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On Divisorial Submodules

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Abstract. This paper is devoted to study the divisorial submodules. We get some equivalent conditions for a submodule to be a divisorial submodule. Also we get equivalent conditions for $(N \cap L)^{-1}$ to be a ring, where N, L are submodules of a module M.

1. Introduction

Throughout this paper all rings are considered commutative rings with identiry and all modules are considered unitary. Let R be a commutative ring with identity and let M be an R-module. M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Let M be a multiplication R-module and N a submodule of M. Then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [8]. Let R be an integral domain, M a faithful multiplication R-module and N, L submodules of M. Then $(N :_R M)(L :_R M) = (NL :_R M)$ [7, Lemma 3.6]. Therefore $NL = (NL :_R M)M$. Thus we get $N^n = (N^n :_R M)M$. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [3]. By [17, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [9].

Let S be the set of all non-zero divisors of R and $T(R) = R_S$ the total quotient ring of R. For a nonzero ideal I of R, let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. An ideal I

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of R is called invertible, if $II^{-1} = R$. Let M be an R-module and set

$$T = \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\}.$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [9, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$ and $N_\nu = (N^{-1})^{-1}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. If I is an invertible ideal of R then IM is invertible in M and the converse is true if M is a finitely generated faithful multiplication R-module [15]. Every invertible submodule N of a finitely generated faithful multiplication R-module M is finitely generated faithful multiplication and the converse is true if R is an integral domain [1]. An R-module M is called a Dedekind module (resp., Prüfer module) if every nonzero submodule (resp., every nonzero finitely generated submodule) of M is invertible [15]. An R-module M is called a valuation module if for all $m, n \in M$, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$. Equivalently, M is a valuation module if for all submodules N and K of M, either $N \subseteq K$ or $K \subseteq N$ [4].

Following [3], a submodule N of M is called a divisorial submodule of M in case $N = N_\nu M$. If M is a finitely generated faithful multiplication R-module, then $N_\nu = (N :_R M)$. Consequently, $M_\nu = R$. Let M be a finitely generated faithful multiplication R-module, N a submodule of M and I an ideal of R. Then N is a divisorial submodule of M if and only if $(N :_R M)$ is a divisorial ideal of R. Also I is divisorial ideal of R if and only if IM is a divisorial submodule of M [2]. If N is an invertible submodule of a faithful multiplication module M over an integral domain R, then $(N :_R M)$ is invertible hence a divisorial ideal of R. So N is a divisorial submodule of M [2]. If R is an integral domain, M a faithful multiplication R-module and N a nonzero submodule of M, then $N_\nu = (N :_R M)_\nu$ [2, Lemma 1]. A submodule N of an R-module M is called an idempotent submodule of M if $N = (N :_R M)N$. It is shown that, if M is a multiplication R-module and N a submodule of M such that $(N :_R M)$ is an idempotent ideal of R, then N is an idempotent submodule of M. The converse is true if M is a finitely generated faithful multiplication R-module [5, Theorem 3]. We say that a submodule N of M is a radical submodule of M if $N = \sqrt{N}$, where $\sqrt{N} = \sqrt{(N :_R M)}M$. If $a \in \sqrt{N} = \sqrt{(N :_R M)}M$, then $a^n \in (N :_R M)M = N$. Also, $N = (N :_R M)M \subseteq \sqrt{(N :_R M)}M = \sqrt{N}$. It is shown that if R is an integral domain, M a faithful multiplication R-module and N a submodule of M, then N is a radical submodule of M if and only if $(N :_R M)$ is a radical ideal of R [6, Lemma 6]. Let R be a ring and M a finitely generated multiplication R-module. Then Q is a P-primary submodule of M if and only if $(Q :_R M)$ is a $(P :_R M)$-primary ideal of R [2, Lemma 4].

Let R be a an integral domain, M a faithful multiplication R-module and N a submodule of M. Then $T(N) = \bigcup_{n=1}^{\infty}(M :_K (N; M)^n M)$. Furthermore, if M is

