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특이시스템의 일반적 안정화

Generalized Stability Condition for Descriptor Systems

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Abstract: In this paper, we propose a generalized index independent stability condition for a descriptor systemwithout any transformations of system matrices. First, the generalized Lyapunov equation with a specific right-handed matrix form is considered. Furthermore, the existence theorem and the necessary and sufficient conditions for asymptotically stable descriptor systems are presented. Finally, some suitable examples are used to show the validity of the proposed method.

Keywords: generalized Lyapunov equation, descriptor systems, index independent stability condition

I. INTRODUCTIUON

The singular system model is a natural representation of dynamical systems and can better describe a large class of systems than state-space ones do. Studying the stability problem for singular systems is much more complicated than that for state space systems because it requires consideration not only of stability but also regularity absence of impulses for continuous-time singular systems and causality for discrete-time singular systems which may affect the stability of the system. For these reasons, many studies have investigated the generalized Lyapunov equation and stability of descriptor systems [1-4]. When the stability and control design problems of a descriptor system are considered, the standard Lyapunov equation is extended to the generalized Lyapunov equation [5-9]. Such descriptor systems naturally occur in many applications, such as multi-body dynamics, electrical circuit simulation, chemical engineering, and semi-discretization of partial differential equations [10-12]. The GALE (Generalized Algebraic Lyapunov Equation), $E^TXA + A^TXE = -G$, has been considered for the stability analysis, wherein E, A, G are given matricesand X is an unknown matrix. Many numerical algorithms have been developed to solve the GALE with a nonsingular matrix E. However, little attention has been paid to the generalized Lyapunov equation for a singular matrix E ([1,13-17]). It is known that the GALE has a unique solution for every G, provided the matrix E is nonsingular, and all of the eigenvalues of the pencil, $\lambda E - A$, have negative real parts. If E is singular, the GALE may have no solutions, even if all of the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane;If the equation does happen to have a solution, it is not unique. To address thesehallenges, various types of generalized Lyapunov equations have been proposed. Bender [6] defined the reachability and observabilityGramians for descriptor systems by Lyapunov-like (Sylvester) equations. Lewis [14] discussed the system properties of regular and non-regular systems, and investigated the duality. In addition, Takaba [15] presented a new generalized Lyapunov theorem for a continuous-time descriptor system by extending the theorem drawing on the work of Lewis [14]. Unfortunately, these equations are limited to pencilswith an index of at most one [1,6,14,15]. Recently,a projected Lyapunov equation,the existence of a solution for stability therein, and its applications have been investigated [17,18]. This method is independent of anindex; however, the primary difficulty in applying these approaches [1,6,14,15,17,18] is that the Weierstrass canonical form [19] needs to be found before aa solution of the Lyapunov equation can be obtained. Therefore, the objective of this paper is to propose a generalized index independentstability condition for a descriptor system without transformation of the system matrices into a Weierstrass canonical form.In this paper, we propose the necessary and sufficient conditions for the asymptotic stability of descriptor systemswith any index, without any transformations. The generalized Lyapunov equation with a specific right-handed matrix form is discussed, and the existence of the solution is proved. In section 2, the descriptor system and the mathematical preliminaries are discussed,and several of the related existing results are introduced. Section 3 proposes the generalized asymptotic stability conditions for a singular descriptor system. Section 4 provides an exampleto show the validity of the proposed result.

II. DESCRIPTOR SYSTEMS AND THE GENERALIZED LYAPUNOV EQUATION

This section gives a preliminary and problem formulation. Also, some theorems are introduced for the technical manipulations of main results in section 3. Consider a linear time-invariant continuous-time system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

