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Existence and Non-Existence of Positive Solutions of BVPs for Singular ODEs on Whole Lines

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Abstract. This paper is concerned with integral type boundary value problems of second order singular differential equations with quasi-Laplacian on whole lines. Sufficient conditions to guarantee the existence and non-existence of positive solutions are established. The emphasis is put on the non-linear term $[\Phi(\rho(t)x'(t))]'$ involved with the nonnegative singular function ρ and the singular nonlinearity term f in differential equations. Two examples are given to illustrate the main results.

1. Introduction

Nonlocal boundary value problems for ordinary differential equations (ODEs) was initiated by Il'in and Moiseev [14]. Since then, more general nonlocal boundary value problems (BVPs) were studied by several authors, see the text books [1, 11, 13], the papers [21], and the survey papers [16, 17] and the references cited there. In recent years, the study on existence of positive solutions of nonlocal boundary value problems for second order ordinary differential equations on whole real lines seem to be developed [2, 4, 3, 5, 6, 22].

Differential equations governed by nonlinear differential operators have been

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widely studied. In this setting the most investigated operator is the classical $p-$Laplacian, that is $\Phi_p(x) = |x|^{p-2}x$ with $p > 1$, which, in recent years, has been generalized to other types of differential operators, that preserve the monotonicity of the $p-$Laplacian, but are not homogeneous. These more general operators, which are usually referred to as $\Phi-$Laplacian (or quasi-Laplacian), are involved in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(x')]' = f(t, x, x'), \quad t \in R,$$

where $\Phi : R \to R$ is an increasing homeomorphism such that $\Phi(0) = 0$. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, e.g. reaction diffusion equations with non-constant diffusivity and porous media equations.

In recent years, the existence of solutions of boundary value problems of the differential equations governed by nonlinear differential operators has been studied by many authors, see [9, 10, 12, 6, 8, 18, 19] and the references therein. Calamai [7], Cupini, Marcelli and Papalini [9, 10], Marcelli [18, 19], liu [15] and Marcelli and Papalini [20] discussed the solvability of some strongly nonlinear boundary value problem.

Motivated by mentioned papers, we consider the following boundary value problem for second order singular differential equation on the whole line

$$(1.1) \qquad \begin{cases} [\Phi(\rho(t)x'(t))]' + f(t, x(t), x'(t)) = 0, \ a.e., t \in \mathbf{R}, \\ \alpha \lim\limits_{t\to-\infty} x(t) - \beta \lim\limits_{t\to-\infty} \rho(t)x'(t) = \int\limits_{-\infty}^{+\infty} g(s, x(s), x'(s))ds, \\ \gamma \lim\limits_{t\to+\infty} x(t) + \delta \lim\limits_{t\to+\infty} \rho(t)x'(t) = \int\limits_{-\infty}^{+\infty} h(s, x(s), x'(s))ds, \end{cases}$$

where

(a) $\alpha \geq 0, \gamma \geq 0, \beta \geq 0, \delta \geq 0$ are constants with

$$\sigma = \alpha\delta + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds + \beta\gamma > 0,$$

(b) f, g, h defined on $R \times [0, +\infty) \times R$ are nonnegative Caratheodory functions,

(c) $\rho \in C^0(R, [0, \infty))$ with $\rho(t) > 0$ for all $t \neq 0$ (may be singular at $t = 0$) satisfying

$$\int\limits_{-\infty}^{+\infty} \tfrac{1}{\rho(s)} ds < +\infty.$$

(d) Φ is an odd increasing continuous function and Φ maps R onto itself and there is an increasing function $\nu : [0, +\infty) \to [0, +\infty)$ such that $\Phi(xy) \geq \nu(x)\Phi(y), \ x, y \geq 0$.

Remark 1.1. If Φ satisfies (d), the inverse function of Φ is denoted by $\Phi^{-1} : R \to R$ and the inverse function of ν denoted by $\nu^{-1} : [0, +\infty) \to [0, +\infty)$, then Φ^{-1} and ν^{-1} satisfy that $\Phi^{-1}(xy) \le \nu^{-1}(x)\Phi^{-1}(y),\ x, y \ge 0$.

