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On Zeros of Polynomials with Restricted Coefficients

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Abstract. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $Re\ a_j=\alpha_j,\ Im\ a_j=\beta_j$. In this paper, we have obtained a zero- free region for polynomials in terms of α_j and β_j and also obtain the bound for number of zeros that can lie in a prescribed region.

1. Introduction

One of the classical result concerning the roots of a polynomial is known in the literature as the Eneström-kakeya theorem [1-2, 4-5]. Applying this result to the polynomial $P(kz),\ k>0$ the following more general result is immediate.

Theorem A. *If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with*

$$k^n a_n \geq k^{n-1}a_{n-1} \geq k^{n-2}a_{n-2} \geq \ldots \geq ka_1 \geq a_0,$$

then $P(z)$ does not vanish in $|z|>k$.

Concerning the number of zeros of polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ in $|z|\leq \frac{1}{2}$, Mohammad [3] proved the following:

Theorem B. *If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0 > 0,$$

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the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

2. Theorems and Proofs

In this direction by relaxing the restrictions on the coefficients of polynomials, we prove some more general results.

Theorem 2.1. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients. If Re $a_j = \alpha_j$ and Im $a_j = \beta_j$, $j = 0, 1, 2, \dots, n$ and for some $\lambda \geq 1$*

$$\alpha_n \leq \alpha_{n-1} \leq ... \leq \lambda\alpha_k \geq \ldots \geq \alpha_1 \geq \alpha_0, \quad 0 \leq k \leq n, \tag{2.1}$$

then the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$, does not exceed

$$\frac{1}{\log 2} \log \frac{(|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2\lambda\alpha_k + 2(\lambda - 1)|\alpha_k| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Proof. Consider the polynomial

$$\begin{aligned} F(z) &= (1 - z)P(z) \\ &= (1 - z)\left(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0\right). \end{aligned}$$

For $|z| = 1$

$$\begin{aligned} |F(z)| &\leq |a_0| + \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_n| \\ &\leq |\alpha_0| + |\beta_0| + |\alpha_n + |\beta_n| + \sum_{j=1}^{n} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}| \\ &\leq |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |\alpha_k - \alpha_{k-1}| + \sum_{j=1}^{k-1} |\alpha_j - \alpha_{j-1}| \\ &\qquad + |\alpha_k - \alpha_{k-1}| + \sum_{j=k+2}^{n} |\alpha_{j-1} - \alpha_j| + |\beta_0| - |\beta_n| + 2\sum_{j=1}^{n} |\beta_j| \\ &= |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |\lambda\alpha_k + \alpha_k - \lambda\alpha_k - \alpha_{k-1}| + \sum_{j=1}^{k-1} |\alpha_j - \alpha_{j-1}| \\ &\qquad + |\lambda\alpha_k + \alpha_k - \lambda\alpha_k - \alpha_{k+1}| + \sum_{j=k+2}^{n} |\alpha_{j-1} - \alpha_j| + |\beta_0| - |\beta_n| + 2\sum_{j=1}^{n} |\beta_j| \end{aligned}$$

$$\leq |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |\lambda\alpha_k - \alpha_{k-1}| + (\lambda - 1)|\alpha_k| + (\alpha_1 - \alpha_0 + \alpha_2 - \alpha_1 + ... + \alpha_{k-1} - \alpha_{k-2}) + |\lambda\alpha_k - \alpha_{k+1}| + (\lambda - 1)|\alpha_k| + (\alpha_{k+1} - \alpha_{k+2} + \alpha_{k+2} - \alpha_{k+3} + ... + \alpha_{n-1} - \alpha_n) + |\beta_0| - |\beta_n| + 2\sum_{j=1}^{n} |\beta_j|$$

$$\leq |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + \lambda\alpha_k - \alpha_{k-1} + (\lambda - 1)|\alpha_k| + \lambda\alpha_k - \alpha_{k+1} + (\lambda - 1)|\alpha_k| - \alpha_0 + \alpha_{k-1} + \alpha_{k+1} - \alpha_n + |\beta_0| - |\beta_n| + 2\sum_{j=1}^{n} |\beta_j|$$

$$\leq (|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2\lambda\alpha_k + 2(\lambda - 1)|\alpha_k| + 2\sum_{j=0}^{n} |\beta_j|.$$