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where $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $x(t)$ is the state vector, $u(t) \in R^m$ is the control input, and $y(t) \in R^p$ is the output. If $E = I$, then (1) is a standard state space system; otherwise, (1) is a descrip tor system or generalized state space system.When the pencil $\lambda E - A$ is assumedregular, that is, $\det(\lambda E - A) \neq 0$ for some $\lambda \in R$, $\lambda E - A$ can be reduced to the Weierstrass canonical form. There exists nonsingular matrices *W* and *T* such that

$$E = W\begin{bmatrix} I_{n_\varepsilon} & 0 \\ 0 & N \end{bmatrix}T, \quad A = W\begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix}T \tag{2}$$

where I_k is the identity matrix with order *k*, J is the Jordan block corresponding to the finite eigenvalues of $\lambda E - A$, and N is thenilpotent,corresponding to the infinite eigenvalues. The index of nilpotencyof *N*, denoted by ν, is the index of the pencil, $\lambda E - A$. The pencil $\lambda E - A$ is referred to as c-stable if it is regular and if all of the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. Clearly, $\lambda E - A$ has an eigenvalue at infinity if and only if the matrix *E* is singular. Representation (2) defines the decomposition of R^n into complementary deflating subspaces of dimensions n_ε and n_∞, corresponding to the finite and infinite eigenvalues of $\lambda E - A$, respectively. The matrices

$$P_r = T^{-1}\begin{bmatrix} I_{n_\varepsilon} & 0 \\ 0 & 0 \end{bmatrix}T, \quad P_l = W\begin{bmatrix} I_{n_\varepsilon} & 0 \\ 0 & 0 \end{bmatrix}W^{-1} \tag{3}$$

arespectral projectionsonto the right and left deflating subspaces of $\lambda E - A$, corresponding to the finite eigenvalues. Descriptor system (1) can be changed to the Weierstrass canonical form by [19]

$$\begin{aligned} \dot{x}_1(t) &= Jx_1(t) + B_1u(t) \\ N\dot{x}_2(t) &= x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) \end{aligned} \tag{4}$$

where, $W^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $CT^{-1} = [C_1 \quad C_2]$. The block, *N*, has the form

$$\begin{aligned} N &= diag(N_{n_1}, N_{n_2}, \cdots, N_{n_t}) \\ N_{n_j} &= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \\ n_\infty &= \sum_{j=1}^{t} n_j \end{aligned} \tag{5}$$

whichis a nilpotent Jordan block with an order of n_∞. Here, $diag$ represents a block diagonal matrix. The number n_∞ is the algebraic multiplicity of the eigenvalue at infinity of $\lambda E - A$, and it defines the dimension of the right and left deflating subspaces of $\lambda E - A$, corresponding to the eigenvalue at infinity. The size of the largest nilpotent block is the index of the pencil, $\lambda E - A$. Clearly, $N^{\nu-1} \neq 0$, and $N^\nu = 0$. If the matrix *E* is nonsingular, then $\lambda E - A$ has an index of zero. The pencil $\lambda E - A$ has an index of one if and only if it has exactly $n_\varepsilon = rank(E)$ finite eigenvalues.

It is well known that the computation of theWeierstrass canonical form using finite precision arithmetic is an ill-conditioned problem; that is,small changes in the data can drastically change the resultingcanonical form. Therefore, the Weierstrass canonical form is only of theoretical interest. Consider the generalized continuous-time algebraic Lyapunov equation (GCALE)

$$E^T XA + A^T XE = -G. \tag{6}$$

In the case of a non-singular *E*, the following Theorem 2.1 is satisfied.

Theorem 1 [1, Theorem 4.4]**:** Let $\lambda E - A$ be a regular pencil. If all of the eigenvalues of $\lambda E - A$ are finite and lie in the open left half-plane, then, for every positive semi-definite matrix *G*, the GALE (6) has a unique positive semi-definite solution *X*. Conversely, if there exist positive definite matrices *X* and *G* that satisfy (6), then all eigenvalues of the pencil $\lambda E - A$ are finite and lie in the open left half-plane.

Many applications of descriptor systems resultin generalized Lyapunov equations with a singular matrix *E*. In the case of a singular *E*, the GALE (6) may have no solutions, even if all of the finite eigenvalues of $\lambda E - A$ have negative real parts. The presence of an eigenvalue at infinity in $\lambda E - A$ may be a reason for the unsolvabilityof the GALE (6). For this singular system, some useful theorems are introduced in the following, and the GALE (6) was evaluated witha special right-handed side. The so-evaluated GALE was only partially useful since, for such an equation, the existence theorems can be stated only for pencils with indices of at most two.