finitely generated, then $T(N) = \bigcup_{n=1}^{\infty}(R :_K (N :_R M)^n) = T((N :_R M))$, [2].
An integral domain R is said to satisfy the trace property (TP) provided that $Tr(M) = \sum_{f \in Hom(M,R)} f(M)$ either equals R or is a prime ideal of R for each R-module M [11]. An R-module M satisfies trace property (TP) if $Tr_M(N) = M$ or $Tr_M(N)$ is a prime submodule for each nonzero submodule N of M, where $Tr_M(N) = \sum_{f \in Hom(N,M)} f(N)$, [2]. It is proved that if R is a ring, M an R-module and N a submodule of M, then $Tr_M(N)^{-1} = (Tr_M(N) :_{R_T} Tr_M(N))$. Moreover if M is a finitely generated faithful multiplication R-module, then $Tr_M(N) = NN^{-1}$, [2, Theorem 15].
In section 2, we generalize some properties of dividorial ideals of an integral domain to modules. In Lemma 2.1, we get equivalent conditions for a submodule to be a divisorial submodule. Let R be an integral domain, M a Prüfer faithful multiplication R-module and P a maximal submodule of M. Then each power of P is a divisorial submodule of M if and only if P^2 is a divisorial submodule of M (Lemma 2.4). In Theorem 2.5, we imply other equivalent conditions for each power of P to be divisorial submodule. An R-module M satisfies radical trace property (RTP) provided $Tr_M(N) = M$ or $Tr_M(N) = \sqrt{NN^{-1}}$ for each submodule N of M. In Theorem 2.8, we show that if R is an integral domain and M a faithful multiplication R-module, then M satisfies RTP if and only if R satisfies RTP. In Theorem 2.9, we give equivalent conditions for M to satisfies RTP. In proposition 2.14, we show that, if R is an integral domain and M a faithful multiplication R-module, then M is a discrete Valuation module if and only if each P-primary submodule of M is a power of P. Let R ba an integral domain and M a faithful multiplication R-module. Let N and K be two submodules of M such that N is invertible and $K \subseteq N$. Then there exists an ideal I of R such that $K = IN$ (Lemma 2.15). Let R be an integral domain, K a quotient field of R such that $R \neq K$ and M a Valuation faithful multiplication R-module. Then the product of two P-primary submodules of M is a P-primary submodule of M (Lemma 2.17). In Theorem 2.18, we prove that, if R is an integral domain, K a quotient field of R such that $R \neq K$ and M a Valuation faithful multiplication R-module and if $P \neq P^2$ is a prime submodule of M, then each P-primary submodule of M is a power of P.
In section 3, we give equivalent conditions for N^{-1} and L^{-1} to be rings, where N, L are submodules of R-module M. let R be a ring and N, L be two submodules of an R-module M. Then N and L are coprime if $N + L = M$ [2]. Let R be an integral domain, M a faithful multiplication R-module and N, L coprime submodules of M. We get equivalent conditions for $(N \cap L)^{-1}$ to be a ring (Theorem 3.1 and Theorem 3.6). In Proposition 3.7, we show that if R is an integral domain, M a faithful multiplication R-module and N a radical submodule of M such that $N = K \cap L$ for submodules K, L of M, then N^{-1} is a ring if and only if there are radical submodules $K_1 \supseteq K$ and $L_1 \supseteq L$ of M such that $N = K_1 \cap L_1$ and K_1^{-1} and L_1^{-1} are rings.

2. Divisorial Submodules

Compare the next results with [11, Lemma 4.1.8, Lemma 4.1.16, Proposition 4.1.17, Corollary 4.1.18, Theorem 4.1.19, Theorem 4.2.13, Lemma 4.2.14, Lemma 4.2.15 and Lemma 5.3.1]. We start with the following lemma.

Lemma 2.1. *Let R be an integral domain and M a Prüfer faithful multiplication R-module. If P is a maximal submodule of M,* then the following are equivalent:
(1) *P is a divisorial submodule of M;*
(2) *P^n is a divisorial submodule of M for each positive integer n;*
(3) *P is a finitely generated submodule of M(i.e. P is an invertible submodule of M).*

Proof. (1) $\Rightarrow$ (3) If P is not an invertible submodule of M, then $P^{-1} = R$, [2, Proposition 13]. Since P is a divisorial submodule of M, then

$$P = (P^{-1})^{-1} = R^{-1} = R$$

which is a contradiction, because R is invertible. Therefore P is an invertible submodule of M.
(3) $\Rightarrow$ (2) If P is a finitely generated submodule of M, then so is P^n for each positive integer n. So P^n is an invertible submodule of M. Thus P^{-n} is not a subring of $T(R)$, [2, Proposition 11]. Therefore P^n is a divisorial submodule of M for each positive integer n, [2, Proposition 14].
(2) $\Rightarrow$ (1) It is clear. □

Lemma 2.2. *Let R be an integral domain, M a faithful multiplication R-module and N a submodule of M. Then $(M :_{R_T} T(N)) = \bigcap_{n=1}^{\infty}(N^n)_\nu$.*

Proof. By [2, Lemma 1], we have $(N :_R M)_\nu = N_\nu$. Therefore

$$\begin{aligned}
(M :_{R_T} T(N)) &= (M :_{R_T} \bigcup_{n=1}^{\infty}(M :_{R_T} (N :_R M)^n M)) \\
&= (M :_{R_T} \bigcup_{n=1}^{\infty}(N :_R M)^{-n} M) \\
&= \bigcap_{n=1}^{\infty}(M :_{R_T} (N :_R M)^{-n} M) \\
&= \bigcap_{n=1}^{\infty}(R :_{T(R)} (N :_R M)^{-n}) = \bigcap_{n=1}^{\infty}((N :_R M)^{-n})^{-1} \\
&= \bigcap_{n=1}^{\infty}(N :_R M)_\nu^n = \bigcap_{n=1}^{\infty} N_\nu^n.
\end{aligned}$$

□

Proposition 2.3. *Let R be an integral domain, M a faithful multiplication R-module and P a maximal submodule of M. Then P^n is divisorial for each positive* *integer n if and only if $(M :_{R_T} T(P)) = P_0$, where $P_0 = \bigcap_{n=1}^{\infty} P^n$.*

Proof. Let P^n be a divisorial submodule of M for each positive integer n. By Lemma 2.2, $\bigcap_{n=1}^{\infty} P^n = \bigcap_{n=1}^{\infty}(P^n)_\nu = (M :_{R_T} T(P))$.
For converse, assume that P is a maximal submodule of M. If $P^{-1} = R$, then by [2, Lemma 1], $(P :_R M)^{-1} = R$. Hence by induction we have

$$\begin{aligned} P^{-n} &= (P :_R M)^{-n} = (R :_{T(R)} (P :_R M)^n) \\ &= (R :_{T(R)} (P :_R M)^{n-1} :_R (P :_R M)) \\ &= ((P :_R M)^{-(n-1)} :_R (P :_R M)) \\ &= (R :_{T(R)} (P :_R M)) = (P :_R M)^{-1} = P^{-1} = R. \end{aligned}$$