Which implies that

$$\frac{|F(z)|}{|F(0)|} \leq \frac{(|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2\lambda\alpha_k + 2(\lambda - 1)|\alpha_k| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Now it is known ([5, pp.171]) that if $f(z)$ is regular, $f(0) \neq \frac{1}{2}$ and $|f(z)| \leq M$ in $|z| \leq 1$, then the number of zeros of of $f(z)$ in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2} \log \frac{M}{|f(0)|}.$$

Thus, the number of zeros of $F(z)$ in $|z| \leq \frac{1}{2}$ are

$$\frac{1}{\log 2} \log \frac{(|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2\lambda\alpha_k + 2(\lambda - 1)|\alpha_k| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

As the number of zeros of $P(z)$ in $|z| \leq \frac{1}{2}$ is also equal to the number of zeros $F(z)$ the theorem follows. □

If we choose $\lambda = 1$ in Theorem 2.1, we obtain the following:

Corollary 2.2. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients. If* $Re\ a_j = \alpha_j$ *and* $Im\ a_j = \beta_j$, $j = 0, 1, 2, \ldots, n$ *and for some* $\lambda \geq 1$

$$\alpha_n \leq \alpha_{n-1} \leq ... \leq \alpha_k \geq \ldots \geq \alpha_1 \geq \alpha_0, \quad 0 \leq k \leq n,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{(|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2\alpha_k + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Taking $\lambda = 1$ and $\alpha_0 > 0$ in (2.1), we have

Corollary 2.3. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients. If Re* $a_j = \alpha_j$ *and Im* $a_j = \beta_j$, $j = 0, 1, 2, \ldots, n$ *and for some* $\lambda \geq 1$

$$\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_k \geq \ldots \geq \alpha_1 \geq \alpha_0 > 0, \quad 0 \leq k \leq n,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{(|\alpha_n| - \alpha_n) + 2\alpha_k + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Along with the above conditions, if we choose $\beta_j = 0$ for $j = 0, 1, 2, \ldots, n$ in Theorem 2.1, we get the following:

Corollary 2.4. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real coefficients such that*

$$a_n \leq a_{n-1} \leq ... \leq a_k \geq \ldots \geq a_1 \geq a_0 > 0, \quad 0 \leq k \leq n,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{(|a_n| - a_n) + 2a_k}{|a_0|}.$$

Remark 2.5. For $\lambda = 1$, $\alpha_0 > 0$, $\beta_j = 0$ for $j = 0, 1, 2, \ldots, n$ and $k = n$, Theorem 2.1 yields Theorem B.

Next, we have the following:

Theorem 2.6. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients. If Re* $a_j = \alpha_j$, *Im* $a_j = \beta_j$, $j = 0, 1, 2, \ldots, n$ *and for some* $\lambda \geq 1$ *and* $\mu \geq 1$

$$\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \lambda\alpha_k \geq \ldots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \leq \beta_{n-1} \leq ... \leq \mu\beta_k \geq \ldots \geq \beta_1 \geq \beta_0, \quad 0 \leq k \leq n,$$

then the number of zeros of $P(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{(|\alpha_n| - \alpha_n) + (|\alpha_0| - \alpha_0) + 2(\lambda\alpha_k + \mu\beta_k) + 2(\lambda - 1)|\alpha_k| + 2(\mu - 1)|\beta_k| + (|\beta_n| - \beta_n) + (|\beta_0| - \beta_0)}{|a_0|}.$$

Proof. Proceeding on the same lines of proof of Theorem 1, the proof of this result follows. □

Theorem 2.7. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients. If* $a_j=\alpha_j+\ i\beta_j$, *and for some* $K\geq 1,\ L\geq 1$

$$K\alpha_n \geq \alpha_{n-1}\geq \ldots \geq \rho\alpha_\lambda \leq \alpha_{k-1}\leq \ldots \leq \alpha_1 \leq \alpha_0,$$
$$L\beta_n \geq \beta_{n-1}\geq \ldots \geq \rho\beta_k \leq \beta_{k-1}\leq \ldots \leq \beta_1 \leq \beta_0 > 0,$$

where α_0 *and* β_0 *are not zero simultaneously, then the maximum number of zeros in* $\frac{|a_0|}{M_1}<|z|\leq\delta, 0<\delta<1$ *does not exceed*

$$\frac{1}{\log(1/\delta)}\log\left[\frac{(\alpha_0+|\alpha_0|)+(\beta_0+|\beta_0|)-(\rho+1)\left(\alpha_\lambda+\beta_\mu\right)+K\left(\alpha_n+|\alpha_n|\right)+L\left(\beta_n+|\beta_n|\right)}{|a_0|}\right].$$

where

$$M_1=(\alpha_0+|\beta_0|)+K\left(\alpha_n+|\alpha_n|\right)+L\left(\beta_n+|\beta_n|\right)$$
$$-(\rho+1)\left(\alpha_\lambda+\beta_\mu\right)+(\rho-1)\left(|\alpha_\lambda|+|\beta_\mu|\right).$$