Remark 1.2. One dimensional $p-$Laplacian operator $\Phi_p(s) = |s|^{p-2}s$ with $p > 1$ satisfies (d). One sees that the inverse function of Φ is $\Phi_p^{-1}(x) = \Phi_q(x)$ with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and $\nu(s) = \Phi_p(s)$ and $\nu^{-1}(s) = \Phi_p^{-1}(s)$. The following function $\Phi(s) = \sum\limits_{i=1}^{m} c_i\Phi_{p_i}(s),\ \ p_m > p_{m-1} > \cdots > p_1 > 1, c_i > 0, i = 1, 2, \cdots, m$ satisfies (d) with $\nu(s) = \min\{s^{p_m+1}, s^{p_1+1}\}$ for $s \ge 0$.

The purpose is to establish sufficient conditions for the existence and non-existence of positive solutions of BVP(1.1). The results in this paper generalize and improve some known ones since f in (1.1) is singular at $t = 0$ and the p-Laplacian term $[\Phi(\rho(t)x'(t))]'$ involved with the nonnegative function ρ that may be singular at $t = 0$ and may satisfy $\rho(0) = 0$, $\rho(-1) = 0$ and $\rho(1) = 0$. Different from [22, 6, 2, 3, 7, 4, 8], the existence and non-existence results on positive solutions are obtained in this paper.

The main features of our paper are as follows. Firstly, compared with [12], we establish the existence results of solutions of second order singular differential equation on the whole line with quasi-Laplacian operator. Secondly, we investigate the existence of positive solutions by a different method and imposing growth conditions on f, g, h. Thirdly, compared with [15], we consider the case $\int\limits_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds < +\infty$ in this paper while $\int\limits_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds = +\infty$ considered in [15]. The boundary conditions in (1.1) generalize the corresponding ones in [15]. One sees that it is easy to define a nonlinear operator T in [12, 15] while in this paper the operator T is defined skillfully.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the existence result of positive solutions of BVP(1.1) is proved in Section 3. Finally the non-existence results on positive solutions of BVP(1.1) are presented in Section 4. Two examples are given to illustrate the main results in Section 5.

2. Preliminary Results

In this section, we present some background definitions in Banach spaces see [11] and state an important fixed point theorem see Theorem 2.2.11 in [13]. The preliminary results are given too.

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \ge 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. F is called a Carathédory function, that is

(i) $t \to f\left(t, x, \frac{y}{\rho(t)}\right)$ is measurable on $\mathbf{R}$ for any $x, y \in \mathbf{R}$,

(ii) $(x, y) \to f\left(t, x, \frac{y}{\rho(t)}\right)$ is continuous on $\mathbf{R}^2$ for a.e. $t \in \mathbf{R}$,

(iii) for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(\mathbf{R})$ such that $|u|, |v| \le r$ implies $\left|f\left(t, x, \frac{y}{\rho(t)}\right)\right| \le \phi_r(t)$, a.e. $t \in \mathbf{R}$.

Lemma 2.1.([11],[13]) *Let X be a real Banach space, P be a cone of X, Ω_1, Ω_2 be two nonempty bounded open sets of P with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. Suppose that $T : \overline{\Omega_2} \to K$ is a completely continuous operator, and*

(E1) *$Tx \ne \lambda x$ for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_1$, $Tx \ne \lambda x$ for all $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$;*

or

(E2) *$Tx \ne \lambda x$ for all $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_1$, $Tx \ne \lambda x$ for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_2$.*

Then T has at least one fixed points $x \in \overline{\Omega_2} \setminus \Omega_1$.

Choose

$$X = \left\{ x : R \to R : \begin{array}{c} x \in\in C^0(R),\ \rho x' \in C^0(R) \\ \text{and there exist the limits} \\ \lim\limits_{t\to-\infty} x(t),\ \lim\limits_{t\to+\infty} x(t) \\ \lim\limits_{t\to-\infty} \rho(t)x'(t),\ \lim\limits_{t\to+\infty} \rho(t)x'(t) \end{array} \right\}.$$

For $x \in X$, define the norm of x by $||x|| = \max\left\{\sup\limits_{t\in R} |x(t)|,\ \sup\limits_{t\in R} \rho(t)|x'(t)|\right\}$. One can prove that X is a Banach space with the norm $||x||$ for $x \in X$.