Thus

$$\begin{aligned} T(P) &= \bigcup_{n=1}^{\infty}(M :_{R_T} (P :_R M)^n M) = \bigcup_{n=1}^{\infty}(R :_{T(R)} (P :_R M)^n) \\ &= \bigcup_{n=1}^{\infty}(P :_R M)^{-n} = \bigcup_{n=1}^{\infty}((P :_R M)^{-1})^n = \bigcup_{n=1}^{\infty}(P^{-1})^n = R. \end{aligned}$$

Then $(M :_{R_T} T(P)) = (M :_{R_T} R) = M$, which is a contradiction. Hence P is a invertible submodule of M, [2, Proposition 13]. So by Lemma 2.1, P is a finitely generated submodule of M. Then, by Lemma 2.1, P^n is a divisorial submodule of M for each positive integer n. □

Lemma 2.4. *Let R be an integral domain, M a Prüfer faithful multiplication R-module and P a maximal submodule of M. Then each power of P is a divisorial submodule of M if and only if P^2 is a divisorial submodule of M.*

Proof. Assume that P^2 is a divisorial submodule of M. If P is not invertible, then $P^{-1} = R$, [2, Proposition 13]. Then, by [2, Lemma 1], $(P :_R M)^{-1} = R$. So, by [2, Lemma 1], we have

$$\begin{aligned} P^{-2} &= (P :_R M)^{-2} = (R :_{T(R)} (P :_R M)^2) \\ &= (R :_{T(R)} (P :_R M) :_R (P :_R M)) \\ &= ((P :_R M)^{-1} :_R (P :_R M)) \\ &= (R :_{T(R)} (P :_R M)) = (P :_R M)^{-1} = R. \end{aligned}$$

Thus P^2 is not a divisorial submodule of M, which is a contradiction. Therefore P is an invertible submodule of M and so P is a finitely generated submodule of M. Thus by Lemma 2.1, P is a divisorial submodule of M and so P^n is a divisorial submodule of M for each $n \geq 1$. The converse is clear. □

It is shown that, if M is a finitely generated multiplication R-module and N a submodule of M, then N is an idempotent submodule of M if and only if $(N :_R M)$ is an idempotent ideal of R.

Theorem 2.5. *Let R be an integral domain, M a Prüfer faithful multiplication R-module and P a maximal submodule of M. The following are equivalent:*
(1) *P^n is a divisorial submodule of M for each $n \geq 1$;*
(2) *$(M : T(P)) = P_0$, where $P_0 = \bigcap_{n=1}^{\infty} P^n$;*
(3) *P^2 is a divisorial submodule of M;*
(4) *Either $P^{-1} \subset T(P)$ or P is a divisorial idempotent submodule of M.*

Proof. (1) $\Leftrightarrow$ (2) Lemma 2.3.
(1) $\Leftrightarrow$ (3) Lemma 2.4.
(1) $\Rightarrow$ (4) Let $P^{-1} = T(P)$. Then, by [2, Lemma 1], $(P :_R M)^{-1} = P^{-1} = T(P) = T((P :_R N))$. Thus, by [11, Theorem 4.1.19], $(P :_R M)$ is an idempotent ideal of R. Hence P is an idempotent submodule of M.
(4) $\Rightarrow$ (1) Let $P^{-1} \subset T(P)$. If $P^{-1} = R$, then by [2, Lemma 1], $(P :_R M)^{-1} = R$. Hence

$$\begin{aligned} T(P) &= \bigcup_{n=1}^{\infty}(M :_{R_T} (P :_R M)^n M) = \bigcup_{n=1}^{\infty}(R :_{T(R)} (P :_R M)^n) \\ &= \bigcup_{n=1}^{\infty}(P :_R M)^{-n} = R \end{aligned}$$

which is a contradiction, because $R \subseteq P^{-1} \subset T(P)$. Therefore P is an invertible submodule of M, [2, Proposition 13]. Then, by Lemma 2.1, P is a finitely generated submodule of M and so P is a divisorial submodule of M. Therefore, by Lemma 2.1, P^n is divisorial submodule of M for each $n \geq 1$. □

Lemma 2.6. *Let R be an integral domain, M a faithful multiplication R-module and N a submodule of M. Then $N^{-1} = (N :_{T(R)} N)$ if and only if $NN^{-1} = N$.*

Proof. Assume that $N^{-1} = (N :_{T(R)} N)$. Then $(N :_R M)^{-1} = ((N :_R M)M :_{T(R)} (N :_R M)M)$ and so $(N :_R M)^{-1}(N :_R M)M = ((N :_R M)M :_{T(R)} (N :_R M)M)(N :_R M)M = (N :_{T(R)} N)N$. Therefore $N^{-1}N = N$. Conversely, if $NN^{-1} = N$, then by [2, Theorem 15], $Tr_M(N) = NN^{-1} = N$. Therefore, by [2, Theorem 15], $N^{-1} = (N :_{T(R)} N)$. □

Recall that a ring R satisfies the radical trace property if $Tr(I) = \sqrt{II^{-1}}$ or $Tr(I) = R$. It is shown that R satisfies the radical trace property if and only if R_S satisfies the radical trace property for each multiplicatively closed subset S of R ([11, Theorem 4.2.13]).

Definition 2.7. We say that an R-module M satisfies radical trace property (RTP) provided that $Tr_M(N) = M$ or $Tr_M(N) = \sqrt{NN^{-1}}$ for each submodule N of M.

