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Achievable Ergodic Capacity of a MIMO System with a MMSE Receiver

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Abstract

This paper considers the multiple-input multiple-output (MIMO) system with linear minimum mean square error (MMSE) detection under ideal fast fading. For N_t transmit and $N_r(\geq N_t)$ receive antennas, we derive the achievable ergodic capacity of MMSE detection exactly. When MMSE detection is considered in a receiver, we introduce a different approach that gives the approximation of a MIMO channel capacity at high signal-to-noise ratio (SNR). The difference between the channel capacity and the achievable capacity of MMSE detection converges to some constant that depends only on the number of antennas. We validate the analytical results by comparing them with Monte Carlo simulated results.

Key Words: Capacity, Multiple-Input Multiple-Output (MIMO), Minimum Mean Square Error (MMSE), Signal to Interference and Noise Ratio (SINR).

I. Introduction

The use of multiple antennas at the transmitter and receiver provides a higher capacity than that of a single antenna system [1]. Multiple-input multiple-output (MIMO) communications have been intensively studied over the last few years and are widely considered as suitable for improving performance of modern wireless communications [2].

Maximum-likelihood (ML) detection gives the maximum receive diversity and has high computational complexity. We can apply linear zero-forcing or minimum mean square error (MMSE) detectors to reduce computational complexity. Unfortunately, linear receivers have a smaller diversity order than that of the ML detector.

In [3], the authors derived the achievable instantaneous capacity of a MMSE detector for a given channel matrix $\mathbf{H}$. However, we have introduced the achievable ergodic capacity for an uncorrelated channel matrix $\mathbf{H}$. In this paper, we investigate the analytical measurement associated with MMSE detection. We use the probability density function (PDF) of signal-to-interference plus noise ratio (SINR) [4] to obtain a precise value for the achievable capacity of the MMSE detector. We also derive the achievable ergodic capacity of the MMSE detector at high SNR. Finally, we confirm for a small number of antennas that the difference between the channel capacity and the achievable capacity of the MMSE detector converges to some constant that depends on the number of transmit and receive antennas.

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II. SYSTEM DESCRIPTION

We consider single-user communications and investigate a point-to-point link, where the transmitter is equipped with N_t antennas and the receiver employs $N_r(\geq N_t)$ antennas. Suppose that no inter-symbol interference (ISI) exists. Let h_{ij} be the complex-valued channel coefficient from transmit antenna j to receive antenna i. If the complex modulated signals $x_1, \cdots, x_{N_t}$ are transmitted via the N_t antennas, then the received signal at antenna i can be represented as $\boldsymbol{y}_i = \sum_{j=1}^{N_r} h_{ij}x_j + \boldsymbol{n}_i$,where $\boldsymbol{n}_i$ represents additive white Gaussian noise. This relation is easily written in a matrix form as

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n}, \tag{1}$$

where $\mathbf{y} = [y_1 \ \cdots \ y_{N_r}]^{\mathrm{T}} \in \mathbb{C}^{N_r \times 1}$ denotes the received complex vector, $\mathbf{x} = [x_1 \ \cdots \ x_{N_t}]^{\mathrm{T}} \in \mathbb{C}^{N_t \times 1}$ is the transmitted vector with the power E_s, $\mathbf{H} = [h_{ij}] \in \mathbb{C}^{N_r \times N_t}$ is an independent and identically distributed (i.i.d) complex Gaussian fading channel matrix with unit variance, and $\boldsymbol{n} \in \mathbb{C}^{N_r \times 1}$ is an additive white Gaussian noise with zero mean and variance N_0.