Theorem 2 [1, Theorem 4.8]**:** Let $\lambda E - A$ be a regular pencilwith an index of at most two. If the $\lambda E - A$ is c-stable, then, for every matrix *G*, the GALE

$$E^T XA + A^T XE = -E^T GE \tag{7}$$

has a solution. If the GALE (7) has a solution *X*, then the pencil $\lambda E - A$ has an index of at most two.

Theorem 3 represents the necessary and sufficient condition for a c-stable descriptor system,especially in the case of Corollary 4, which is derived for an index of one under a more general condition for the positive definite solution.

Theorem 3 [1, Theorem 4.11]**:** Let $\lambda E - A$ be a regular pencil with an index of at most two. Let *G* be a positive definite matrix. The solution of GALE (7) is positive definite on the range space of P_l, $\mathrm{Im}(P_l)$ if and only if $\lambda E - A$ is c-stable.

Corollary 1 [1, Corollary 4.12]**:** Let $\lambda E - A$ be a regular pencilwith an index of at most one. Let *G* be a symmetric positive definite matrix. The GALE (7) has a symmetric positive definite solution X if and only if the index of the pencil $\lambda E - A$ is at most one and $\lambda E - A$ is c-stable.

The projected Lyapunov equation

$$E^T XA + A^T XE = -P_r^T G P_r \tag{8}$$

hasalso been investigated [17 and referenced therein].A necessary and sufficient condition for the existence of a solution to the GALE was proposed for an asymptotically stable descriptor system. This condition isindependent of the index, but, unfortunately, it is difficult to apply because the Weierstrass canonical form is required to solve the Lyapunov equation.

III. THE GENERALIZED STABILITY CONDITION

This section proposesa specific type of the generalized Lyapunov equation and the existence theorem for asymptotic stability. Lemma 1 describes the necessary and sufficient conditions for the unique solvability of the generalized Sylvester equation.

Lemma 1 [20]**:** The generalized Sylvester equation

$$BXA - FXE = -G \tag{9}$$

has a unique solution X if and only if the pencils $\lambda B - F$ and $\lambda E - A$ are regular and have no common eigenvalues. $A, B, E, F, G \in R^{n\times n}$ are given matrices, and $X \in R^{n\times n}$ is an unknown matrix.

Theorem 4 provides the necessary and sufficient condition for a c-stable descriptor system via a specific type of the generalized Lyapunov equation with a specific right-handed form.

Theorem 4: Let $\lambda E - A$ be a regular pencilof any index, $\nu \ge 3$. If $\lambda E - A$ is c-stable, then, for every matrix G the GCALE

$$\begin{gathered} E^T XA + A^T XE = -(M^{\nu-2})^T E^T G E M^{\nu-2} \\ M = (E-A)^{-1} E \end{gathered} \tag{10}$$

has a solution for the descriptor system with each index . Herein, it is assumed that $E - A$ is a non-singular matrix.

Proof: If $\nu = 2$, then it is the same as (7) in Theorem 2. Let the pencil $\lambda E - A$ be in the Weierstrass canonical form (2), wherein the eigenvalues of J lie in the open left half-plane. Let the matrices,

$$G_W := W^T G W = \begin{bmatrix} G_{W1} & G_{W2} \\ G_{W2}^T & G_{W4} \end{bmatrix}, \quad P := W^T X W = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} \tag{11}$$

be defined and partitioned into blocks conformably to E and A, with the Weierstrass canonical transform matrices, T and W. Hene,

$$\begin{aligned} &(W^{-1}ET^{-1})^T W^T XW(W^{-1}AT^{-1}) + (W^{-1}AT^{-1})^T W^T XW(W^{-1}ET^{-1}) \\ &= -T^{-T}\left\{(M^{\nu-2})^T E^T GE(M^{\nu-2})\right\}T^{-1} \\ &= -T^{-T}\left\{E^T(E-A)^{-T}\right\}\cdots\left\{E^T(E-A)^{-T}\right\} \\ &\quad \times E^T GE\left\{(E-A)^{-1}E\right\}\cdots\left\{(E-A)^{-1}E\right\}T^{-1} \end{aligned}$$