Theorem 2.8. *Let R be an integral domain and M a faithful multiplication R-module. Then M satisfy RTP if and only if R satisfies RTP.*

Proof. Let M satisfy RTP and I an ideal of R. Then IM is a submodule of M. Therefore $(IM)(IM)^{-1} = M$ or $(IM)(IM)^{-1} = \sqrt{(IM)(IM)^{-1}}$. Hence, by

[6, Lemma 6], $II^{-1} = R$ or $II^{-1} = \sqrt{II^{-1}}$ and so R satisfies RTP. Conversely, assume that R satisfies RTP and let N be a submodule of M. Then $(N :_R M)$ is an ideal of R. Thus $(N :_R M)(N :_R M)^{-1} = R$ or $(N :_R M)(N :_R M)^{-1} = \sqrt{(N :_R M)(N :_R M)^{-1}}$. So, by [6, Lemma 6], $(N :_R M)(N :_R M)^{-1}M = M$ or $(N :_R M)(N :_R M)^{-1}M = \sqrt{(N :_R M)(N :_R M)^{-1}M}$. Thus, by [2, Lemma 1], $NN^{-1} = M$ or $NN^{-1} = \sqrt{NN^{-1}}$. Therefore M satisfies RTP. □

Corollary 2.9. *Let R be an integral domain, M a faithful multiplication R-module and S a multiplicatively closed subset of R. Then M_S satisfies RTP if and only if R_S satisfies RTP.*

Theorem 2.10. *Let M be a faithful multiplication R-module. Then the following are equivalent:*
(1) *M satisfies RTP;*
(2) *For each multiplicatively closed subset S of R, M_S satisfies RTP;*
(3) *For each prime submodule P of M, $\frac{M}{P}$ satisfies RTP.*

Proof. (1) ⇒ (2) Assume that M satisfies RTP. Then, by Theorem 2.8, R satisfies RTP. So, by [11, Theorem 4.2.13], R_S satisfies RTP. Therefore, by Corollary 2.9, M_S satisfies RTP.
(2) ⇒ (1) Set $S = \{1\}$, then $M = M_S$.
(1) ⇒ (3) If P is a maximal submodule of M, then $\frac{M}{P}$ is a simple module. Therefore assume that P is a nonmaximal prime submodule of M. Let N be a submodule of M containing P and $K' := (\frac{N}{P})(\frac{N}{P})^{-1} \neq \frac{M}{P}$. We show that K' is a radical submodule of $\frac{M}{P}$. Write $K' = \frac{K}{P}$ for some submodule K of M, where K contains P. By [2, Proposition 15], we have

$$\begin{aligned} \frac{K^{-1}}{P} &= (\frac{M}{P} :_{R_T} \frac{K}{P}) = (\frac{M}{P} :_{R_T} K') = (K')^{-1} = Tr_{\frac{M}{P}}(\frac{N}{P})^{-1} \\ &= (Tr_{\frac{M}{P}}(\frac{N}{P}) :_{T(R)} Tr_{\frac{M}{P}}(\frac{N}{P})) = (K' :_{T(R)} K'). \end{aligned}$$

Let $u \in K^{-1}$, then $(u + P)\frac{K}{P} \subseteq \frac{K}{P}$ and so $uK \subseteq K$. Thus $K^{-1} \subseteq (K :_{T(R)} K)$. On the other hand, by [2, Lemma 1], $(K :_{T(R)} K) \subseteq K^{-1}$. Hence $K^{-1} = (K :_{T(R)} K)$ and so by Lemma 2.6, $KK^{-1} = K (\neq M)$. Since M satisfies RTP, then K is a radical submodule of M. Consequently $\sqrt{K'} = \sqrt{\frac{K}{P}} = \frac{\sqrt{K}}{P} = \frac{K}{P} = K'$.
(3) ⇒ (1) Set $P = 0$, then $M \simeq \frac{M}{P}$. □

It is obvious that if I and J are ideals of R, then $(I :_R J)M \subseteq (IM :_{T(R)} JM)$.

Lemma 2.11. *Let R be an integral domain, M a faithful multiplication R-module and Q a P-primary submodule of M. If N is a submodule of M containing Q which is not contained in P, then $N^{-1} \subseteq (Q :_{T(R)} Q)$.*

Proof. Since Q is a P-primary submodule of M, then by [2, Lemma 4], $(Q :_R M)$ is a $(P :_R M)$-primary ideal of R. Therefore, by [11, Lemma 4.2.14], $(N :_R M)^{-1} \subseteq$

$((Q :_R M) :_{T(R)} (Q :_R M))$. Since, by [2, Lemma 1], $N^{-1} = (N :_R M)^{-1}$, so $N^{-1} \subseteq ((Q :_R M) :_{T(R)} (Q :_R M))M \subseteq (Q :_{T(R)} Q)$. □

Lemma 2.12. *Let R be an integral domain, M a faithful multiplication R-module and Q a P-primary submodule of M. Let N is a submodule of M such that $Q \subseteq N \subseteq QQ^{-1}$ and $N \nsubseteq P$, Then $N^{-1} = (QQ^{-1})^{-1} = (QQ^{-1} :_{T(R)} QQ^{-1}) = (Q :_{T(R)} Q)$.*
Proof. Since $Q \subseteq N \subseteq QQ^{-1}$, then, by [2, Proposition 15] and Lemma 2.11, we have

$$(Q :_{T(R)} Q) \subseteq (QQ^{-1} :_{T(R)} QQ^{-1}) = (QQ^{-1})^{-1} \subseteq N^{-1} \subseteq (Q :_{T(R)} Q)$$

Therefore $N^{-1} = (QQ^{-1})^{-1} = (QQ^{-1} :_{T(R)} QQ^{-1}) = (Q :_{T(R)} Q)$. □