Let σ be defined in (a). Denote

$$A_0 = \frac{\gamma \int\limits_{-\infty}^{+\infty} g(r, x(r), x'(r))dr - \alpha \int\limits_{-\infty}^{+\infty} h(r, x(r), x'(r))dr}{\sigma},$$

$$A_1 = -\int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds + \Phi(-A_0), \quad A_2 = \Phi(-A_0).$$

Lemma 2.2. *Suppose that* (a)-(d) *hold and* $x \in X$. *Then there exists a unique* $A_x \in [A_1, A_2]$ *such that*

$$\alpha\delta\Phi^{-1}(A_x) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right) ds$$

$$+\beta\gamma\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr\right) \tag{2.1}$$

$$+\gamma \int_{-\infty}^{+\infty} g(r, x(r), x'(t))dr - \alpha \int_{-\infty}^{+\infty} h(r, x(r), x'(t))dr = 0.$$

Furthermore, it holds that

$$|A_x| \le \int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds + \Phi\left(\frac{\gamma \int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr + \alpha \int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr}{\sigma}\right). \tag{2.2}$$

Proof. Since $x \in X$ and f, g, h are Carathéodory functions, we have $||x|| \le r < +\infty$, and

$$\int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr,\ \int_{-\infty}^{+\infty} g(r, x(r), x'(r))dr,\ \int_{-\infty}^{+\infty} h(r, x(r), x'(r))dr$$

converge. Let

$$G(c) = \alpha\delta\Phi^{-1}(c) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(c + \int_s^{+\infty} f(r, x(r), x'(r))dr\right) ds$$

$$+\beta\gamma\Phi^{-1}\left(c + \int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr\right)$$

$$+\gamma \int_{-\infty}^{+\infty} g(r, x(r), x'(t))dr - \alpha \int_{-\infty}^{+\infty} h(r, x(r), x'(t))dr.$$

It is easy to see from (a) that $G(c)$ is strictly increasing on $\mathbf{R}$. We find that

$$G(A_1) \le \alpha\delta\Phi^{-1}(\Phi(-A_0)) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}(\Phi(-A_0))\, ds + \beta\gamma\Phi^{-1}(\Phi(-A_0))$$

$$+\gamma \int_{-\infty}^{+\infty} g(r, x(r), x'(t))dr - \alpha \int_{-\infty}^{+\infty} h(r, x(r), x'(t))dr = 0.$$

Similarly we find that

$$G(A_2) \geq \alpha\delta\Phi^{-1}(\Phi(-A_0)) + \alpha\gamma \int_{-\infty}^{\infty} \frac{1}{\rho(s)}\Phi^{-1}(\Phi(-A_0))\,ds + \beta\gamma\Phi^{-1}(\Phi(-A_0))$$

$$+\gamma \int_{-\infty}^{+\infty} g(r, x(r), x'(t))dr - \alpha \int_{-\infty}^{+\infty} h(r, x(r), x'(t))dr = 0.$$

Hence there exists a unique $A_x \in [A_1, A_2]$ such that (2.1) holds. It is easy to see that (2.2) holds. The proof is complete. □

Fix $k > 0$. Denote $\mu = \int_{-\infty}^{-k} \frac{1}{\rho(s)}ds \left(2\int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)^{-1}$. Let

$$P = \left\{x \in X : \ x(t) \geq 0 \text{ for all } t \in \mathbf{R}, \ \min_{t\in[-k,k]} x(t) \geq \mu \sup_{t\in\mathbf{R}} x(t) \ \right\}.$$

From (a), we have either $\alpha > 0$ or $\gamma > 0$. Define the operator T on X by

$$(Tx)(t) = B_x + \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(r, x(r), x'(r))dr\right)ds, \tag{2.3}$$

where A_x satisfies (2.1) and

(2.4)

$$B_x = \begin{cases} \dfrac{\int_{-\infty}^{+\infty} g(r,x(r),x'(t))dr + \beta\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right)}{\alpha} & \text{for } \alpha > 0, \\ \dfrac{\int_{-\infty}^{+\infty} h(s,x(s),x'(s))ds - \gamma\int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r))dr\right)ds - \delta\Phi^{-1}(A_x)}{\gamma} & \text{for } \alpha = 0. \end{cases}$$

