A MULTIVARIABLE MAYER-ERDÖS PHENOMENON

Xianchang Meng and Alexandru Zaharescu

ABSTRACT. In this paper we consider a generalization of the Mayer-Erdös phenomenon discussed in [12] to linear forms in a larger number of variables.

1. Introduction

Let Q be a positive integer. The Farey series $\mathcal{F}_Q$ of order Q consists of reduced fractions between 0 and 1 whose denominators do not exceed Q. For the basic properties of Farey series, the reader is referred to Hardy and Wright [9], Chapter 3. Let $N(Q)$ denote the number of elements of $\mathcal{F}_Q$. We write these elements in increasing order,

$$\gamma_1 = \frac{a_1}{q_1} < \gamma_2 = \frac{a_2}{q_2} < \cdots < \gamma_{N(Q)} = \frac{a_{N(Q)}}{q_{N(Q)}}.$$

Hardy, Littlewood and Pólya [8] proposed the concept of similar ordering for pairs of rational numbers, as follows. If all the reduced fractions $r = \frac{a}{q}$ and $r' = \frac{a'}{q'}$ satisfy

$$(a - a')(q - q') \geq 0,$$

then r and r' are said to be similarly ordered.

Mayer [10] proved that the first and second neighbors in any Farey series are similarly ordered, and he also proved the similar ordering of the third neighbors with the exception of $\mathcal{F}_4$. Hardy suggested and Mayer [11] confirmed that the k-th neighbors are similarly ordered, with a finite number (depending only on k) of exceptions. Mayer showed the following:

Given a positive integer k, there exists a number $Q(k)$ such that for any $Q > Q(k)$, and any $1 \leq j < j' \leq N(Q)$ with $j' - j \leq k$, the numbers $\frac{a_j}{q_j}$ and $\frac{a_{j'}}{q_{j'}}$ are similarly ordered.

Erdös [7] established a vast improvement of Mayer's result:

There exists an absolute constant $C > 0$ such that, for any positive integer Q and any $1 \leq j < j' \leq N(Q)$ for which $j' - j < \frac{Q}{C}$, the fractions $\frac{a_j}{q_j}$ and $\frac{a_{j'}}{q_{j'}}$ are similarly ordered.

One of the authors [12] extended the result of Erdös by considering a general linear form in two variables

$$L(X, Y) = AX + BY,$$

and proving the following result:

There exists an absolute constant $C > 0$ such that, for any positive integers D, Q, any integer numbers $|A|, |B| \leq D$, and any $1 \leq j < j' \leq N(Q)$ satisfying the inequality $j' - j \leq \frac{Q}{CD^2}$, we have

$$(Aa_j + Ba_{j'})(Aq_j + Bq_{j'}) \geq 0.$$

Note that the result by Erdös in [7] is for the case $D = 1$.

In the present paper, our goal is to extend the results further by considering a general linear form in v variables. Let $v \geq 3$ be an integer and let D be a positive integer. Let $\{A_i\}_{i=1}^v$ be integers with $|A_i| \leq D$ for all $1 \leq i \leq v$. Consider the linear form,

$$L(\mathbf{x}) := \sum_{i=1}^v A_i x_i = A_1 x_1 + A_2 x_2 + \cdots + A_v x_v,$$

where $\mathbf{x} := (x_1, x_2, \ldots, x_v)$.

For any v elements in the Farey series of order Q,

$$\frac{a_{r_1}}{q_{r_2}}, \; \frac{a_{r_2}}{q_{r_2}}, \ldots, \frac{a_{r_v}}{q_{r_v}},$$

where $1 \leq r_i \leq N(Q)$ $(1 \leq i \leq v)$, we let $\mathbf{a} := (a_{r_1}, a_{r_2}, \ldots, a_{r_v})$ and $\mathbf{q} := (q_{r_1}, q_{r_2}, \ldots, q_{r_v})$. Then, we have the following result:

Theorem 1. *There exists a positive constant $C = C(v)$ which only depends on v such that, for any positive integers D and Q, any set of integers $\{A_i\}_{i=1}^v$ with $|A_i| \leq D$ satisfying:*
1) $\sum_{i=1}^v A_i \neq 0$, and
2) $\sum_{i \neq k} A_i \neq 0$ for any $1 \leq k \leq v$,
and for any v-tuple $1 \leq r_1 < r_2 < \cdots < r_v \leq N(Q)$, with

$$r_v - r_1 \leq \frac{Q}{CD^2},$$

we have

$$L(\mathbf{a})L(\mathbf{q}) = \left(\sum_{i=1}^v A_i a_{r_i} \right) \left(\sum_{i=1}^v A_i q_{r_i} \right) \geq 0.$$

Remark 1. If we let $d = \sum_{i=1}^{v} A_i$, the conditions 1) and 2) are equivalent to: $d \neq 0$ and $d \notin \{A_i\}_{i=1}^{v}$. One can use this statement to check if the A_i's satisfy the conditions from the statement of the theorem in concrete numerical examples. In our proof, the conditions 1) and 2) are more convenient to use.

Remark 2. Surprisingly, as we shall see below, for $v \geq 3$, counterexamples do exist if one does not assume any one of the above two conditions. See Section 3 for the details.