Suppose that the transmitter does not know the channel realization. Minimizing the mean squared error (MSE) between the actually transmitted symbol $\mathbf{x}_i$ and the output of a linear MMSE detector leads to the filter vector $\mathbf{g}_i = (\mathbf{HH}^{\mathrm{H}} + s\mathbf{I})^{-1}\mathbf{h}_i$, where $s = \frac{N_t N_0}{E_s} = N_t/SNR$ and $\mathbf{h}_i$ is the *i*th column of $\mathbf{H}$. Applying this filter vector into Eq. (1) yields $z_i = \mathbf{g}_i^{\mathrm{H}}\mathbf{y} = \beta_i x_i + w$, where $\beta_i = \mathbf{g}_i^{\mathrm{H}}\mathbf{h}_i$ and the interference-plus-noise term w is defined as $\sum_{j \neq i} \mathbf{g}_i^{\mathrm{H}}\mathbf{h}_i \mathrm{x}_j + \mathbf{g}_i^{\mathrm{H}}\mathbf{n}$. The variance of w is computed as $\sigma_w^2 = \frac{E_s}{N_t}(\beta_i - \beta_i^2)$. The SINR of a linear MMSE detector on the *i*th spatial stream can be computed as $\mathrm{SINR}_i = \frac{\beta_i}{1-\beta_i}$. For the uncorrelated channel, the statistical property of SINR_i is the same for all $1 \leq i \leq N_t$. Then we omit the subscript *i* of SINR_i.

III. ACHIEVABLE CAPACITY OF MMSE RECEIVER

We introduce the PDF of SINR at output of linear MMSE detector to calculate the ergodic capacity. We use it for analytical derivation of the ergodic capacity obtained by the MMSE detector. For a high SNR, we will prove that the difference between the channel capacity and the capacity of the MMSE detector converges to the constant that depends only on N_t and $d = N_r - N_t + 1$.

1. PDF of SINR

For small N_r and N_t, we summarize the PDF $f_{N_r \times N_t}(\gamma)$ of SINR [4] as following:

$$\begin{aligned}
f_{2\times2}(\gamma) &= se^{-s\gamma}\left[1 + s - \frac{1}{(1+\gamma)^2} - \frac{s}{(1+\gamma)}\right],\\
f_{3\times2}(\gamma) &= s^2\gamma e^{-s\gamma}\left[1 + \frac{s}{2} - \frac{1}{2(1+\gamma)^2} - \frac{s+1}{2(1+\gamma)}\right],\\
f_{3\times3}(\gamma) &= se^{-s\gamma}\left[1 + 2s + \frac{s^2}{2} + \frac{s+2}{(1+\gamma)^3} + \frac{s^2-6}{2(1+\gamma)^2} - \frac{s^2+3s}{(1+\gamma)}\right],\\
f_{4\times2}(\gamma) &= s^3\gamma^2 e^{-s\gamma}\left[\frac{1}{2} + \frac{s}{6} - \frac{1}{6(1+\gamma)^2} - \frac{s+2}{6(1+\gamma)}\right],\\
f_{4\times3}(\gamma) &= s^2\gamma e^{-s\gamma}\left[1 + s + \frac{s^2}{6} + \frac{s+3}{3(1+\gamma)^3} + \frac{s^2+2s-6}{6(1+\gamma)^2} - \frac{s^2+5s+3}{3(1+\gamma)}\right],\\
f_{4\times4}(\gamma) &= se^{-s\gamma}\left[\begin{array}{l} 1 + 3s + \frac{3s^2}{2} + \frac{s^3}{6} - \frac{s^2+6s+6}{2(1+\gamma)^4} - \frac{s^3-36s-48}{6(1+\gamma)^3} \\ \quad + \frac{s^3+6s^2-12}{2(1+\gamma)^2} - \frac{s^3+8s^2+12s}{2(1+\gamma)} \end{array}\right]
\end{aligned} \tag{2}$$

2. Achievable Capacity

For the uncorrelated MIMO channel, the ergodic capacity $C_{N_r \times N_t}^{\mathrm{MMSE}}$ achievable by the MMSE detector can be defined as [3]

$$C_{N_r\times N_t}^{MMSE} = N_t \int_0^{\infty} \log_2(1+\gamma) f_{N_r\times N_t}(\gamma) d\gamma. \tag{3}$$

For example, in $N_r = 2$ and $N_t = 2$, we compute the integral directly, as follows:

$$\begin{aligned}
C_{2\times2}^{MMSE} &= \frac{2}{\ln 2}\int_0^{\infty} \ln(1+\gamma) e^{-s\gamma}\left[1 + s - \frac{1}{(1+\gamma)^2} - \frac{s}{(1+\gamma)}\right] d\gamma\\
&= \frac{2}{\ln 2}\left[\begin{array}{c} \underbrace{(1+s)\int_0^{\infty} \ln(1+\gamma) se^{-s\gamma} d\gamma}_{=A} \\ \underbrace{-\int_0^{\infty} \frac{\ln(1+\gamma)}{(1+\gamma)^2} se^{-s\gamma} d\gamma - s\int_0^{\infty} \frac{\ln(1+\gamma)}{1+\gamma} se^{-s\gamma} d\gamma}_{=B} \end{array}\right]
\end{aligned} \tag{4}$$

