with respect to* x^1 *or* y^1;

(ii) $f(u,v)+(u^2)^T w$ be η_1-bonvex in u^2 for each (u^1,v) and $f(x,y)-(y^2)^T z$ be η_2-boncave in y^2 for each (x,y^1);

(iii) $f(x,y)$ be thrice differentiable in x^2 and y^2;

(iv) $\nabla_{y^2 y^2} f(\bar{x},\bar{y})$ be non-singular;

(v) one of the matrices $\frac{\partial}{\partial y_i{}^2}(\nabla_{y^2 y^2} k(\bar{x},\bar{y}))$, $i = m_1 + 1, m_2 + 1, \ldots, m$ be positive or negative definite;

(vi) $\eta_1(x^2,u^2) + u^2 \geq 0$ and $\eta_2(v^2,y^2) + y^2 \geq 0$ for any feasible solution (x,y,z,p) in (SP) and any feasible solution (u,v,w,r) in (SD).

Then, $\bar{p} = 0$ and there exist $\bar{w}$ such that $(\bar{x},\bar{y},\bar{w},\bar{r} = 0)$ is an optimal solution for (SD) and the values of two objective functions are equal.

Proof. Let

$$q = \text{Max}_{x^1} \text{Min}_{x^2,y,z} \ \{f(x,y) + S(x^2|C) - (y^2)^T \nabla_{y^2} f(x,y) - (y^2)^T \nabla_{y^2 y^2} f(x,y)p$$
$$- \frac{1}{2}p^T \nabla_{y^2 y^2} f(x,y)p : \ (x,y,z,p) \in S\}$$

and

$$t = \text{Min}_{v^1} \text{Max}_{u,v^2,w} \ \{f(u,v) - S(v^2|D) - (u^2)^T \nabla_{x^2} f(u,v) - (u^2)^T \nabla_{x^2 x^2} f(u,v)r$$
$$- \frac{1}{2}r^T \nabla_{x^2 x^2} f(u,v)r : \ (u,v,w,r) \in T\},$$

where S and T are feasible regions of primal (SP) and dual (SD) respectively. Since $f(x,y)$ is additively separable with respect to x^1 or y^1 (say with respect to x^1), it follows that

$$(7) \qquad\qquad f(x,y) = f^1(x^1) + f^2(x^2,y).$$

Therefore $\nabla_{y^2} f(x,y) = \nabla_{y^2} f^2(x^2,y)$ and q can be written as

$$q = \text{Max}_{x^1} \text{Min}_{x^2,y,z} \ \{ \ f^1(x^1) + f^2(x^2,y) + S(x^2|C) - (y^2)^T \nabla_{y^2} f^2(x^2,y)$$
$$- (y^2)^T \nabla_{y^2 y^2} f^2(x^2,y)p - \frac{1}{2}p^T \nabla_{y^2 y^2} f^2(x^2,y)p :$$
$$\nabla_{y^2} f^2(x^2,y) - z + \nabla_{y^2 y^2} f^2(x^2,y)p \leq 0,$$
$$z \in D, \ x^1 \in U, \ y^1 \in V\},$$

or

$$(8) \qquad q = \text{Max}_{x^1} \ \text{Min}_{y^1} \ \{f^1(x^1) + \phi(y^1) : \ x^1 \in U, \ y^1 \in V\},$$

where

$$\phi(y^1) = \ \text{Min}_{x^2,y^2,z} \ \{ \ f^2(x^2,y) + S(x^2|C) - (y^2)^T \nabla_{y^2} f^2(x^2,y)$$
$$(9) \qquad\qquad - (y^2)^T \nabla_{y^2 y^2} f^2(x^2,y)p - \frac{1}{2}p^T \nabla_{y^2 y^2} f^2(x^2,y)p :$$
$$\nabla_{y^2} f^2(x^2,y) - z + \nabla_{y^2 y^2} f^2(x^2,y)p \leq 0, \ z \in D\}.$$

Similarly,

$$(10) \qquad t = \text{Min}_{v^1} \ \text{Max}_{u^1} \ \{f^1(u^1) + \psi(v^1) : \ u^1 \in U, \ v^1 \in V\},$$

where
(11)

$$\psi(v^1) = \text{Max}_{u^2,v^2,w} \{ f^2(u^2,v) - S(v^2|D) - (u^2)^T \nabla_{x^2} f^2(u^2,v)$$
$$- (u^2)^T \nabla_{x^2 x^2} f^2(u^2,v)r - \frac{1}{2} r^T \nabla_{x^2 x^2} f^2(u^2,v)r :$$
$$\nabla_{x^2} f^2(u^2,v) + w + \nabla_{x^2 x^2} f^2(u^2,v)r \geq 0, \ w \in C\}.$$