We end this section with a discussion of a geometric interpretation of the Mayer-Erdös phenomenon. There is a natural one-to-one correspondence between the set of Farey fractions of order Q and the set of visible lattice points inside the triangle with vertices $(0,0)$, $(Q,0)$, (Q,Q), where a visible point (q,a) corresponds to the Farey fraction a/q. In this correspondence, Farey fractions are arranged in increasing order, and the visible points are arranged in increasing order of the slope of the ray from the origin that passes through them. Various problems where visible points play an important role have been studied by a number of authors, see for example [1], [2], [3], [4], [5], [6], and the references therein. A peculiar geometric aspect of the Mayer-Erdös phenomenon, which to our knowledge has not been exploited so far, is the following. Consider, for any visible point (q,a) the coordinate system with horizontal and vertical axes parallel to the initial ones, centered at the point (q,a). Then, by the theorem of Erdös mentioned earlier, the first about a constant times Q many visible points following (q,a) lie inside the first or third quadrant. The same holds for the previous about a constant times Q many visible points which appear before (q,a) in above ordering. The generalizations of Erdös' theorem obtained in [12] and in the present paper have similar geometric interpretations.

2. Proof of Theorem 1

If the A_i's are all positive or all negative, the result is trivial. Let's assume that at least one of them is positive, and at least one of them is negative. Let M be the number of positive coefficients, and N be the number of negative coefficients, so $M + N = v$. We collect all the positive coefficients together, and all the negative coefficients together. We then rearrange the indices, let $s_j = r_{M+j}$, and write

$$L(\mathbf{a})L(\mathbf{q}) = \left(\sum_{i=1}^{M} A_i a_{r_i} - \sum_{j=1}^{N} B_j a_{s_j} \right) \left(\sum_{i=1}^{M} A_i q_{r_i} - \sum_{j=1}^{N} B_j q_{s_j} \right),$$

where $A_1, \ldots, A_M, B_1, \ldots, B_N$ are all positive integers.

Let us suppose that $L(\mathbf{a})L(\mathbf{q}) < 0$. Let $r = \min\{r_i : 1 \leq i \leq v\}$ and $s = \max\{r_i : 1 \leq i \leq v\}$. We want to show that $s - r > \frac{Q}{CD^2}$ for some constant $C = C(v)$. We distinguish two cases:

Case I:

$$
\begin{cases}
\sum\limits_{i=1}^{M} A_i a_{r_i} - \sum\limits_{j=1}^{N} B_j a_{s_j} \leq -1, \\
\sum\limits_{i=1}^{M} A_i q_{r_i} - \sum\limits_{j=1}^{N} B_j q_{s_j} \geq 1.
\end{cases}
$$

Case II:

$$
\begin{cases}
\sum\limits_{i=1}^{M} A_i a_{r_i} - \sum\limits_{j=1}^{N} B_j a_{s_j} \geq 1, \\
\sum\limits_{i=1}^{M} A_i q_{r_i} - \sum\limits_{j=1}^{N} B_j q_{s_j} \leq -1.
\end{cases}
$$

Denote

$$
\Delta(r_i, s_j) := \frac{q_{r_i} a_{s_j} - a_{r_i} q_{s_j}}{q_{r_i} q_{s_j}} = \frac{a_{s_j}}{q_{s_j}} - \frac{a_{r_i}}{q_{r_i}},
$$

and

$$
\Delta := \frac{a_s}{q_s} - \frac{a_r}{q_r},
$$

where $r = \min\{r_1, \ldots, r_M, s_1, \ldots, s_N\}$, and $s = \max\{r_1, \ldots, r_M, s_1, \ldots, s_N\}$.
We need an estimate for Δ.

2.1. A lower bound for Δ

We deal with **Case I** first. We have

$$
\sum_{i=1}^{M} A_i q_{r_i} \geq \sum_{j=1}^{N} B_j q_{s_j} + 1,
$$

and

$$
\sum_{j=1}^{N} B_j a_{s_j} \geq \sum_{i=1}^{M} A_i a_{r_i} + 1.
$$

Multiplying both sides of the above two inequalities, one has

$$
\left(\sum_{i=1}^{M} A_i q_{r_i} \right) \left(\sum_{j=1}^{N} B_j a_{s_j} \right) \geq \left(\sum_{j=1}^{N} B_j q_{s_j} + 1 \right) \left(\sum_{i=1}^{M} A_i a_{r_i} + 1 \right).
$$

Then,

$$
(2.1) \qquad \sum_{i=1}^{M} \sum_{j=1}^{N} A_i B_j (q_{r_i} a_{s_j} - a_{r_i} q_{s_j}) \geq \sum_{i=1}^{M} A_i a_{r_i} + \sum_{j=1}^{N} B_j q_{s_j} + 1.
$$

We divide by the product $\prod\limits_{i,j} A_i B_j q_{r_i} q_{s_j}$ on both sides of (2.1) to obtain

$$
\sum_{i=1}^{M} \sum_{j=1}^{N} \frac{\Delta(r_i, s_j)}{\prod\limits_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}} > \sum_{i=1}^{M} \frac{\gamma_{r_i}}{\prod\limits_{k \neq i, l} A_k B_l q_{r_k} q_{s_l}} + \sum_{j=1}^{N} \frac{1}{\prod\limits_{k, l \neq j} A_k B_l q_{r_k} q_{s_l}}
$$

$$:= S_1 + S_2,$$

where $\gamma_{r_i} = \frac{a_{r_i}}{q_{r_i}}$, and S_1 and S_2 denote the first and respectively the second sum on the right side above. Let