We can easily check the first term in Eq. (4) by [5, 4.337]:

$$A = \frac{2}{\ln 2}(1+s)e^s E_1(s), \tag{5}$$

where $E_1(s) = \int_s^{\infty} \frac{exp(-t)}{t} dt$ and $E_1(s) = -Ei(-s)$ [5]. The second term in Eq. (4) is calculated by integration by parts and [5, 3.352] and then the third one in Eq. (4) is eliminated:

$$B = \frac{2}{\ln 2}\left[-s + s^2 e^s E_1(s)\right] \tag{6}$$

We can obtain the exact ergodic capacity as

$$C_{2\times2}^{\mathrm{MMSE}} = \frac{2}{\ln 2}\left[-s + (1+s+s^2)e^s E_1(s)\right]. \tag{7}$$

In a similar way, $C_{N_r \times N_t}^{\mathrm{MMSE}}$ can be calculated as the following:

$$\begin{aligned}
C_{3\times2}^{\mathrm{MMSE}} &= \frac{2}{\ln 2}\left[1 + \frac{s}{2} + \frac{s^2}{2} + \left(1 - s - s^2 - \frac{s^3}{2}\right)e^s E_1(s)\right],\\
C_{4\times2}^{\mathrm{MMSE}} &= \frac{2}{\ln 2}\left[\frac{3}{2} - \frac{s}{3} - \frac{s^2}{3} - \frac{s^3}{6} + \left(1 - s + \frac{s^2}{2} + \frac{s^3}{2} + \frac{s^4}{6}\right)e^s E_1(s)\right],\\
C_{3\times3}^{\mathrm{MMSE}} &= \frac{3}{\ln 2}\left[-\frac{5s}{2} - \frac{5s^2}{4} - \frac{s^3}{4} + \left(1 + 2s + \frac{7s^2}{2} - \frac{s^3}{2}\right)e^s E_1(s)\right],\\
C_{4\times3}^{\mathrm{MMSE}} &= \frac{3}{\ln 2}\left[1 + s + \frac{23s^2}{12} + \frac{2s^3}{3} + \frac{s^4}{12} + \left(1 - s - \frac{5s^2}{2} - \frac{5s^3}{2} - \frac{3s^4}{4} - \frac{s^5}{12}\right)e^s E_1(s)\right],
\end{aligned}$$

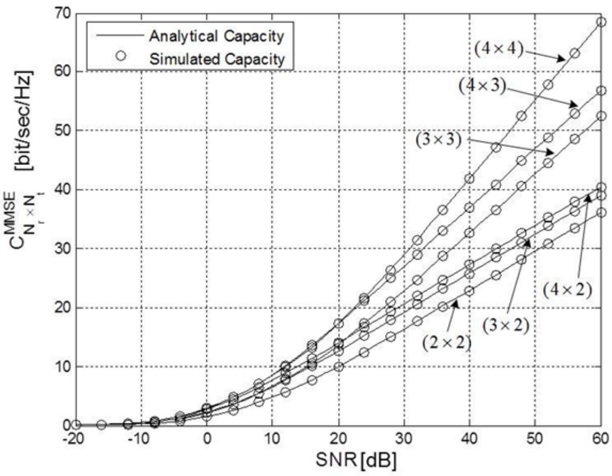

Fig. 1. The ergodic capacities obtained by the minimum mean square error (MMSE) detector. SNR = signal-to-noise ratio.

$$C_{4\times4}^{\text{MMSE}} = \frac{4}{\ln 2}\left[\begin{array}{c} -\frac{13s}{3}-\frac{53s^2}{12}-\frac{37s^3}{18}-\frac{7s^4}{18}-\frac{7s^5}{36} \\ +\left(1+3s+\frac{15s^2}{2}+\frac{37s^3}{6}-\frac{29s^4}{12}+\frac{5s^5}{12}+\frac{s^6}{36}\right)e^s E_1(s)\end{array}\right], \quad (8)$$

where $s = N_t/SNR$. Note that $C_{N_r\times N_t}^{\text{MMSE}}$ is a function of SNR.