For any given y^1 and v^1, programs (9) and (11) are a pair of Wolfe type second order symmetric dual non-differentiable programs of Gulati and Gupta [7] and hence in view of hypotheses (ii)-(vi), Theorem 3.2 in [7] becomes applicable. Therefore, for $y^1 = \bar{y}^1 = v^1$ we obtain

$$\bar{p} = 0 \text{ and } \phi(\bar{y}^1) = \psi(\bar{y}^1).$$

Now, we need only to show that $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ is optimal for (SD). If this is not the case, then there exist $y^{*1} \in V$ such that $\psi(y^{*1}) < \psi(\bar{y}^1)$. But then, in view of assumptions (iv) and (v), we have

$$\phi(\bar{y}^1) = \psi(\bar{y}^1) > \psi(y^{*1}) = \phi(y^{*1}),$$

contradicting the optimality of $(\bar{x}, \bar{y}, \bar{z}, \bar{p} = 0)$ for (SP). Hence $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ is an optimal solution for (SD). $\square$

4. Self duality

A mathematical problem is said to be self dual if it is formally identical with its dual, that is, if the dual is recast in the form of the primal, the new problem so obtained is the same as the primal. In general, (SP) and (SD) are not self dual without an added restriction on f. The vector function $f : \mathbb{R}^n \times R^m \mapsto \mathbb{R}$ is said to be skew symmetric if for all $x, y \in \mathbb{R}^n$, $f(y,x) = -f(x,y)$.

Theorem 2. *Let* $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ *be skew symmetric and* $D = C$. *Then* (SP) *is a self dual. Furthermore, if* (SP) *and* (SD) *are dual programs and* $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ *is an optimal solution for* (SP), *then* $\bar{p} = 0$, $(\bar{y}, \bar{x}, \bar{z}, \bar{r} = 0)$ *is an optimal solution for* (SD). *Moreover, if* $y^2 \geq 0$ *and* $u^2 \geq 0$, *then the values of two objective functions are equal to zero.*

Proof. The dual problem (SD) can be written as

$$\text{Max}_{v^1} \text{Min}_{u,v^2,w} - f(u,v) + S(v^2|D) + (u^2)^T \nabla_{x^2} f(u,v)$$
$$+ (u^2)^T \nabla_{x^2 x^2} f(u,v)r + \frac{1}{2} r^T \nabla_{x^2 x^2} f(u,v)r$$

subject to

$$-\nabla_{x^2} f(u,v) - w - \nabla_{x^2 x^2} f(u,v)r \leq 0,$$
$$w \in C,$$
$$u^1 \in U, \ v^1 \in V.$$

Since k is skew symmetric, $\nabla_{x^2} f(u,v) = -\nabla_{y^2} f(v,u)$ and $\nabla_{x^2 x^2} f(u,v) = -\nabla_{y^2 y^2} f(v,u)$.

Also, since $D = C$, the above problem becomes

$$\text{Max}_{v^1} \text{Min}_{u,v^2,w} f(v,u) + S(v^2|C) - (u^2)^T \nabla_{y^2} f(v,u) - (u^2)^T \nabla_{y^2 y^2} f(v,u) r$$
$$- \frac{1}{2} r^T \nabla_{y^2 y^2} f(v,u) r$$

subject to

$$\nabla_{y^2} f(v,u) - w + \nabla_{y^2 y^2} f(v,u) r \leq 0,$$
$$w \in D,$$
$$u^1 \in U, \ v^1 \in V,$$

which is the primal problem (SP). Thus (SP) is a self dual problem. Hence if $(\bar{x}, \bar{y}, \bar{w}, \bar{p})$ is optimal for (SP), then $\bar{p} = 0$ and $(\bar{y}, \bar{x}, \bar{w}, \bar{r} = 0)$ is optimal for (SD). Also, $M(\bar{x}, \bar{y}, \bar{p}) = N(\bar{y}, \bar{x}, \bar{r})$. Now we show that $M(\bar{x}, \bar{y}, \bar{p}) = 0$.

$$M(\bar{x}, \bar{y}, \bar{p}) = f(\bar{x}, \bar{y}) + S(\bar{x}^2|C) - (\bar{y}^2)^T \nabla_{y^2} f(\bar{x}, \bar{y}) - (\bar{y}^2)^T \nabla_{y^2 y^2} f(\bar{x}, \bar{y}) \bar{p}$$
$$- \frac{1}{2} \bar{p}^T \nabla_{y^2 y^2} f(\bar{x}, \bar{y}) \bar{p}$$
$$\geq f(\bar{x}, \bar{y}) + S(\bar{x}^2|C) - (\bar{y}^2)^T z \ \text{(using (1), } \bar{y}^2 \geq 0 \text{ and } \bar{p} = 0)$$
$$\geq f(\bar{x}, \bar{y}) + S(\bar{x}^2|C) - S(\bar{y}^2|D) \ \text{(since } (\bar{y}^2)^T z \leq S(\bar{y}^2|D))$$
$$\geq f(\bar{x}, \bar{y}). \ \text{(using } C = D)$$

Similarly, $N(\bar{x}, \bar{y}, \bar{r}) \leq f(\bar{x}, \bar{y})$. Hence by Theorem 1,

$$f(\bar{x}, \bar{y}) \leq M(\bar{x}, \bar{y}, \bar{p}) = N(\bar{x}, \bar{y}, \bar{r}) \leq f(\bar{x}, \bar{y}),$$

which implies

$$M(\bar{x}, \bar{y}, \bar{p}) = N(\bar{y}, \bar{x}, \bar{r}) = f(\bar{x}, \bar{y}) = f(\bar{y}, \bar{x}) = -f(\bar{x}, \bar{y}).$$

and therefore $M(\bar{x}, \bar{y}, \bar{p}) = 0$. $\qquad \square$

5. Special cases

(i) Let U and V be empty sets. Then (SP) and (SD) are reduced to the second order nondifferentiable symmetric dual programs of Gulati and Gupta [7].