Lemma 2.13.([7, Lemma 3.6]) *Let R be an integral domain, M a faithful multiplication R-module and N, L submodules of M. Then $(N :_R M)(L :_R M) = (NL :_R M)$.*

Proposition 2.14. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a discrete Valuation module if and only if each P-primary submodule of M is a power of P.*

Proof. Assume that M be a discrete Valuation module and Q a P-primary submodule of M. Then, by [2, Lemma 4], $(Q :_R M)$ is $(P :_R M)$-primary ideal of R. So, By [12, Theorem 1], R is a discrete valuation domain. Thus, by [11, Lemma 5.3.1] and Lemma 2.13, there exists a positive integer n such that $(Q :_R M) = (P :_R M)^n = (P^n :_R M)$. Therefore $Q = P^n$.
Conversely, suppose that Q be a P-primary submodule of M such that for some positive integer n, $Q = P^n$. So, by [2, Lemma 4], $(Q :_R M)$ is $(P :_R M)$-primary ideal of R. If M is not discrete Valuation module, then by [4, Theorem 1], R is not a discrete Valuation domain. Thus, by [11, Lemma 5.3.1] and Lemma 2.13, $(Q :_R M) \neq (P :_R M)^n = (P^n :_R M)$ for each positive integer n. Therefore $Q \neq P^n$ for each positive integer n, which is a contradiction. □

Compare the next results with [12, Theorem 7.2,Theorem 17.1 and Theorem 17.3]

Lemma 2.15. *Let R ba an integral domain and M a faithful multiplication R-module. Let N and K be two submodules of M such that N is invertible and $K \subseteq N$. Then there exists an ideal I of R such that $K = IN$.*

Proof. Let N and K be two submodules of M such that N is invertible and $K \subseteq N$. Then $(N :_R M)$ and $(K :_R M)$ are ideals of R and $(N :_R M)$ is an invertible ideal of R and $(K :_R M) \subseteq (N :_R M)$. Thus, by [12, Theorem 7.2], there exists an ideal I of R such that $(K :_R M) = I(N :_R M)$. Therefore $K = (K :_R M)M = I(N :_R M)M = IN$. □

Lemma 2.16. *Let R be an integral domain, K a quotient field of R such that $R \neq K$, M a Valuation faithful multiplication R-module and N a proper submodule of M. Then $P = \bigcap_{n=1}^{\infty} N^n$ is a prime submodule of M.*

Proof. It is obvious that $P = \bigcap_{n=1}^{\infty} N^n$ is a submodule of M. Since $N \neq M$, then $[N : M]$ is a proper ideal of R. Since M is Valuation module, then R is Valuation domain and so by [12, Theorem 17.1] and Lemma 2.13, $P_0 = \bigcap_{n=1}^{\infty}[N : M]^n = \bigcap_{n=1}^{\infty}[N^n : M]$ is a prime ideal of R. Therefore, by [9, Lemma 2.10], $P = \bigcap_{n=1}^{\infty}[N^n : M]M = \bigcap_{n=1}^{\infty} N^n$ is a prime submodule of M. □

Lemma 2.17. *Let R be an integral domain, K a quotient field of R such that $R \neq K$ and M a Valuation faithful multiplication R-module. Then the product of two P-primary submodules of M is a P-primary submodule of M.*

Proof. Let N_1 and N_2 be two P-primary submodules of M. Clearly N_1N_2 is a submodule of M. Then, by [2, Lemma 1], $(N_1 :_R M)$ and $(N_2 :_R M)$ are $(P :_R M)$-primary ideals of R. Since M is Valuation module, then R is Valuation domain and so by [12, Theorem 17.3] and Lemma 2.13, $(N_1 :_R M)(N_2 :_R M) = (N_1N_2 :_R M)$ is a $(P :_R M)$-primary ideal of R. Therefore by [2, Lemma 4], $N_1N_2 = (N_1N_2 :_R M)M$ is a $P = (P :_R M)M$-primary submodule of M. □

Theorem 2.18. *Let R be an integral domain, K a quotient field of R such that $R \neq K$ and M a Valuation faithful multiplication R-module. If $P \neq P^2$ is a prime submodule of M, then each P-primary submodule of M is a power of P.*

Proof. Since P is a prime submodule of M, then P is a primary submodule of M. Also $(P :_R M)$ is a prime ideal of R, so $(P :_R M)$ is $(P :_R M)$-primary ideal of R. Then, by [2, Lemma 4], P is P-primary submodule of M. So, by Lemma 2.17, each power of P is a P-primary submodule of M.
Now, let Q be a P-primary submodule of M. Then by [2, Lemma 4], $(Q :_R M)$ is a $(P :_R M)$-primary ideal of R. If $Q \subseteq P^n$ for each positive integer n, then $Q \subseteq \bigcap_{n=1}^{\infty} P^n$ and by Lemma 2.16, $P_0 = \bigcap_{n=1}^{\infty} P^n$ is a prime submodule of M. Since $P_0 \subseteq P^2 \neq P$, then $P_0 \subsetneq P$. Since $Q \subseteq \bigcap_{n=1}^{\infty} P^n = P_0$, then $(Q :_R M) \subseteq (P_0 :_R M)$ and so $\sqrt{(Q :_R M)} \subseteq \sqrt{(P_0 :_R M)}$. Thus $(P :_R M) \subseteq (P0 :_R M)$ and so $P = (P :_R M)M \subseteq (P_0 :_R M)M = P_0$, which is a contradiction. Therefore there exists a positive integer n such that $Q \nsubseteq P^n$. Since M is a Valuation module, then $P^n \subseteq Q$. Suppose that m is the smallest positive integer such that $P^m \subseteq Q$. Thus $Q \subsetneq P^{m-1}$. choose $x \in M$ such that $x \in P^{m-1}$ and $x \notin Q$. Since M is a Valuation module, then $Q \subseteq (x)$. On the other hand (x) is a principal submodule of M and so is an invertible submodule of M. Therefore, by Lemma 2.15, there exists an ideal I of R such that $Q = I.(x)$. Thus $Q \subseteq I$. So for each $a \in I$, $ax \in I.(x) = Q$. We know that Ra is an ideal of R and $a \in Ra \subseteq R$. Since $ax \in Q$ and Q is a P-primary submodule of M and $x \notin Q$, so $a \in \sqrt{(Q :_R M)} = P$. Thus $I \subseteq P$. Then $Q = I.(x) \subseteq PP^{m-1} = P^m$. Therefore $Q = P^m$. □