We consider an independent and identically distributed Rayleigh fading channel with N_t transmit antennas and N_r receive antennas. The simulated capacity performance of the MMSE detector under a typical realization is displayed with respect to SNR. This is validated by the simulation results shown in Fig. 1.

3. Constant Gap between the Channel Capacity and Achievable Capacity of the MMSE Detector at High SNR

At a high SNR regime, the capacity of the MMSE detector for a MIMO system can be approximated as

$$C_{N_r\times N_t}^{\text{MMSE}} \approx N_t \int_0^{\infty} \log_2\gamma\, f(\gamma)d\gamma. \quad (9)$$

Up to $N_r = 4$ and $N_t = 4$, by integrating Eq. (9), the approximated ergodic capacities of MMSE detector can be computed easily as

$$C_{N_r\times N_t}^{\text{MMSE}} \approx -N_t \log_2 s + \frac{1}{\ln 2}\psi(d), \quad (10)$$

where $d = N_r - N_t + 1$ and $\psi(k)$ is expressed as [6]

$$\psi(1) = -\mu,\ \psi(k) = -\mu + \sum_{p=1}^{k-1}\frac{1}{p}\ (k \geq 2). \quad (11)$$

Note that μ is Euler's constant. In [6], the authors found the approximation for ergodic channel capacity at high SNR as

$$C_{N_r\times N_t} \approx -N_t \log_2 s + \frac{1}{\ln 2}\sum_{i=0}^{N_t-1}\psi(d+i). \quad (12)$$

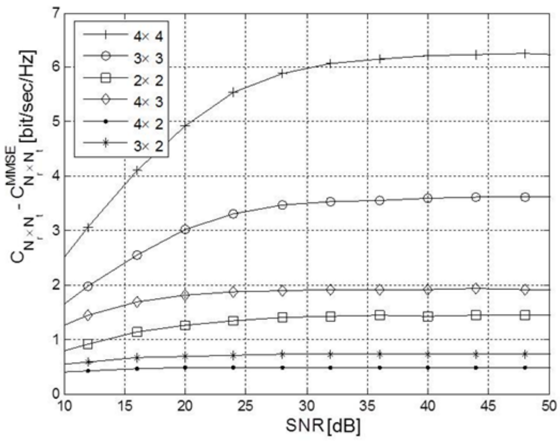

Fig. 2. Convergence of the difference between the channel capacity and the capacity obtained by the minimum mean square error (MMSE) detector. SNR = signal-to-noise ratio.

The difference between the MIMO channel capacity and the capacity of the MMSE detector converges to some constant as

$$\lim_{s\to0}\left[C_{N_r\times N_t} - C_{N_r\times N_t}^{\text{MMSE}}\right] = \frac{1}{\ln 2}\sum_{i=1}^{N_t-1}\left(\psi(d+i) - \psi(d)\right) \quad (13)$$

The Eq. (13) shows that the achievable capacity of the MMSE detector has constant capacity degradation. We will validate the capacity degradation by the simulation results shown in Fig. 2.

IV. Conclusion

In this paper, we have studied the achievable ergodic capacity of a MIMO system with a MMSE detector for a small number of antennas. We have also shown that the difference between the channel capacity and achievable capacity of the MMSE detector converges to some constant at high SNR.

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References

[1] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585-595, Nov. 1999.
[2] G. J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs Technical Journal*, vol. 1, no. 2, pp. 41-59, 1996.

[3] X. Zhang and S. Y. Kung, "Capacity analysis for parallel and sequential MIMO equalizers," *IEEE Transactions on Signal Processing*, vol. 51, no 11, pp. 2989-3002, Nov. 2003.

[4] N. Kim, Y. Lee, and H. Park, "Performance analysis of MIMO system with linear MMSE receiver," *IEEE Transactions on Wireless Communications*, vol. 7, no. 11, pp. 4474-4478, Nov. 2008.

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. Boston, MA: Elsevier, 2007.

[6] O. Oyman, E. U. Nabar, H. Bolcskei, and A. J. Paulraj, "Tight lower bounds on the ergodic capacity of Rayleigh fading MIMO channels," in *Proceedings of IEEE Global Telecommunications Conference (GLOBECOM)*, Taipei, Taiwan, 2002, pp. 1172-1176.