(ii) Let U and V be empty sets and $C = \{Ay : y^T Ay \leq 1\}$, $D = \{Bx : x^T Bx \leq 1\}$ and $\eta_1(x,u) = x - u$, $\eta_2(v,y) = v - y$, where A and B are positive semidefinite matrices with $p = 0$ and $r = 0$, then $(x^T Ax)^{1/2} = S(x|C)$ and $(y^T By)^{1/2} = S(y|D)$. In this case (SP) and (SD) reduce to the problems (WP) and (WD) as considered in Ahmad and Husain [1].

(iii) Let $p = 0$ and $r = 0$, $\eta_1(x,u) = x - u$, $\eta_2(v,y) = v - y$. Then we obtain the mixed integer primal (P) and dual (D) considered in Chandra and Abha [4].

(iv) If we take $C = \{0\}$ and $D = \{0\}$, then our programs reduce to the problem (NP) and (ND) studied in Mishra [11].

6. Examples

Let $n = m = 3$, $x^1 = (x_1, x_2)$, $x^2 = x_3$, $y^1 = (y_1, y_2)$, and $y^2 = y_3$.

1. Let $f(x, y) = x_1^2 + x_2^2 + x_3^2 - e^{y_1} - e^{y_2} - e^{y_3}$, $C = [0, 1]$ and $D = \{0\}$. Then

$$S(x^2|C) = \frac{x_3 + |x_3|}{2}, \ S(v^2|D) = 0$$

and our problems (SP)and (SD) reduce to

Primal problem (SP1):

$$\text{Max}_{x^1}\text{Min}_{x^2,y,z} \ x_1^2 + x_2^2 + x_3^2 - e^{y_1} - e^{y_2} - e^{y_3} + \frac{x_3 + |x_3|}{2}$$
$$+ e^{y_3}(y_3 + y_3 p + \frac{1}{2}p^2)$$

subject to

$$e^{y_3}(1 + p) \geq 0,$$
$$x^1 \in U, \ y^1 \in V.$$

Dual problem (SD1):

$$\text{Min}_{v^1}\text{Max}_{u,v^2,w} \ u_1^2 + u_2^2 + u_3^2 - e^{v_1} - e^{v_2} - e^{v_3} - 2(u^3)^2 - 2u^3 r - r^2$$

subject to

$$2x_3 + w + 2r \geq 0,$$
$$w \in [0, 1],$$
$$u^1 \in U, \ v^1 \in V.$$

Therefore, our results give the duality relations for (SP1) and (SD1), which cannot be obtained from the work in Chandra and Abha [4] as the above dual pair is a second order minimax mixed integer problem involving a nondifferentiable term.

2. Let $f(x, y) = e^{x_1} + e^{x_2} + e^{x_3} - e^{y_1} - e^{y_2} - e^{y_3}$, $C = D = [0, 1]$. Then

$$S(x^2|C) = \frac{x_3 + |x_3|}{2}, \ S(v^2|D) = \frac{v_2 + |v_2|}{2}$$

and problems (SP)and (SD) become

Primal problem (SP2):

$$\text{Max}_{x^1}\text{Min}_{x^2,y,z} \ e^{x_1} + e^{x_2} + e^{x_3} - e^{y_1} - e^{y_2} - e^{y_3} + \frac{x_3 + |x_3|}{2}$$
$$+ e^{y_3}(y_3 + y_3 p + \frac{1}{2}p^2)$$

subject to

$$e^{y_3}(1+p) + z \geq 0,$$
$$z \in [0,1],$$
$$x^1 \in U, \ y^1 \in V.$$

Dual problem (SD2):

$$\text{Min}_{v^1}\text{Max}_{u,v^2,w} \ e^{u_1} + e^{u_2} + e^{u_3} - e^{v_1} - e^{v_2} - e^{v_3} - \frac{v_2 + |v_2|}{2}$$
$$- e^{u_3}(u_3 + u_3 p + \frac{1}{2}r^2)$$

subject to

$$e^{u_3}(1+r) + w \geq 0$$
$$w \in [0,1],$$
$$u^1 \in U, \ v^1 \in V.$$

Clearly $f(x,y) = -f(y,x)$. Therefore, the problem (SP2) is a self dual and hence Theorem 2 is applicable for this pair.

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