Proof. Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)\left(a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0\right).$$

For $|z|=1$

$$|F(z)|\leq|a_0|+\sum_{j=1}^{n}|a_j-a_{j-1}|+|a_n|$$
$$\leq|\alpha_0+|\beta_0|+|\alpha_n|+|\beta_n|+\sum_{j=1}^{n}|\alpha_j-\alpha_{j-1}|+\sum_{j=1}^{n}|\beta_j-\beta_{j-1}|$$
$$\leq|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+|\alpha_n-\alpha_{n-1}|+|\alpha_{\lambda+1}-\alpha_\lambda|$$
$$+\sum_{j=1}^{\lambda}|\alpha_j-\alpha_{j-1}|+\sum_{j=\lambda+2}^{n-1}|\alpha_j-\alpha_{j-1}|+|\beta_n-\beta_{n-1}|+|\beta_{\mu+1}-\beta_\mu|$$
$$+\sum_{j=1}^{\mu}|\beta_j-\beta_{j-1}|+\sum_{\mu+2}^{n-1}|\beta_j-\beta_{j-1}|$$
$$=|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+|K\alpha_n+\alpha_n-K\alpha_n-\alpha_{n-1}|+|\rho\alpha_\lambda$$
$$+\alpha_{\lambda+1}-\alpha_\lambda-\rho\alpha_\lambda|+\sum_{j=1}^{\lambda}|\alpha_j-\alpha_{j-1}|+\sum_{j=\lambda+2}^{n-1}|\alpha_j-\alpha_{j-1}|+|L\beta_n$$
$$+\beta_n-L\beta_n-\beta_{n-1}|+|\rho\beta_\mu+\beta_{\mu+1}-\rho\beta_\mu-\beta_\mu|+\sum_{j=1}^{\mu}|\beta_j-\beta_{j-1}|$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{j=\mu+2}^{n-1}|\beta_j-\beta_{j-1}|\\
&=|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+|K\alpha_n-\alpha_{n-1}|+(K-1)|\alpha_n|+|\alpha_{\lambda+1}-\rho\alpha_\lambda|\\
&\quad+(\rho-1)|\alpha_\lambda|+\sum_{j=1}^{\lambda}|\alpha_{j-1}-\alpha_j|+\sum_{j=\lambda+2}^{n-1}|\alpha_j-\alpha_{j-1}|+|L\beta_n-\beta_{n-1}|\\
&\quad+(L-1)|\beta_n|+|\beta_\mu-\rho\beta_\mu|+(\rho-1)|\beta_\mu|+\sum_{j=1}^{\mu}|\beta_{j-1}-\beta_j|+\sum_{j=\mu+2}^{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad |\beta_j-\beta_{j-1}|\\
&\le|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+|K\alpha_n-\alpha_{n-1}|+(K-1)|\alpha_n|+|\alpha_{\lambda+1}-\rho\alpha_\lambda|\\
&\quad+(\rho-1)|\alpha_\lambda|+\alpha_0-\alpha_1+\alpha_1-\alpha_2+...+\alpha_{\lambda-2}-\alpha_{\lambda-1}+\alpha_{\lambda-1}-\alpha_\lambda\\
&\quad+\alpha_{\lambda+2}-\alpha_{\lambda+1}+\alpha_{\lambda+3}-\alpha_{\lambda+2}+...+\alpha_{n-1}-\alpha_{n-2}+\beta_0-\beta_1\\
&\quad+\beta_1-\beta_2+...+\beta_{\mu-2}-\beta_{\mu-1}+\beta_{\mu-1}-\beta_\mu+\beta_{\mu+2}-\beta_{\mu+1}+\beta_{\mu+3}\\
&\quad-\beta_{\mu+2}+...+\beta_{n-1}-\beta_{n-2}+|L\beta_n-\beta_{n-1}|+(L-1)|\beta_n|\\
&\qquad\qquad\qquad\qquad+|\beta_{\mu+1}-\rho\beta_\mu|+(\rho-1)|\beta_\mu|\\
&=|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+|K\alpha_n-\alpha_{n-1}|+K|\alpha_n|-|\alpha_n|+\alpha_{\lambda+1}-\rho\alpha_\lambda\\
&\quad+\rho|\alpha_\lambda|+\alpha_0-\alpha_\lambda-\alpha_{\lambda+1}+\alpha_{n-1}+\beta_0-\beta_\mu+\beta_{\mu+2}-\beta_{\mu+1}+\beta_{n-1}\\
&\quad-\beta_{n-2}+L\beta_n-\beta_{n-1}+\beta_{\mu+1}-\rho\beta_\mu+L|\beta_n|-\beta_n+\rho|\beta_\mu|-\beta_\mu|\\
&=|\alpha_0|+|\beta_0|+K|\alpha_n|+K|\alpha_n|+\alpha_0-\alpha_\lambda+\rho|\alpha_\lambda|-|\alpha_\lambda|-\rho\alpha_\lambda+\beta_0-\beta_\mu\\
&\quad+L\beta_n+L|\beta_n|-\rho\beta_\mu-|\beta_n|+\rho\beta_\mu\\
&=(|\alpha_0|+\alpha_0)+(|\beta_0|+\beta_0)+K(|\alpha_n|+\alpha_n)+L(|\beta_n|+\beta_n)-(|\alpha_\lambda|+\alpha_\lambda)\\
&\quad+\rho(|\alpha_\lambda|-\alpha_\lambda)-(\beta_\mu+|\beta_\mu|)+\rho(|\beta_\mu|-\beta_\mu)\\
&=(|\alpha_0|+\alpha_0)+(|\beta_0|+\beta_0)+K(|\alpha_n|+\alpha_n)+L(|\beta_n|+\beta_n)\\
&\quad[(\rho-1)|\alpha_\lambda|-(\rho+1)\alpha_\lambda]+[(\rho-1)|\beta_\mu|-(\rho+1)\beta_\mu]=M_1.
\end{aligned}
$$