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{\prod_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}}.$$

Since $\Delta(r_i, s_j) \leq \Delta$ for any i, j, we have $\Delta \cdot S > S_1 + S_2$, so

$$(2.2) \qquad \Delta > \frac{S_1 + S_2}{S}.$$

For each $1 \leq i \leq M$,

$$S_2 = \sum_{j=1}^{N} \frac{1}{\prod_{k,l \neq j} A_k B_l q_{r_k} q_{s_l}} \geq \frac{1}{DQ} \sum_{j=1}^{N} \frac{1}{\prod_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}}.$$

We sum over i to obtain

$$MS_2 \geq \frac{1}{DQ} \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{1}{\prod_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}} = \frac{1}{DQ} S.$$

Thus,

$$(2.3) \qquad \frac{S_2}{S} \geq \frac{1}{DQM}.$$

Similarly, for each $1 \leq j \leq N$,

$$S_1 = \sum_{i=1}^{M} \frac{\gamma_{r_i}}{\prod_{k \neq i, l} A_k B_l q_{r_k} q_{s_l}} \geq \frac{1}{DQ} \sum_{i=1}^{M} \frac{\gamma_{r_i}}{\prod_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}},$$

and summing over j, we have

$$NS_1 \geq \frac{1}{DQ} \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{\gamma_{r_i}}{\prod_{k \neq i, l \neq j} A_k B_l q_{r_k} q_{s_l}}$$

$$:= \frac{1}{DQ} S_\gamma.$$

So $S_1 \geq \frac{1}{DQN} S_\gamma$, and

$$(2.4) \qquad \frac{S_1}{S} \geq \frac{1}{DQN} \frac{S_\gamma}{S} = \frac{1}{DQN} \tilde{\gamma},$$

where $\tilde{\gamma} = \frac{S_\gamma}{S}$. Since $\gamma_{r_i} < 1$, $S_\gamma < S$, so $0 < \tilde{\gamma} < 1$.

Combining (2.2), (2.3) and (2.4), we obtain

$$(2.5) \qquad \Delta > \frac{1}{DQ} \left(\frac{1}{M} + \frac{\tilde{\gamma}}{N} \right).$$

For **Case II**, we only need to change the positions of a_{r_i} and q_{r_i}, and change the positions of a_{s_j} and q_{s_j}. The calculations are similar to those in **Case I**. We find that

$$(2.6) \qquad \Delta > \frac{1}{DQ}\left(\frac{1}{N} + \frac{\tilde{\gamma}'}{M}\right),$$

where

$$\tilde{\gamma}' = \frac{S_{\gamma'}}{S}, \quad S_{\gamma'} = \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\gamma_{s_j}}{\prod_{k\neq i, l\neq j} A_k B_l q_{r_k} q_{s_l}}, \quad \gamma_{s_j} = \frac{a_{s_j}}{q_{s_j}},$$

and by the definition of S we see that $0 < \tilde{\gamma}' < 1$.

Hence, combining (2.5) and (2.6), in all cases we have,

$$(2.7) \qquad \Delta > \frac{1}{DQ(M+N)} = \frac{1}{vDQ}.$$

2.2. An upper bound for Δ

Let $r = \min\{r_1, r_2, \ldots, r_v\}$, and $s = \max\{r_1, r_2, \ldots, r_v\}$. We have

$$(2.8) \qquad \Delta = \frac{a_s}{q_s} - \frac{a_r}{q_r} = \sum_{j=r+1}^{s}\left(\frac{a_j}{q_j} - \frac{a_{j-1}}{q_{j-1}}\right) = \sum_{j=r+1}^{s} \frac{1}{q_j q_{j-1}}.$$

Here, in the last equality, we used the property of Farey neighbors which states that $a_j q_{j-1} - a_{j-1} q_j = 1$.

Denote by j^* the smallest element of the set $\{r, r+1, \ldots, s\}$ for which the equality

$$q_{j^*} = \min\{q_j : r \leq j \leq s\}$$

holds. In [12], part of the proof is independent of the linear form. We apply that method to deal with some of our cases. Let $T \geq 1$ be a parameter, whose size will be determined later, and denote by $r'_1, \ldots, r'_t$ those elements from the set $\{r, r+1, \ldots, s\}$ for which

$$q_{r'_1}, \ q_{r'_2}, \ldots, \ q_{r'_t} \leq \frac{Q}{T}.$$

We arrange the above numbers in increasing order, $r'_1 < r'_2 < \cdots < r'_t$. Next, we distinguish several cases.

For the cases $t = 0$ and $t \geq 2$, we apply the method from [12].