$$\begin{aligned} &= -\left\{(W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1}W^{-1}ET^{-1}\right\}^T \cdots \\ &\quad \times\left\{(W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1}W^{-1}ET^{-1}\right\}^T \\ &\quad \times(W^{-1}ET^{-1})^T W^T GW(W^{-1}ET^{-1}) \\ &\quad \times\left\{(W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1}W^{-1}ET^{-1}\right\}\cdots \\ &\quad \times\left\{(W^{-1}ET^{-1} - W^{-1}AT^{-1})^{-1}W^{-1}ET^{-1}\right\} \\ &= -\begin{bmatrix} \left\{(I-J)^{\nu-2}\right\}^T & 0 \\ 0 & \left\{\left\{(N-I)^{-1}N\right\}^{\nu-2}\right\}^T N^T \end{bmatrix} W^T GW \\ &\quad \times\begin{bmatrix} \left\{(I-J)^{\nu-2}\right\} & 0 \\ 0 & N\left\{(N-I)^{-1}N\right\}^{\nu-2} \end{bmatrix} \end{aligned} \tag{12}$$

$$\begin{aligned} &P_1 J + J^T P_1 = -(J_r^{\nu-2})^T G_{W1} J_r^{\nu-2} \\ &P_2 + J^T P_2 N = -(J_r^{\nu-2})^T G_{W2} N N_r^{\nu-2} \\ &N^T P_2^T J + P_2^T = -(N_r^{\nu-2})^T N^T G_{W2}^T J_r^{\nu-2} \\ &N^T P_4 + P_4 N = -(N_r^{\nu-2})^T N^T G_{W4} N N_r^{\nu-2}, \\ &\quad \text{where } J_r = (I-J)^{-1}, \quad N_r = (N-I)^{-1}N. \end{aligned} \tag{13}$$

Since all of the eigenvalues of J have negative real parts, the first Lyapunov equation of (13) has a unique positive definite solution, P_1, for every positive definite G_{W1} in Theorem 3.2. If the left-handed side matrices of the Sylvester equation (9), that is, $B = I$, $A = I$, $F = J^T$, and $E = -N$, then the second equation of (13) has the same form as the Sylvester equation, and $(\lambda I - J^T)$ and $(-\lambda N - I)$ are regular with no common eigenvaluessince $(\lambda I - J^T)$ has finite eigenvalues and $(-\lambda N - I)$ has infinite eigenvalues. Therefore, as per Lemma 1, the second equation of (13) has a unique solution. If we assume that

$$P_2 = -(J_r^{\nu-2})^T G_{W2} N N_r^{\nu-2}. \tag{14}$$

we can see that

$$\begin{aligned} P_2 + J^T P_2 N &= -(J_r^{\nu-2})^T G_{W2} N N_r^{\nu-2} - J^T (J_r^{\nu-2})^T G_{W2} N N_r^{\nu-2} N \\ &= -(J_r^{\nu-2})^T G_{W2} N N_r^{\nu-2}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\text{where } N N_r^{\nu-2} N = N\left\{(N-I)^{-1}N\right\}^{\nu-2} N \\ &\Rightarrow N\left\{(N-I)^{-1}N\right\}N = 0_{3\times3}, \quad \nu = 3 \\ &\Rightarrow N\left\{(N-I)^{-1}N\right\}^2 N = 0_{4\times4}, \quad \nu = 4 \\ &\qquad \vdots \\ &\Rightarrow N\left\{(N-I)^{-1}N\right\}^{\nu-2} N = N N_r^{\nu-2} N = 0_{\nu\times\nu}, \quad \nu = \nu. \end{aligned}$$