3. When $(N \cap L)^{-1}$ is a Ring?

Theorem 3.1. *Let R be an integral domain, M a faithful multiplication R-module and N, L coprime radical submodules of M. Then the following are equivalent:*
(1) *N^{-1} and L^{-1} are rings.*

(2) $(N \cap L)^{-1}$ *and* $(N+L)^{-1}$ *are rings.*
Moreover, $(N+L)^{-1} = (N+L :_{T(R)} N+L)$.

Proof. (1) $\Rightarrow$ (2) Let N^{-1} and L^{-1} be rings. Then, by [2, Lemma 1], $(N :_R M)^{-1}$ and $(L :_R M)^{-1}$ are rings. So $(N :_R M)^{-1} + (L :_R M)^{-1}$ is a ring. Thus $N^{-1} + L^{-1}$ is a ring. Since $N + L = M$, therefore by [2, Lemma 2], $(N \cap L)^{-1} = N^{-1} + L^{-1}$ is a ring. Also, by [2, Lemma 2] we have

$$\begin{aligned}(N+L)^{-1} &= ((N :_R M)M + (L :_R M)M)^{-1} = (((N :_R M) + (L :_R M))M)^{-1} \\ &= ((N :_R M) + (L :_R M))^{-1} = (N :_R M)^{-1} \cap (L :_R M)^{-1} \\ &= N^{-1} \cap L^{-1}.\end{aligned}$$

Since $N^{-1} \cap L^{-1}$ is a ring, then $(N+L)^{-1}$ is a ring.
(2) $\Rightarrow$ (1) Let $(N \cap L)^{-1}$ and $(N+L)^{-1}$ are rings. Then, by [2, Lemma 1], $(N \cap L :_R M)^{-1}$ and $(N + L :_R M)^{-1}$ are rings. Therefore, by [10, Theorem 3.4], $(N :_R M)^{-1}$ and $(L :_R M)^{-1}$ are rings. So, by [2, Lemma 1], N^{-1} and L^{-1} are rings.
Now we show that $(N+L)^{-1} = (N+L :_{T(R)} N+L)$. By [2, Lemma 1], $(N + L :_{T(R)} N+L) \subseteq (N+L)^{-1}$. For the other inclusion, by [2, Lemma 2], we have $(N+L)^{-1} = N^{-1} \cap L^{-1}$. Let $x \in N^{-1} \cap L^{-1}$, then by [2, Proposition 11] $x \in N^{-1} = (N :_{T(R)} N)$ and so $xN \subseteq N$. Similary $x \in L^{-1} = (L :_{T(R)} L)$, so $xL \subseteq L$ and thus $x(N+L) = xN + xL \subseteq N + L$ and therefore $x \in (N + L :_{T(R)} N+L)$. □

By induction we have the following corollary.

Corollary 3.2. *Let R be an integral domain and M a faithful multiplication R-module. Let $N_1, ..., N_n$ be radical submodules of M such that $N_i + N_j = M$ for $1 \leq i, j \leq n$ and $i \neq j$. If $N_1^{-1}, ..., N_n^{-1}$ are rings, then $(N_1 \cap ... \cap N_n)^{-1}$ is a ring.*

Proposition 3.3. *Let R be an integral domain, M a faithful multiplication R-module and N a nonzero submodule of M such that N^{-1} is a ring. Then $(\sqrt{N})^{-1}$ is a ring and $(\sqrt{N})^{-1} = (\sqrt{N} :_{T(R)} \sqrt{N})$.*

Proof. Suppose that $x \in (\sqrt{N})^{-1}$. For each $a \in \sqrt{N} = \sqrt{(N :_R M)}M$ there exists a positive integer number n such that $a^n \in (N :_R M)M = N$. Since $N = (N :_R M)M \subseteq \sqrt{(N :_R M)}M = \sqrt{N}$, then $(\sqrt{N})^{-1} \subseteq N^{-1}$ and so $x \in N^{-1}$. Since N^{-1} is a ring, then $x^{2n} \in N^{-1}$. So $a^n x^{2n} \in NN^{-1} \subseteq M$. Thus $(ax)^{2n} = a^n(a^n x^{2n}) \in NM \subseteq N$ and it follows that $ax \in \sqrt{N}$. Then $x\sqrt{N} \subseteq \sqrt{N}$ and so $x \in (\sqrt{N} :_{T(R)} \sqrt{N})$. On the other hand, by [2, Lemma 1], $(\sqrt{N} :_{T(R)} \sqrt{N}) \subseteq (\sqrt{N})^{-1}$. Therefore $(\sqrt{N})^{-1} = (\sqrt{N} :_{T(R)} \sqrt{N})$ is a ring.