Thus, $|F(z)|\le M_1$, for $|z|=1$.

Also, $|F(0)|=|a_0|\neq 0$ as α_0 and β_0 are not zeros simultaneously.
Now it is known [5, pp.171] that if $f(z)$ is regular, $f(0)\neq 0$ and $|f(z)|\le M$ in $|z|\le 1$; then the number of zeros of $f(z)$ in $|z|\le\delta<1$ does not exceed. Applying this result to $F(z)$, we get the number of zeros of $F(z)$ and hence of $p(z)$ in $|z|<\delta$ does not exceed

$$
\begin{aligned}
&\tfrac{1}{\log(1/\delta)}\log[[(|\alpha_0|+\alpha_0)+(|\beta_0|+\beta_0)+K\left(|\alpha_n|+\alpha_n\right)+L\left(|\beta_n|+\beta_n\right)\\
&\qquad+[(\rho-1)\,|\alpha_\lambda|-(\rho+1)\,\alpha_\lambda]+[(\rho-1)\,|\beta_\mu|-(\rho+1)\,\beta_\mu]]/a_0]
\end{aligned}
$$

This proves the one part of the theorem.
Now we show that no zero lie in

$$|z| \leq \frac{|a_0|}{M_1}.$$

For this we have

$$F(z) = a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + ... + (a_n - a_{n-1})z^n - a_n z^{n+1}$$

$$F(z) = a_0 + h(z), \tag{2.2}$$

where

$$h(z) = -a_n z^{n+1} + (a_1 - a_0)z + (a_2 - a_1)z^2 + ... + (a_n - a_{n-1})z^n,$$

that is,

$$h(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=1}^{n-1}(a_j - a_{j-1})z^j.$$