If $t = 0$, then $q_r, q_{r+1}, \ldots, q_s > \frac{Q}{T}$, and since $\max\{q_j, q_{j-1}\} \geq \frac{q_j + q_{j-1}}{2}$, we have

$$\frac{1}{q_j q_{j-1}} \leq \frac{2}{(q_j + q_{j-1})\min\{q_j, q_{j-1}\}} < \frac{2}{Q\min\{q_j, q_{j-1}\}}.$$

Using (2.8) together with the above inequalities, we find that

$$\Delta \leq \frac{2T}{Q^2}(s - r).$$

By (2.7), it follows in this case that

$$(2.9) \qquad s - r > \frac{Q}{2vTD}.$$

If $t = 1$, then $r_1' = j^*$, and by (2.11) of [12],

$$(2.10) \qquad \Delta \leq \frac{2}{q_{j^*} \max\{q_{j^*}, Q - q_{j^*}\}} + \frac{2T(s - r - 1)}{Q^2}.$$

If $t \geq 2$, assume $T > 8$, and then by (2.19) of [12], we have

$$\Delta \cdot \frac{Q}{2}\left(1 - \frac{8}{T}\right) \leq \frac{(s - r)T}{Q}.$$

Using this formula and (2.7), we derive that

$$(2.11) \qquad s - r > \frac{Q}{2vTD}\left(1 - \frac{8}{T}\right).$$

By (2.9), (2.10) and (2.11), the proof of the theorem is finished in cases $t = 0$ and $t \geq 2$ if we let T be a suitable constant. Moreover, case $t = 1$ is also done provided that q_{j^*} is not very small.

2.3. Case $t = 1$

In the following, we deal with the case $t = 1$. Let $T' \geq 1$ be another parameter, and assume that $q_{j^*} \geq T'D$. By (2.21) in [12],

$$\frac{2}{q_{j^*} \max\{q_{j^*}, Q - q_{j^*}\}} \leq \frac{2T}{T'(T - 1)DQ}.$$

Combining the above inequality, (2.7), and (2.10), we have

$$\frac{2T(s - r)}{Q^2} \geq \frac{1}{vDQ} - \frac{2T}{T'(T - 1)DQ} = \frac{1}{DQ}\left(\frac{1}{v} - \frac{2T}{T'(T - 1)}\right).$$

Then,

$$(2.12) \qquad s - r > \frac{Q}{D}\left(\frac{1}{2vT} - \frac{1}{T'(T - 1)}\right).$$

We now assume that $q_{j^*} < T'D$. Consider the neighbors of $\frac{a_{j^*}}{q_{j^*}}$ in the set of Farey fractions of order Q with

$$\frac{u}{v} < \frac{a_{j^*}}{q_{j^*}} < \frac{z}{w}.$$

By the argument in [12], we know that the first right neighbors of $\frac{z}{w}$ are

$$(2.13) \qquad \frac{z - ka_{j^*}}{w - kq_{j^*}}, \quad k = 0, 1, 2, \ldots, \left[\frac{Q + 1}{2q_{j^*}}\right] - 1,$$

and the first left neighbors of $\frac{u}{v}$ are

$$(2.14) \qquad \frac{u - k'a_{j^*}}{v - k'q_{j^*}}, \quad k' = 0, 1, 2, \ldots, \left[\frac{Q + 1}{2q_{j^*}}\right] - 1.$$

If $\frac{a_{j*}}{q_{j*}}$ does not coincide with $\frac{a_r}{q_r}$, and $\frac{a_r}{q_r}$ does not coincide with any of the fractions from (2.14), then all these fractions lie between $\frac{a_r}{q_r}$ and $\frac{a_s}{q_s}$. We have

$$(2.15) \qquad s - r \geq \left[\frac{Q+1}{2q_{j*}}\right] \geq \left[\frac{Q+1}{2T'D}\right],$$

in this case, the conclusion of the theorem is satisfied. Similarly, if $\frac{a_{j*}}{q_{j*}}$ does not coincide with $\frac{a_s}{q_s}$, and $\frac{a_s}{q_s}$ does not coincide with any fractions from (2.13), then again, all these fractions lie between $\frac{a_r}{q_r}$ and $\frac{a_s}{q_s}$. We obtain the same result as above.

Next we assume that $\frac{a_r}{q_r}$ coincides with $\frac{a_{j*}}{q_{j*}}$ or one of the fractions from (2.14), and $\frac{a_s}{q_s}$ coincides with $\frac{a_{j*}}{q_{j*}}$ or one of the fractions from (2.13). There are two cases:

1) $\frac{a_{j*}}{q_{j*}} \notin \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}$ or   2) $\frac{a_{j*}}{q_{j*}} \in \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}$.

2.3.1. *Case* 1). If $\frac{a_{j*}}{q_{j*}} \notin \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}$, then each fraction $\frac{a_{r_i}}{q_{r_i}}$ $(1 \leq i \leq v)$ has the form

$$\frac{z - k_i a_{j*}}{w - k_i q_{j*}} \quad \text{or} \quad \frac{u - k_i a_{j*}}{v - k_i q_{j*}}$$

for some integer $k_i \geq 0$. So, a_{r_i} has the form

$$z - k_i a_{j*} \quad \text{or} \quad u - k_i a_{j*},$$

and q_{r_i} has the form

$$w - k_i q_{j*} \quad \text{or} \quad v - k_i q_{j*}.$$

In the following, we want to estimate the size of a_{r_i} and q_{r_i}.