Corollary 3.4. *Let R be a integral domain, M a faithful multiplication R-module and N, L coprime submodules of M. If N^{-1} and L^{-1} are rings, then $(\sqrt{N} \cap \sqrt{L})^{-1}$ and $(\sqrt{N} + \sqrt{L})^{-1}$ are rings.*

Proof. Let N^{-1} and L^{-1} be rings. Then, by Proposition 3.3, $(\sqrt{N})^{-1}$ and $(\sqrt{L})^{-1}$ are rings. Since $M = N + L \subseteq \sqrt{N} + \sqrt{L} \subseteq M$, then $\sqrt{N} + \sqrt{L} = M$. Moreover $\sqrt{N}$ and $\sqrt{L}$ are radical submodules. Therefore, by Theorem 3.1, we are done. □

Lemma 3.5. *Let R be a ring and M an R-module. Then $(N_\nu)^{-1} = N^{-1}$.*

Proof. Since $NN^{-1} \subseteq M$, then $N \subseteq (M :_{R_T} N^{-1}) = (N^{-1})^{-1} = N_\nu$. So $(N_\nu)^{-1} \subseteq N^{-1}$. For the other inclusion, let $x \in N_\nu = (M :_{R_T} N^{-1})$. Then $xN^{-1} \subseteq M$ and hence $N_\nu N^{-1} \subseteq M$. Thus $N^{-1} \subseteq (M :_{R_T} N_\nu) = (N_\nu)^{-1}$. Therefore $(N_\nu)^{-1} = N^{-1}$. □

Theorem 3.6. *Let R be an integral domain, M a faithful multiplication R-module and N, K coprime submodules of M such that $N^{-1} \cap K^{-1} = R$. Then the following are equivalent:*
(1) *N^{-1} and L^{-1} are rings.*
(2) *$(N \cap L)^{-1}$ is a ring.*
(3) *$(N_\nu \cap L_\nu)^{-1}$ is a ring.*
Moreover, $(N \cap L)^{-1} = (N_\nu \cap L_\nu)^{-1} = (NL)^{-1} = (N_\nu L_\nu)^{-1}$.

Proof. (1) $\Rightarrow$ (2) Let N^{-1} and L^{-1} be rings. Then $N^{-1} + L^{-1}$ is a ring. Therefore, by [2, Lemma 2], $(N \cap L)^{-1} = N^{-1} + L^{-1}$ is a ring.
(2) $\Rightarrow$ (1) Let $(N \cap L)^{-1}$ be a ring. Then, by [2, Lemma 1], $(N \cap L :_R M)^{-1} = ((N :_R M) \cap (L :_R M))^{-1}$ is a ring. Hence, by [10, Theorem 3.7], $(N :_R M)^{-1}$ and $(L :_R M)^{-1}$ are rings. Therefore, by [2, Lemma 1], N^{-1} and L^{-1} are rings.
If N^{-1} and L^{-1} are rings, then, by Lemma 3.5, $(N_\nu)^{-1}$ and $(L_\nu)^{-1}$ are rings. Since (1) and (2) are equivalent, it follows that $(N_\nu \cap L_\nu)^{-1}$ is a ring.
For the last equality, by [2, Lemma 1] we have $N^{-1} = (N :_R M)^{-1}$ and $N_\nu = (N :_R M)_\nu$. Therefore $(N \cap L)^{-1} = (N_\nu \cap L_\nu)^{-1} = (NL)^{-1} = (N_\nu L_\nu)^{-1}$. □

Proposition 3.7. *Let R be an integral domain, M a faithful multiplication R-module and N a radical submodule of M such that $N = K \cap L$ for submodules K, L of M. Then N^{-1} is a ring if and only if there are radical submodules $K_1 \supseteq K$ and $L_1 \supseteq L$ of M such that $N = K_1 \cap L_1$ and K_1^{-1} and L_1^{-1} are rings.*

Proof. Let N be a radical submodule of M such that $N = K \cap L$ for submodules K, L of M. Then $(N :_R M)$ is a radical ideal of R and $(N :_R M) = (K \cap L :_R M) = (K :_R M) \cap (L :_R M)$. Now, if N^{-1} is a ring, then by [2, Lemma 1], $(N :_R M)^{-1}$ is a ring and so by [10, Corollary 3.12], there are radical ideals $A \supseteq (K :_R M)$ and $B \supseteq (L :_R M)$ such that $(N :_R M) = A \cap B$ and A^{-1} and B^{-1} are rings. Therefore, there exist radical submodules $AM \supseteq (K :_R M)M = K$ and $BM \supseteq (L :_R M)M = L$ such that $N = AM \cap BM$ and A^{-1} and B^{-1} are rings, by [2, Lemma 1].
Conversely, suppose that there are radical submodules $K_1 \supseteq K$ and $L_1 \supseteq L$ of M such that $N = K_1 \cap L_1$ and K_1^{-1} and L_1^{-1} are rings. Then, by [2, Lemma 1] and [6, Lemma 6], there are radical ideals $(K_1 :_R M) \supseteq (K :_R M)$ and $(L_1 :_R M) \supseteq (L :_R M)$ of R such that $(N :_R M) = (K_1 :_R M) \cap (L_1 :_R M)$ and $(K_1 :_R M)^{-1}$ and $(L_1 :_R M)^{-1}$ are rings. Therefore, by [10, Corollary 3.12], $(N :_R M)^{-1}$ is a ring and so by [2, Lemma 1], N^{-1} is a ring. □

Definition 3.8. Let R be an ring, M an R-module and $\{K_\alpha\}_{\alpha \in \Lambda}$ a non-empty set of prime submodules of M. We say that $N = \bigcap_{\alpha \in \Lambda} K_\alpha$ is irredundant, if for each $\beta \in \Lambda$, $\bigcap_{\alpha \neq \beta} K_\alpha \nsubseteq K_\beta$.