For $|z| = 1$,

$$\max_{|z|=1} |h(z)| \leq |a_n| + |a_n - a_{n-1} + \sum_{j=1}^{n-1} |a_j - a_{j-1}|$$

$$\leq (|\alpha_n| + |\beta_n|) + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + \sum_{j=1}^{n-1} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n-1} |\beta_j - \beta_{j-1}|$$

$$\leq (\alpha_n + |\beta_n|) + |K\alpha_n - \alpha_{n-1} - K\alpha_n + \alpha_n| + |L\beta_n - \beta_{n-1} - L\beta_n + \beta_n|$$
$$+ \sum_{j=1}^{\lambda} |\alpha_j - \alpha_{j-1}| + \sum_{\lambda+1}^{n-1} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{\lambda} |\beta_j - \beta_{j-1}| + \sum_{j=\lambda+1}^{n-1} |\beta_j - \beta_{j-1}|$$

$$\leq (|\alpha_n + |\beta_n|) + |K\alpha_n - \alpha_{n-1} - K\alpha_n + \alpha_n| + |L\beta_n - \beta_{n-1} - L\beta_n + \beta_n|$$
$$+ \sum_{j=1}^{\lambda} |\alpha_{j-1} - \alpha_j| + |\alpha_{\lambda+1} - \alpha_\lambda| + \sum_{j=\lambda+2}^{n-1} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{\lambda} |\beta_{j-1} - \beta_j| + |\beta_{\lambda+1}$$
$$- \beta_\lambda| + \sum_{j=\lambda+2}^{n-1} |\beta_j - \beta_{j-1}|$$

$$= (|\alpha_n + |\beta_n|) + K\alpha_n - \alpha_{n-1} + (K-1)|\alpha_n| + L\beta_n - \beta_{n-1} + (L-1)|\beta_n| + \alpha_0 - \alpha_1$$
$$+\alpha_1 - \alpha_2 + ... + \alpha_{\lambda-2} - \alpha_{\lambda-1} + \alpha_{\lambda-1} - \alpha_\lambda + |\rho\alpha_\lambda + \alpha_{\lambda+1} - \alpha_\lambda - \rho\alpha_\lambda| + \alpha_{\lambda+2}$$

$$-\alpha_{\lambda+1}+\alpha_{\lambda+3}-\alpha_{\lambda+2}+...+\alpha_{n-2}-\alpha_{n-3}+\alpha_{n-1}-\alpha_{n-2}+\beta_0-\beta_1+\beta_1-\beta_2$$
$$+...+\beta_{\mu-2}-\beta_{\mu-1}+\beta_{\mu-1}-\beta_\mu+|\rho\beta_\mu+\beta_{\mu+1}-\rho\beta_\mu-\beta_\mu|+\beta_{\mu+2}-\beta_{\mu+1}$$
$$+\beta_{\mu+3}-\beta_{\mu+2}+...+\beta_{n-2}-\beta_{n-3}+\beta_{n-1}-\beta_{n-2}$$
$$=(|\alpha_n|+|\beta_n|)+K\alpha_n-\alpha_{n-1}+(K-1)|\alpha_n|+L\beta_n-\beta_{n-1}+(L-1)|\beta_n|+\alpha_0-\alpha_1$$
$$+\alpha_1-\alpha_2+...+\alpha_{\lambda-2}-\alpha_{\lambda-1}-\alpha_\lambda+|\alpha_{\lambda+1}-\rho\alpha_\lambda|+(\rho-1)|\alpha_\lambda|+\alpha_{\lambda+2}$$
$$-\alpha_{\lambda+1}+\alpha_{\lambda+3}-\alpha_{\lambda+2}+...+\alpha_{n-2}-\alpha_{n-3}+\alpha_{n-1}-\alpha_{n-2}+\beta_0-\beta_1$$
$$+\beta_1-\beta_2+...+\beta_{\mu-2}-\beta_{\mu-1}+\beta_{\mu-1}-\beta_\mu+|\beta_{\mu+1}-\rho\beta_\mu|+(\rho-1)|\beta_\mu|$$
$$+\beta_{\mu+2}-\beta_{\mu+1}+\beta_{\mu+3}-\beta_{\mu+2}+...+\beta_{n-2}-\beta_{n-3}+\beta_{n-1}-\beta_{n-2}$$
$$=(|\alpha_n|+|\beta_n|)+K\alpha_n-\alpha_{n-1}+(K-1)|\alpha_n|+\alpha_0-\alpha_\lambda-\rho\alpha_\lambda+(\rho-1)|\alpha_\lambda|+\alpha_{n-1}$$
$$+L\beta_n-\beta_{n-1}+(L-1)|\beta_n|+\beta_0-\beta_\mu-\rho\beta_\mu+(\rho-1)|\beta_\mu|+\beta_{n-1}$$
$$=(\alpha_0+|\alpha_0|)+K(\alpha_n+|\alpha_n|)+L(\beta_n+|\beta_n|)-(\rho+1)(\alpha_\lambda+\beta_\mu)+(\rho-1)(|\alpha_\lambda|+|\beta_\mu|)$$
$$=M_1.$$