Therefore, (14) is a unique solution of the second equation of (13). The solution of the last equation of (13) is not unique for every G_{W4}. For example, the matrix

$$\begin{aligned} P_4 = -\frac{1}{2}\{&N^T (N_r^{\nu-2})^T G_{W4} N N_r^{\nu-3} (N-I)^{-1} \\ &+ (N-I)^{-T} (N_r^{\nu-3})^T N^T G_{W4} N N_r^{\nu-2}\} \end{aligned} \tag{16}$$

can be a solution. From the last equation of (13) and from equation (15) we have

$$
\begin{aligned}
N^T P_4 + P_4 N = &-\frac{1}{2} N^T \{(N_r^{v-2})^T N^T G_{W4} N N_r^{v-3} (N-I)^{-1} \\
&+ (N-I)^{-T} (N_r^{v-3})^T N^T G_{W4} N N_r^{v-2}\} \\
&-\frac{1}{2} \{(N_r^{v-2})^T N^T G_{W4} N N_r^{v-3} (N-I)^{-1} \\
&+ (N-I)^{-T} (N_r^{v-3})^T N^T G_{W4} N N_r^{v-2}\} N \\
=& -\frac{1}{2} N^T (N-I)^{-T} (N_r^{v-3})^T N^T G_{W4} N N_r^{v-2} \\
&-\frac{1}{2} (N_r^{v-2})^T N^T G_{W4} N N_r^{v-3} (N-I)^{-1} N \\
=& -\frac{1}{2} (N_r^{v-2})^T N^T G_{W4} N N_r^{v-2} \\
&-\frac{1}{2} (N_r^{v-2})^T N^T G_{W4} N N_r^{v-2} \\
=& -(N_r^{v-2})^T N^T G_{W4} N N_r^{v-2}.
\end{aligned} \tag{17}
$$

Thus, we complete the proof.

Theorem 5: Let $\lambda E - A$ be a regular pencilof any index. Let G be a positive definite matrix. The solution of the GALE (10) is positivedefinite on the subspace $\mathrm{Im}(P_l)$ if and only if $\lambda E - A$ is c-stable. $E - A$ is a nonsingular matrix, and $\mathrm{Im}(P_l)$ represents the range space of P_l.

Proof: The symmetric matrix

$$
X := W^{-T} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} W^{-1} \tag{18}
$$

satisfiesthe GALE (10). If $\lambda E - A$ is c-stable, then P_1 is positive definite for positive definite matrices G and G_{W1}. Therefore, X is positive definite on $\mathrm{Im}(P_l)$; that is, for $z \in \mathrm{Im}(P_l)$, we have

$$
\begin{aligned}
z^T X z &= x^T P_l^T X P_l x \\
&= x^T W^{-T} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^T W^{-T} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} W^{-1} W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} x \\
&= x^T W^{-T} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} W^{-1} x, \quad W^{-1} x = \eta = \begin{bmatrix} \eta_\varepsilon \\ \eta_\infty \end{bmatrix} \\
&= \begin{bmatrix} \eta_\varepsilon^T & \eta_\infty^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_\varepsilon \\ \eta_\infty \end{bmatrix} = \eta_\varepsilon^T P_1 \eta_\varepsilon > 0.
\end{aligned} \tag{19}
$$

Conversely, if X is a positive definite on $\mathrm{Im}(P_l)$, then P_1 is positive definite, and hence, the matrix

$$
E^T X E = T^T \begin{bmatrix} P_1 & P_2 N \\ N^T P_2^T & N^T P_4 N \end{bmatrix} T \tag{20}
$$

is positive definite on the subspace $\mathrm{Im}(P_r)$. As mentioned in section 2, let $\zeta \neq 0$ be an eigenvector of the pencil $\lambda E - A$ that corresponds to a finite eigenvalue λ ; that is, $\lambda E \zeta = A \zeta$, and ζ is a vector in $\mathrm{Im}(P_r)$. The multiplication of (10) on the right and left sides by ζ and ζ^T, respectively, and by (12) produce