Since $w + q_{j*} > Q$, $v + q_{j*} > Q$ (see [9], Chapter 3), and by the assumption $q_{j*} < T'D$, we have

$$w - k_i q_{j*} > Q - q_{j*} - k_i q_{j*} > Q - (k_i + 1)T'D.$$

Similarly,

$$v - k_i q_{j*} > Q - q_{j*} - k_i q_{j*} > Q - (k_i + 1)T'D.$$

So, we obtain, for any $1 \leq i \leq v$,

$$(2.16) \qquad q_{r_i} > Q - (k_i + 1)T'D.$$

Let $\gamma = \frac{a_{j*}}{q_{j*}}$. We know that

$$(2.17) \qquad \frac{u}{v} < \frac{u + a_{j*}}{v + q_{j*}} < \frac{a_{j*}}{q_{j*}} < \frac{z + a_{j*}}{w + q_{j*}} < \frac{z}{w}.$$

Then,

$$\frac{a_{j*}}{q_{j*}} - \frac{u + a_{j*}}{v + q_{j*}} = \frac{1}{q_{j*}(v + q_{j*})} < \frac{1}{Q}.$$

Therefore

$$u + a_{j*} > (v + q_{j*})(\gamma - \frac{1}{Q}) > Q(\gamma - \frac{1}{Q}) = Q\gamma - 1,$$

and so

(2.18) $$u > Q\gamma - a_{j^*} - 1.$$

By (2.17),

$$\frac{z + a_{j^*}}{w + q_{j^*}} > \gamma,$$

so we have

(2.19) $$z > (w + q_{j^*})\gamma - a_{j^*} > Q\gamma - a_{j^*}.$$

Thus, by (2.18) and (2.19), for any $1 \le i \le v$, we obtain

(2.20) $\quad a_{r_i} > Q\gamma - a_{j^*} - 1 - k_i a_{j^*} = Q\gamma - (k_i+1)\gamma q_{j^*} - 1 > Q\gamma - (k_i+1)\gamma T'D - 1.$

We also need an upper bound for a_{r_i}. Since

$$\frac{a_{j^*}}{q_{j^*}} - \frac{u}{v} = \frac{1}{vq_{j^*}},$$

we have

$$u = v\left(\gamma - \frac{1}{vq_{j^*}}\right) = v\gamma - \frac{1}{q_{j^*}} < Q\gamma,$$

and

$$\frac{z}{w} - \frac{a_{j^*}}{q_{j^*}} = \frac{1}{wq_{j^*}},$$

so,

$$z = w\left(\gamma + \frac{1}{wq_{j^*}}\right) = w\gamma + \frac{1}{q_{j^*}} < Q\gamma + 1.$$

Thus, for any $1 \le i \le v$,

(2.21) $$a_{r_i} < Q\gamma + 1.$$

In **Case I**, by condition 1), we have

$$\sum_{i=1}^{M} A_i \ne \sum_{j=1}^{N} B_j.$$

If $\sum_{i=1}^{M} A_i < \sum_{j=1}^{N} B_j$, by (2.16), and the second inequality in **Case I**, we obtain

$$\left(\sum_{i=1}^{M} A_i\right) Q > \sum_{i=1}^{M} A_i q_{r_i} \ge \sum_{j=1}^{N} B_j q_{s_j} + 1 > \sum_{j=1}^{N} B_j(Q - (k_j+1)T'D)$$

$$\ge \left(1 + \sum_{i=1}^{M} A_i\right) Q - T'D^2 \sum_{j=1}^{N} (k_j+1).$$

Therefore

(2.22) $$\sum_{j=1}^{N} (k_j+1) > \frac{Q}{T'D^2}.$$

If $\sum_{i=1}^{M} A_i > \sum_{j=1}^{N} B_j$, we use the first inequality in **Case I**, and by (2.20) and (2.21), we have

$$\sum_{j=1}^{N} B_j(Q\gamma + 1) > \sum_{j=1}^{N} B_j a_{s_j} \geq \sum_{i=1}^{M} A_i a_{r_i} + 1$$

$$> \sum_{i=1}^{M} A_i(Q\gamma - (k_i+1)\gamma T'D - 1)$$

$$> Q\gamma \left(\sum_{i=1}^{M} A_i \right) - \gamma T'D^2 \sum_{i=1}^{M}(k_i+1) - \sum_{i=1}^{M} A_i$$

$$\geq Q\gamma \left(1 + \sum_{j=1}^{N} B_j \right) - \gamma T'D^2 \sum_{i=1}^{M}(k_i+1) - \sum_{i=1}^{M} A_i.$$

Then,

$$T'D^2 \sum_{i=1}^{M}(k_i+1) + \frac{1}{\gamma} \left(\sum_{i=1}^{M} A_i + \sum_{j=1}^{N} B_j \right) > Q.$$

Since $\gamma = \frac{a_{j*}}{q_{j*}} > \frac{1}{T'D}$, $\frac{1}{\gamma} < T'D$, we have

$$(2.23) \qquad \frac{1}{\gamma} \left(\sum_{i=1}^{M} A_i + \sum_{j=1}^{N} B_j \right) < T'D^2(M+N) = T'D^2 v.$$

It follows that

$$(2.24) \qquad \sum_{i=1}^{M}(k_i+1) + v > \frac{Q}{T'D^2}.$$

For **Case II**, in the above calculations, we only need to change the positions of a_{r_i} and q_{r_i}, and change the positions of a_{s_j} and q_{s_j}.