Thus,

$$\max_{|z|=1}|h(z)|\le M_1.$$

Therefore, by Schwartz lemma

$$|h(z)|\le M_1|z| \quad \text{for} \quad |z|\le 1.$$

Now from inequality (4)

$$F(z)=a_0+h(z)$$
$$\begin{aligned}|F(z)|&\ge|a_0|-|h(z)|\\ &\ge|a_0|-M_1|z| \quad for \quad |z|\le 1,\\ &>0,\end{aligned}$$

if

$$|z|\le\frac{|a_0|}{M_1}.$$

Therefore no zeros of $F(z)$ and hence of $P(z)$ lie in

$$|z|\le\frac{|a_0|}{M_1}.$$

This completes the proof of the Theorem 2.7. □

For $\rho=1$, in Theorem 2.7 we obtain :

Corollary 2.8. *If* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial with complex coefficients. If* $a_j=\alpha_j+\ i\beta_j$, *and for some* $K\geq 1,\ L\geq 1$

$$K\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq ... \leq \alpha_1 \leq \alpha_0 \tag{2.3}$$

and

$$L\beta_n \geq \beta_{n-1} \geq ... \geq \beta_k \leq ... \leq \beta_1 \leq \beta_0,$$

where α_0 *and* β_0 *are not zero simultaneously, then the maximum number of zeros in* $\frac{|a_0|}{M_2} < |z| \leq \delta,\ 0<\delta<1$ *does not exceed*

$$\frac{1}{\log(1/\delta)}\log\left[\frac{(\alpha_0+|\alpha_0|)+(\beta_0+|\beta_0|)-2\left(\alpha_\lambda+\beta_\mu\right)+K\left(\alpha_n+|\alpha_n|\right)+L\left(\beta_n+|\beta_n|\right)}{|a_0|}\right]$$

where

$$M_2=(\alpha_0+\beta_0)-2\left(\alpha_\lambda+\beta_\mu\right)+K\left(\alpha_n+|\alpha_n|\right)+L\left(\beta_n+|\beta_n|\right).$$

If we choose $\rho=1$ and $\lambda=\mu=0$ in Theorem 2.7, we get the following:

Corollary 2.9. *If* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial with complex coefficients. If* $a_j=\alpha_j+\ i\beta_j$, *and for some* $K\geq 1,\ L\geq 1$

$$K\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0 \tag{2.4}$$

$$L\beta_n \geq \beta_{n-1} \geq ... \geq \beta_0,$$

where α_0 *and* β_0 *are not zero simultaneously, then the maximum number of zeros in* $\frac{|a_0|}{M_3} < |z| \leq \delta,\ 0<\delta<1$ *does not exceed*

$$\frac{1}{\log(1/\delta)}\log\left[\frac{(|\alpha_0|+|\beta_0|)-(\alpha_0+\beta_0)+K\left(\alpha_n+|\alpha_n|\right)+L\left(\beta_n+|\beta_n|\right)}{|a_0|}\right]$$

where

$$M_3=K\left(\alpha_0+|\alpha_0|\right)+L\left(\beta_0+|\beta_0|\right)-(\alpha_0+\beta_0).$$

For $\rho=1$, $\alpha_j>0$ and $\beta_j>0$, $j=0,1,2,...,n$ in Theorem 2.7, we have

Corollary 2.10. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial with complex coefficients. If* $a_j=\alpha_j+\ i\beta_j$ *and for some* $K\geq 1,\ L\geq 1$

$$K\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0 > 0$$

and

$$L\beta_n \geq \beta_{n-1} \geq ... \geq \beta_0 > 0,$$

where α_0 *and* β_0 *are not zero simultaneously, then the maximum number of zeros in* $\frac{|a_0|}{M_4} < |z| \leq \delta,\ 0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log(1/\delta)} \log \left[\frac{2\,(\ K\alpha_n + L\beta_n)}{|a_0|} \right],$$

where

$$M_4 = 2\,(K\alpha_n + L\beta_n) - (\alpha_0 + \beta_0)\,.$$

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