$$
\begin{aligned}
&-\zeta^T (M^{v-2})^T E^T G E M^{v-2} \zeta \\
&= -\zeta^T \left[\left\{ E^T (E-A)^{-T} \right\}^{v-2} E^T G E \left\{ (E-A)^{-1} E \right\}^{v-2} \right] \zeta \\
&= -x^T P_r^T \left[\left\{ E^T (E-A)^{-T} \right\}^{v-2} E^T G E \left\{ (E-A)^{-1} E \right\}^{v-2} \right] P_r x \\
&= -x^T \left[T^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-T} \left\{ E^T (E-A)^{-T} \right\}^{v-2} E^T G E \right. \\
&\quad \left. \times \left\{ (E-A)^{-1} E \right\}^{v-2} T^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T \right] x \\
&= -x^T \left[T^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left\{ (I-J)^{v-2} \right\}^T G_{W1} \left\{ (I-J)^{v-2} \right\} & * \\ * & * \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T \right] x \\
&= -x^T T^T \begin{bmatrix} \left\{ (I-J)^{v-2} \right\}^T G_{W1} \left\{ (I-J)^{v-2} \right\} & 0 \\ 0 & 0 \end{bmatrix} T x, \qquad T x = \begin{bmatrix} \mu_\varepsilon \\ \mu_\infty \end{bmatrix} \\
&= -\begin{bmatrix} \mu_\varepsilon^T & \mu_\infty^T \end{bmatrix} \begin{bmatrix} \left\{ (I-J)^{v-2} \right\}^T G_{W1} \left\{ (I-J)^{v-2} \right\} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_\varepsilon \\ \mu_\infty \end{bmatrix} \\
&= -\mu_\varepsilon^T \left\{ (I-J)^{v-2} \right\}^T G_{W1} \left\{ (I-J)^{v-2} \right\} \mu_\varepsilon < 0,
\end{aligned} \tag{21}
$$

andas a result,

$$
\begin{aligned}
&-\zeta^T (M^{v-2})^T E^T G E M^{v-2} \zeta \\
&= \zeta^T (E^T X A + A^T X E) \zeta \\
&= \lambda \zeta^T E^T X E \zeta + \bar{\lambda} \zeta^T E^T X E \zeta \\
&= 2 \operatorname{Re}\{\lambda\} \zeta^T E^T X E \zeta < 0.
\end{aligned} \tag{22}
$$

Since $E^T X E$ is positive definite on the subspace $\mathrm{Im}(P_r)$, we obtain that $2\operatorname{Re}\{\lambda\} < 0$, that is, all of the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane.

Corollary 2: Let $\lambda E - A$ be a regular pencilwith an indexof $v-1$, $v \geq 3$. Let G be a symmetric positive definite matrix. The GALE (3.2), which is applied with the index v, has a symmetric positive semi-definite solution X if and only if $\lambda E - A$ is c-stable. Moreover, if the solution Xsatisfies $X = X P_l$, then it is unique.

Proof: If the pencil $\lambda E - A$ has an index of $v-1 \geq 2$ and is c-stable, then, from the proof of Theorem 4, we obtain

$$
\begin{aligned}
&P_2 + J^T P_2 N = -(J_r^{v-2})^T G_{W2} N N_r^{v-2} = 0 \;\Rightarrow\; P_2 = 0, \\
&N^T P_4 + P_4 N = -(N_r^{v-2})^T N^T G_{W4} N N_r^{v-2} = 0 \;\Rightarrow\; P_4 = 0 \\
&X = W^{-T} \begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix} W
\end{aligned} \tag{23}
$$

which satisfies the GALE (10). Here, P_1 is a unique symmetric positive definite solution of the first equation of (13). If one available solution for P_4 is the zero matrix, then X must be symmetric and positive semi-definite. Conversely, we cannow assume that the GALE (10) has a symmetric positive semi-definite solution X. From assumption we also know that $\lambda E - A$ hasan indexof $v-1$. The matrix $P_2 = 0$ is a unique solution, and P_4 cannot be a positive definite because some of leading principle minors are always zero matrices. Therefore, P_1 should be positive definite and P_4 should be a positive semi-definite matrix, such that X is a symmetric positive semi-definite solution of (10) using the Schur complement [21]

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} \geq 0 \Leftrightarrow P_1 > 0,\ P_4 - P_2^T P_1^{-1} P_2 \geq 0, \quad \text{or} \\ \Leftrightarrow P_4 > 0,\ P_1 - P_2 P_4^{-1} P_2^T \geq 0. \tag{24}$$

From the proof of Theorem 5, all of the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. Moreover, if the solution of (10) satisfies $X = XP_l$, then

$$W^{-T}\begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix} W = W^{-T}\begin{bmatrix} P_1 & 0 \\ 0 & P_4 \end{bmatrix} W W^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W = W^{-T}\begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} W, \tag{25}$$

Therefore, $P_4 = 0$ and the solution is unique.