If $\sum_{i=1}^{M} A_i < \sum_{j=1}^{N} B_j$, by (2.20), (2.21), and the first inequality in **Case II**, we obtain

$$(2.25) \qquad \sum_{j=1}^{N}(k_j+1) + v > \frac{Q}{T'D^2}.$$

If $\sum_{i=1}^{M} A_i > \sum_{j=1}^{N} B_j$, by (2.16), and the second inequality in **Case II**, we have

$$(2.26) \qquad \sum_{i=1}^{M}(k_i+1) > \frac{Q}{T'D^2}.$$

By the definitions of k_i and k_j, we can see that

$$s - r \geq \max\{k_i, k_j : 1 \leq i \leq M, 1 \leq j \leq N\} + 1.$$

Hence, combining (2.22), (2.24), (2.25), and (2.26), we obtain

$$s - r > \frac{Q}{vT'D^2} - 1,$$

and since $s - r$ is an integer, one has

(2.27) $$s - r \geq \left[\frac{Q}{vT'D^2}\right], \text{ if } \frac{a_{j*}}{q_{j*}} \notin \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}.$$

2.3.2. *Case* 2). Next, we consider the case $\frac{a_{j*}}{q_{j*}} \in \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}$ which can be divided into two subcases:

$$\frac{a_{j*}}{q_{j*}} \in \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq M\right\} \quad \text{or} \quad \frac{a_{j*}}{q_{j*}} \in \left\{\frac{a_{s_j}}{q_{s_j}} : 1 \leq j \leq N\right\}.$$

Let j' be the index of the coefficient associated to a_{j*} and q_{j*}.

1) $\frac{a_{j*}}{q_{j*}} \in \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq M\right\}$. By condition 2), we may assume that $\sum_{i \neq j'} A_i \neq \sum_{j=1}^{N} B_j$.

In **Case I**, if $\sum_{i \neq j'} A_i < \sum_{j=1}^{N} B_j$, we use the second inequality in **Case I**, and (2.16), to obtain

$$\left(\sum_{i \neq j'} A_i\right) Q + T'D^2 > \sum_{i \neq j'} A_i q_{r_i} + A_{j'} q_{j*} \geq \sum_{j=1}^{N} B_j q_{s_j} + 1$$

$$> \sum_{j=1}^{N} B_j (Q - (k_j + 1)T'D)$$

$$\geq \left(\sum_{i \neq j'} A_i + 1\right) - T'D^2 \sum_{j=1}^{N} (k_j + 1).$$

So, we have

(2.28) $$\sum_{j=1}^{N} (k_j + 1) + 1 > \frac{Q}{T'D^2}.$$

If $\sum_{i \neq j'} A_i > \sum_{j=1}^{N} B_j$, we use the first inequality in **Case I**, together with (2.20) and (2.21), to derive

$$\sum_{j=1}^{N} B_j(Q\gamma + 1) \geq \sum_{i=1}^{M} A_i a_{r_i} + 1 > \sum_{i \neq j'} A_i a_{r_i}$$

$$> \sum_{i \neq j'} A_i (Q\gamma - (k_i + 1)T'D\gamma - 1)$$

$$\geq \left(\sum_{j=1}^{N} B_j + 1\right) Q\gamma - T'D\gamma \sum_{i \neq j'} A_i(k_i + 1) - \sum_{i \neq j'} A_i$$

$$\geq \left(\sum_{j=1}^{N} B_j + 1 \right) Q\gamma - T'D^2\gamma \sum_{i \neq j'} (k_i + 1) - \sum_{i \neq j'} A_i.$$

Therefore

$$T'D^2 \sum_{i \neq j'} (k_i + 1) + \frac{1}{\gamma} \left(\sum_{i \neq j'} A_i + \sum_{j=1}^{N} B_j \right) > Q.$$

Similarly to (2.23), we have

$$\frac{1}{\gamma} \left(\sum_{i \neq j'} A_i + \sum_{j=1}^{N} B_j \right) < T'D^2(v - 1).$$

Thus,

(2.29)
$$\sum_{i \neq j'} (k_i + 1) + v - 1 > \frac{Q}{T'D^2}.$$

In **Case II**, similar to **Case I**, if $\sum_{i \neq j'} A_i < \sum_{j=1}^{N} B_j$, we use the first inequality in **Case II**, and (2.20) and (2.21), to derive

(2.30)
$$\sum_{j=1}^{N} (k_j + 1) + v > \frac{Q}{T'D^2}.$$

If $\sum_{i \neq j'} A_i > \sum_{j=1}^{N} B_j$, we use the second inequality in **Case II**, and (2.16), to obtain

(2.31)
$$\sum_{i \neq j'} (k_i + 1) > \frac{Q}{T'D^2}.$$

2) $\frac{a_{j*}}{q_{j*}} \in \left\{ \frac{a_{s_j}}{q_{s_j}} : 1 \leq j \leq N \right\}$. Because of symmetry, one can get results similar to the above in this case as well.