Lemma 2.3. *Suppose that* (a)-(d) *hold. Then*

(i) $T : P \to X$ *is well defined,*

(ii) *it holds that*

$$\begin{cases} [\Phi(\rho(t)(Tx)'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\ \alpha \lim\limits_{t\to-\infty}(Tx)(t) - \beta \lim\limits_{t\to-\infty} \rho(t)(Tx)'(t) = \int_{-\infty}^{+\infty} g(s, x(s), x'(s))ds, \\ \gamma \lim\limits_{t\to+\infty}(Tx)(t) + \delta \lim\limits_{t\to+\infty} \rho(t)(Tx)'(t) = \int_{-\infty}^{+\infty} h(s, x(s), x'(s))ds, \end{cases} \tag{2.5}$$

(iii) $T : P \to P$ *is completely continuous;*

(iv) $x \in X$ *is a positive solution of* $BVP(1.1)$ *if and only if* x *is a fixed point of* T *in* P.

Proof. **(i)** For $x \in P \subset X$, by Lemma 2.2, both A_x and B_x are uniquely determined respectively. So Tx is defined. We need to prove $Tx \in X$. Form (2.3), $Tx \in C^0(R)$ and there exist the limits

$$\lim_{t\to-\infty}(Tx)(t) = B_x,$$

$$\lim_{t\to+\infty}(Tx)(t) = B_x + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right) ds.$$

Furthermore,

$$\rho(t)(Tx)'(t) = \Phi^{-1}\left(A_x + \int_t^{+\infty} f(r, x(r), x'(r))dr\right). \tag{2.6}$$

It is easy to see that then $t \to \rho(t)(Tx)'(t)$ is continuous on R. Then $\rho(Tx)' \in C^0(R)$. Furthermore, there exist the limits

$$\lim_{t\to-\infty}\rho(t)(Tx)'(t) = \Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr\right),$$

$$\lim_{t\to+\infty}\rho(t)(Tx)'(t) = \Phi^{-1}(A_x).$$

It follows that $Tx \in X$. Hence $T : P \to X$ is well defined.

(ii) From (2.3), (2.1) and (2.4), we get (2.5) easily.

(iii) First, we prove that $T : P \to P$ is well defined.

From (i), for $x \in P$, we have $Tx \in X$. From (ii), since f, g and h are nonnegative, then

$$\begin{cases} [\Phi(\rho(t)(Tx)'(t)]' = -f(t, x(t), x'(t)) \le 0, \ t \in R, \\ \alpha \lim\limits_{t\to-\infty}(Tx)(t) - \beta \lim\limits_{t\to-\infty}\rho(t)(Tx)'(t) \ge 0, \\ \gamma \lim\limits_{t\to+\infty}(Tx)(t) + \delta \lim\limits_{t\to+\infty}\rho(t)(Tx)'(t) \ge 0. \end{cases} \tag{2.7}$$

Then $\rho(t)(Tx)'(t)$ is decreasing on R. If $\rho(t)(Tx)'(t) > 0$ for all $t \in R$, then Tx is increasing on R. Hence

$$\alpha \lim_{t\to-\infty}(Tx)(t) \ge \beta \lim_{t\to-\infty}\rho(t)(Tx)'(t) \ge 0.$$

Then $\lim\limits_{t\to-\infty}(Tx)(t) \ge 0$. Since Tx is increasing, then $(Tx)(t) \ge 0$ for all $t \in R$. If $\rho(t)(Tx)'(t) < 0$ for all $t \in R$, then Tx is decreasing on R. Hence

$$\gamma \lim_{t\to+\infty}(Tx)(t) \ge -\delta \lim_{t\to+\infty}\rho(t)(Tx)'(t) \ge 0$$

Then $\lim\limits_{t\to+\infty}(Tx)(t) \ge 0$. Since Tx is decreasing, then $(Tx)(t) \ge 0$ for all $t \in R$. If there exists $\tau_0 \in R$ such that $\rho(\tau_0)(Tx)'(\tau_0) = 0$, then

$$[\Phi(\rho(t)(Tx)'(t))]' = -f(t, x(t), x'(t)) \le 0$$

implies that $\rho(t)(Tx)'(t)$ is decreasing on R. Hence $\rho(t)(Tx)'(t) \ge 0$ for all $t \in (-\infty, \tau_0]$ and $\rho(t)(Tx)'(t) \le 0$ for all $t \in [\tau_0, +\infty)$. Thus

$$\alpha \lim_{t\to-\infty}(Tx)(t) \ge \beta \lim_{t\to-\infty}\rho(t)(Tx)'(t) \ge 0$$

implies $\lim_{t\to-\infty}(Tx)(t) \ge 0$. Similarly we get

$$\gamma \lim_{t\to+\infty}(Tx)(t) \ge -\delta \lim_{t\to+\infty}\rho(t)(Tx)'(t) \ge 0.$$