Lemma 3.9. *Let R be an integral domain, M a faithful multiplication R-module and $\{K_\alpha\}_{\alpha\in\Lambda}$ a non-empty set of prime submodules of M. Then $N = \bigcap_{\alpha\in\Lambda} K_\alpha$ is an irredundant submodule of M if and only if $(N :_R M) = (\bigcap_{\alpha\in\Lambda} K_\alpha :_R M)$ is an irredundant ideal of R.*

Proof. Let $N = \bigcap_{\alpha\in\Lambda} K_\alpha$ be an irredundant submodule of M. Since K_α is a prime submodule of M, then $(K_\alpha :_R M)$ is a prime ideal of R. If there exists $\beta \in \Lambda$ such that $(\bigcap_{\alpha\in\Lambda} K_\alpha :_R M) \subseteq (K_\beta :_R M)$, then $\bigcap_{\alpha\neq\beta} K_\alpha = (\bigcap_{\alpha\neq\beta} K_\alpha :_R M)M \subseteq (K_\beta :_R M)M = K_\beta$, which is a contradiction. Therefore, $(N :_R M) = (\bigcap_{\alpha\in\Lambda} K_\alpha :_R M)$ is an irredundant ideal of R. The converse is similar. □

Theorem 3.10. *Let R be an integral domain, M a faithful multiplication R-module and $\{K_\alpha\}_{\alpha\in\Lambda}$ a non-empty set of prime submodules of M. If $N = \bigcap_{\alpha\in\Lambda} K_\alpha$ is a nonzero and irredundant submodule of M, then the following are equivalent:*
(1) *N^{-1} is a ring;*
(2) *For each $\alpha \in \Lambda$, K_α^{-1} is a ring;*
(3) *For each non-empty subset Γ of Λ, $(\bigcap_{\alpha\in\Gamma} K_\alpha)^{-1}$ is a ring.*

Proof. Let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a non-empty set of prime submodules of M. Then $\{(K_\alpha :_R M)\}_{\alpha\in\Lambda}$ is a non-empty set of prime ideals of R. If $N = \bigcap_{\alpha\in\Lambda} K_\alpha$ is a nonzero and irredundant submodule of M, then by Lemma 3.9, $(N :_R M) = (\bigcap_{\alpha\in\Lambda} K_\alpha :_R M) = \bigcap_{\alpha\in\Lambda}(K_\alpha :_R M)$ is a nonzero irredundant ideal of R.
(1) $\Rightarrow$ (2) Let N^{-1} be a ring. Then, by [2, Lemma 1], $(N :_R M)^{-1}$ is a ring. Therefore, by [10, Proposition 3.13], $(K_\alpha :_R M)^{-1}$ is a ring and so by [2, Lemma 1], K_α^{-1} is a ring.
(2) $\Rightarrow$ (3) Let Γ be a non-empty subset of Λ and $L = \bigcap_{\alpha\in\Gamma} K_\alpha$. Then $(L :_R M) = (\bigcap_{\alpha\in\Gamma} K_\alpha :_R M) = \bigcap_{\alpha\in\Gamma}(K_\alpha :_R M)$. Therefore, by [10, Proposition 3.13], $(L :_R M)^{-1}$ is a ring and so by [2, Lemma 1], L^{-1} is a ring.
(3) $\Rightarrow$ (2) It is obvious. □

References

[1] M. M. Ali, *Invertibility of multiplication modules*, New Zealand J. Math., **35**(2006), 17-29.

[2] M. M. Ali, *Invertibility of multiplication modules* Π, New Zealand J. Math., **39**(2009), 45-64.

[3] M. M. Ali, *Some remarks on generalized GCD domains*, Comm. Algebra, **36**(2008), 142-164.

[4] M. M. Ali, *Invertibility of multiplication modules III*, New Zealand J. Math., **39**(2009), 139-213.

[5] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra, **36**(2008), 4620-4642.

[6] M. M. Ali, *The transform formula for submodules of multiplication modules*, New Zealand J. Math., **41**(2011), 25-37.

[7] R. Ameri, *On the prime submodules of multiplication modules*, Internat. J. Math. Math. Sci., **27**(2003), 1715-1724.

[8] A. Barnard, *Multiplication modules*, J. Algebra, **71**(1981), 174-178.

[9] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, **16**(1998), 755-799.

[10] E. Houston, S. Kabbaj, T. Lucas and A. Mimouni *When is the dual of an ideal a ring?*, J. Algebra, **225**(2000), 429-450.

[11] M. Fontana, J. Hukaba and I. Papick, *Prüfer Domains*, Marcel Dekker, (1997).

[12] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker: New York, (1972).

[13] M. D. Larsen and P. J. MacCaarthy, *Multiplication theory of ideal*, Academic Press: New York, (1971).

[14] A. G. Naoum, *Flat modules and multiolication modules*, Periodica. Math. Hungar., **21**(1990), 309-317.

[15] A. G. Naoum and F. H. Al-Alwan, *Dedekind modules*, Comm. Algebra, **24**(1996), 225-230.

[16] P. F. Smith, *Multiplication modules*, Comm. Algebra, **16**(1988), 755-799.

[17] P. F. Smith, *Some remarks on multiplication modules*, Arch. der. Math., **50**(1988), 223-235.