IV. NUMERICAL EXAMPLE

Consider the following c-stable descriptor system with a nilpotent index of three. We can easily see that there is no solution using the standard (6) and generalized Lyapunov equations (7).

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix}$$

From (6), (7), and (10), we have

$$p_{11}(-1) + (-1)p_{11} = -2 \Rightarrow p_{11} = 1 \quad \text{for (6) and (7)}$$

$$p_{11}(-1) + (-1)p_{11} = -\frac{1}{4}(2) = -\frac{1}{2} \Rightarrow p_{11} = \frac{1}{4} \quad \text{for (10)}$$

$$\begin{bmatrix} p_{12} & p_{13} & p_{14} \end{bmatrix} + (-1)\begin{bmatrix} p_{12} & p_{13} & p_{14} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_{12} & p_{13} - p_{12} & p_{14} - p_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

for (6), (7), and (10)

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} p_{22} & p_{23} & p_{24} \\ p_{23} & p_{33} & p_{34} \\ p_{24} & p_{34} & p_{44} \end{bmatrix} + \begin{bmatrix} p_{22} & p_{23} & p_{24} \\ p_{23} & p_{33} & p_{34} \\ p_{24} & p_{34} & p_{44} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & p_{22} & p_{23} \\ p_{22} & p_{23} + p_{23} & p_{24} + p_{33} \\ p_{23} & p_{33} + p_{24} & p_{34} + p_{34} \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for (6)}$$

$$= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for (7)}$$

$$= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for (10)}$$

For (6) and (7), there is no solution, and,in the case of (10)

$$P = \begin{bmatrix} p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{24} \\ 0 & 0 & p_{33} & -0.5 \\ 0 & p_{24} & -0.5 & p_{44} \end{bmatrix},$$

$$p_{11} = 0.25,\quad p_{24} + p_{33} = 0,\ p_{34} = -0.5$$

If (16) is applied to obtain P_4, we can get $p_{24} = p_{33} = 0$, $p_{34} = -0.5$, $p_{44} = -1$. If $\lambda E - A$ is c-stable, then the solution P always exists, even though it is not unique, and X is positive definite on the subspace $\mathrm{Im}(P_l)$. Now, consider a descriptor system with an index of two to show the validity of Corollary 2 as follows:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

$$p_{11}(-1) + (-1)p_{11} = -2 \Rightarrow p_{11} = 1 \quad \text{for (6) and (7)}$$

$$p_{11}(-1) + (-1)p_{11} = -\frac{1}{4}(2) = -\frac{1}{2} \Rightarrow p_{11} = \frac{1}{4} \quad \text{for (10)}$$

$$\begin{bmatrix} p_{12} & p_{13} \end{bmatrix} + (-1)\begin{bmatrix} p_{12} & p_{13} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_{12} & p_{13} - p_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

for (6), (7), and (10)

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{22} & p_{23} \\ p_{23} & p_{33} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p_{22} \\ p_{22} & p_{23} + p_{23} \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for (6)}$$

$$= -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for (7)}$$

$$= -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for (10)}$$

There is no solution for (6). In the case of (7), the solution is

$$P = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & 0 & -0.5 \\ 0 & -0.5 & p_{33} \end{bmatrix},$$

where P_4 cannot be positive definite and is indefinite for every p_{33}. In the case of (10), we have

$$P = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$$

If we select the arbitrary elements p_{33}, such that $p_{33} \geq 0$, we can obtain a positive semi-definite solution, P and X. Under the conditionof $X = XP_1$, $p_{33} = 0$, and the solution is unique. Conversely, if the semi-definite solution X exists, then P_1 is necessarily a positive definite matrix for X, and, hence, the descriptor system is c-stable from the proof of Theorem 5.

V. CONCLUSIONS

In this paper, we have proposed the generalized asymptotic stability condition for descriptor systems with a singularmatrix E. The proposed method uses thegeneralized Lyapunov equation witha specificright-handed side matrix. Especially, the existence theorem and the necessary and sufficient conditions were derived for asymptotically stable descriptor systems with every possible index. To show the validity of the proposed method, some suitable examples have been demonstrated.

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