In **Case I**, if $\sum_{i=1}^{M} A_i < \sum_{j \neq j'}^{N} B_j$, we have

(2.32)
$$\sum_{j \neq j'} (k_j + 1) > \frac{Q}{T'D^2}.$$

If $\sum_{i=1}^{M} A_i > \sum_{j \neq j'}^{N} B_j$, then

(2.33)
$$\sum_{i=1}^{M} (k_i + 1) + v > \frac{Q}{T'D^2}.$$

In **Case II**, if $\sum_{i=1}^{M} A_i < \sum_{j \neq j'}^{N} B_j$, one has

(2.34)
$$\sum_{j \neq j'} (k_j + 1) + v - 1 > \frac{Q}{T'D^2}.$$

If $\sum_{i=1}^{M} A_i > \sum_{j \neq j'}^{N} B_j$, then

$$(2.35) \qquad \sum_{i=1}^{M}(k_i + 1) + 1 > \frac{Q}{T'D^2}.$$

Similar to (2.27), combining (2.28-2.35), we find that

$$(2.36) \qquad s - r \geq \left[\frac{Q}{vT'D^2}\right], \text{ if } \frac{a_{j^*}}{q_{j^*}} \in \left\{\frac{a_{r_i}}{q_{r_i}} : 1 \leq i \leq v\right\}.$$

Thus, for $t = 1$ and $q_{j^*} < T'D$, by (2.15), (2.27), and (2.36), we get,

$$(2.37) \qquad s - r \geq \left[\frac{Q}{vT'D^2}\right].$$

Therefore, by (2.9), (2.11), (2.12), and (2.37), there exists a constant $C = C(v)$, such that

$$s - r > \frac{Q}{CD^2}.$$

We now provide an admissible choice for the constant $C(v)$ in our theorem. Take $T = 16$, and $T' = 4v$. In case $t = 0$, by (2.9), we get

$$s - r > \frac{Q}{32vD}.$$

In case $t \geq 2$, by (2.11), we get

$$s - r > \frac{Q}{64vD}.$$

In case $t = 1$, and $q_{j^*} \geq T'D$, by (2.12), we obtain

$$s - r > \frac{Q}{D}\left(\frac{1}{32v} - \frac{1}{60v}\right) > \frac{Q}{69vD}.$$

In case $t = 1$, and $q_{j^*} < T'D$, by (2.37), we get

$$s - r \geq \left[\frac{Q}{4v^2D^2}\right].$$

Hence, one can take $C = C(v) = 69v^2$.
This completes the proof of the theorem.

3. Examples

In this section we provide counterexamples in some cases when one does not assume the conditions on coefficients from the statement of the theorem. First, we assume that

$$\sum_{i=1}^{v} A_i = 0, \ i.e., \ \sum_{i=1}^{M} A_i = \sum_{j=1}^{N} B_j.$$

Consider the neighbors of $\frac{1}{2}$. If Q is even, let $Q = 2K$, then the neighbors of $\frac{1}{2}$ are

$$\dots, \frac{K-2}{2K-3}, \frac{K-1}{2K-1}, \frac{1}{2}, \frac{K}{2K-1}, \frac{K-1}{2K-3}, \dots$$

Let $\gamma_0 = \frac{1}{2}$, and let γ_{-i} ($i > 0$) be the i-th fraction to the left of $\frac{1}{2}$, and γ_i be the i-th fraction to the right of $\frac{1}{2}$.

Then, we have the following examples.

Example 1. Let $v = 3$ and $D = 2$. Denote as usual $\mathbf{x} = (x_1, x_2, x_3)$, and consider the linear form

$$L(\mathbf{x}) = x_1 + x_2 - 2x_3.$$

Take the fractions γ_{-2}, γ_{-1}, and γ_2,

$$\gamma_{-2} = \frac{K-2}{2K-3}, \ \gamma_{-1} = \frac{K-1}{2K-1}, \ \gamma_2 = \frac{K-1}{2K-3}.$$

Then,

$$L(\mathbf{a}) = (K-2) + (K-1) - 2(K-1) = -1,$$

and

$$L(\mathbf{q}) = (2K-3) + (2K-1) - 2(2K-3) = 2.$$

So, for any even Q, we always have

$$L(\mathbf{a})L(\mathbf{q}) < 0.$$

But, here $s - r = 4$. Thus, there does not exist a constant C such that, for any Q and any three fractions satisfying $s - r \leq \frac{Q}{CD^2}$, we have

$$L(\mathbf{a})L(\mathbf{q}) \geq 0.$$

Actually, for any two positive integers m and n satisfying $m + n \leq D$, one may consider the linear form

$$L(\mathbf{x}) = mx_1 + nx_2 - (m+n)x_3.$$

Then taking the same three fractions γ_{-2}, γ_{-1}, and γ_2, as above, one sees that

$$L(\mathbf{a}) = m(K-2) + n(K-1) - (m+n)(K-1) = -m,$$

and

$$L(\mathbf{q}) = m(2K-3) + n(2K-1) - (m+n)(2K-3) = 2n.$$

Thus, we always have

$$L(\mathbf{a})L(\mathbf{q}) < 0.$$

Example 2. Let $v = 4$, $D = 4$, and consider the linear form

$$L(\mathbf{x}) = x_1 + x_2 + 2x_3 - 4x_4.$$

Take the fractions

$$\gamma_{-3} = \frac{K-3}{2K-5}, \ \gamma_{-2} = \frac{K-2}{2K-3}, \ \gamma_{-1} = \frac{K-1}{2K-1}, \ \gamma_2 = \frac{K-1}{2K-3}.$$

Here $s - r = 5$. But,

$$L(\mathbf{a}) = (K - 3) + (K - 2) + 2(K - 1) - 4(K - 1) = -3,$$

and

$$L(\mathbf{q}) = (2K - 5) + (2K - 3) + 2(2K - 1) - 4(2K - 3) = 2.$$

Thus, $L(\mathbf{a})L(\mathbf{q}) < 0$. Hence, there is no constant C such that, for any four fractions, if $s - r \leq \frac{Q}{CD^2}$, we have

$$L(\mathbf{a})L(\mathbf{q}) \geq 0.$$

Example 3. Let $v = 5$, $D = 2$, and consider the linear form

$$L(\mathbf{x}) = x_1 + x_2 + 2x_3 - 2x_4 - 2x_5.$$

Let us choose the fractions

$$\gamma_{-3} = \frac{K - 3}{2K - 5}, \ \gamma_{-2} = \frac{K - 2}{2K - 3}, \ \gamma_{-1} = \frac{K - 1}{2K - 1}, \ \gamma_2 = \frac{K - 1}{2K - 3}, \ \gamma_3 = \frac{K - 2}{2K - 5}.$$

Then,

$$L(\mathbf{a}) = (K - 3) + (K - 2) + 2(K - 1) - 2(K - 1) - 2(K - 2) = -1,$$

and

$$L(\mathbf{q}) = (2K - 5) + (2K - 3) + 2(2K - 1) - 2(2K - 3) - 2(2K - 5) = 6.$$

Thus, for any even Q, $L(\mathbf{a})L(\mathbf{q}) < 0$. Here $s - r = 6$.

Example 4. Let $v = 6$, $D = 3$, and consider the linear form

$$L(\mathbf{x}) = x_1 + 2x_2 + 3x_3 - 2x_4 - x_5 - 3x_6.$$

Take the fractions

$$\gamma_{-3} = \frac{K - 3}{2K - 5}, \ \gamma_{-2} = \frac{K - 2}{2K - 3}, \ \gamma_{-1} = \frac{K - 1}{2K - 1},$$

$$\gamma_1 = \frac{K}{2K - 1}, \ \gamma_2 = \frac{K - 1}{2K - 3}, \ \gamma_3 = \frac{K - 2}{2K - 5}.$$

Then,

$$L(\mathbf{a}) = (K - 3) + 2(K - 2) + 3(K - 1) - 2K - (K - 1) - 3(K - 2) = -3,$$

and

$$L(\mathbf{q}) = (2K - 5) + 2(2K - 3) + 3(2K - 1) - 2(2K - 1) - (2K - 3) - 3(2K - 5) = 6.$$

Thus, for any even Q, $L(\mathbf{a})L(\mathbf{q}) < 0$. Here $s - r = 6$.

Note. If Q is odd, $Q = 2K + 1$ say, then the neighbors of $\frac{1}{2}$ are

$$\ldots, \frac{K - 1}{2K - 1}, \frac{K}{2K + 1}, \frac{1}{2}, \frac{K + 1}{2K + 1}, \frac{K}{2K - 1}, \ldots$$

Then, the above linear forms also provide counterexamples in this case. We remark that one can also construct counterexamples using the neighbors of $\frac{1}{3}$.

References

[1] E. Alkan, M. S. Xiong, and A. Zaharescu, *A bias phenomenon on the behavior of Dedekind sums*, Math. Res. Lett. **15** (2008), no. 5, 1039–1052.

[2] ———, *Quotients of values of the Dedekind eta function*, Math. Ann. **342** (2008), no. 1, 157–176.

[3] J. Athreya and Y. Cheung, *A Poincaré section for horocycle flow on the space of lattices*, to appear in Int. Math. Res. Not. IMRN 2013.

[4] F. P. Boca, C. Cobeli, and A. Zaharescu, *Distribution of lattice points visible from the origin*, Comm. Math. Phys. **213** (2000), no. 2, 433–470.

[5] ———, *A conjecture of R. R. Hall on Farey points*, J. Reine Angew. Math. **535** (2001), 207–236.

[6] F. P. Boca and A. Zaharescu, *Farey fractions and two-dimensional tori*, Noncommutative geometry and number theory, 57–77, Aspects Math., E37, Vieweg, Wiesbaden, 2006.

[7] P. Erdös, *A note on Farey series*, Quart. J. Math., Oxford Ser. **14** (1943), 82–85.

[8] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1988, Reprint of the 1952 edition. Cambridge Mathematical Library, xii+324pp.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1938.

[10] A. E. Mayer, *A mean value theorem concerning Farey series*, Quart. J. Math., Oxford Ser. **13** (1942), 48–57.

[11] ———, *On neighbours of higher degree in Farey series*, Quart. J. Math., Oxford Ser. **13** (1942), 185–192.

[12] A. Zaharescu, *The Mayer-Erdös phenomenon*, Indag. Math. (N.S.) **17** (2006), no. 1, 147–156.

Xianchang Meng
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street, Urbana, Illinois 61801, USA
E-mail address: xmeng13@illinois.edu

Alexandru Zaharescu
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street, Urbana, Illinois 61801, USA
E-mail address: zaharesc@math.uiuc.edu