Then $\lim_{t\to+\infty}(Tx)(t) \ge 0$. Since $Tx(t)$ is increasing on $(-\infty, \tau_0]$ and decreasing on $[\tau_0, +\infty)$, then $(Tx)(t) \ge 0$ for all $t \in R$. From above discussion, we see

$$(Tx)(t) \ge 0 \text{ for all } t \in R. \tag{2.8}$$

Now, we prove that $(Tx)(t)$ is concave with respect to

$$\tau = \tau(t) = \int_{-\infty}^{t} \frac{1}{\rho(s)}ds.$$

It is easy to see that $\tau \in C\left(R, \left(0, \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)\right)$ and

$$\frac{d\tau}{dt} = \frac{1}{\rho(t)} > 0.$$

Thus

$$\frac{d(Tx)}{dt} = \frac{d(Tx)}{d\tau}\frac{d\tau}{dt} = \frac{d(Tx)}{d\tau}\frac{1}{\rho(t)}. \tag{2.9}$$

It follows that

$$\Phi\left(\rho(t)\frac{d(Tx)}{dt}\right) = \Phi\left(\frac{d(Tx)}{d\tau}\right).$$

Since $\frac{d[\Phi(\rho(t)(Tx)'(t))]}{dt} = -f(t, x(t), x'(t)) \le 0$ and $\frac{d\tau}{dt} > 0$, we get that $\frac{d[\Phi(\rho(t)(Tx)'(t))]}{d\tau}$ ≤ 0 on $\left(0, \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)$. Then $\frac{d(Tx)}{d\tau}$ is decreasing with respect to $\tau \in \left(0, \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)$. Hence $(Tx)(t)$ is concave with respect to $\tau \in \left(0, \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)$.

We need to prove that

$$\min_{t\in[-k,k]}(Tx)(t) \ge \mu \sup_{t\in R}(Tx)(t). \tag{2.10}$$

Since $\frac{d\tau}{dt} > 0$ for all $t \in R$, there exists the inverse function of $\tau = \tau(t)$. Denote the inverse function of $\tau = \tau(t)$ by $t = t(\tau)$.

Case 1. there exists $\tau_0 \in R$ such that $\sup\limits_{t\in R}(Tx)(t) = (Tx)(\tau_0)$. One sees

$$\min_{t\in[-k,k]}(Tx)(t) = \min\left\{(Tx)(-k),\ (Tx)(k)\right\}.$$

Denote

$$\tau(+\infty) = \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds.$$

If $\min\{(Tx)(-k),\ (Tx)(k)\} = (Tx)(k) = (Tx)(t(\tau(k)))$, note $\tau(-k) < 1$, then for $t \in [-k, k]$, one has

$$\begin{aligned}&(Tx)(t) \geq (Tx)(t(\tau(k)))\\&= (Tx)\left(t\left(\tfrac{\tau(+\infty)-\tau(k)+\tau(\tau_0)}{\tau(+\infty)+\tau(\tau_0)}\tfrac{\tau(k)\tau(+\infty)}{\tau(+\infty)-\tau(-k)+\tau(\tau_0)} + \tfrac{\tau(k)}{\tau(+\infty)+\tau(\tau_0)}\tau(\tau_0)\right)\right).\end{aligned}$$

Noting that $\tau(+\infty) > \tau(k)$ and $(Tx)(t)$ is concave with respect to τ, then, for $t \in [-k, k]$,

$$\begin{aligned}(Tx)(t) &\geq \tfrac{\tau(+\infty)-\tau(k)+\tau(\tau_0)}{\tau(+\infty)+\tau(\tau_0)}(Tx)\left(t\left(\tfrac{\tau(k)\tau(+\infty)}{\tau(+\infty)-\tau(k)+\tau(\tau_0)}\right)\right)\\&\quad+ \tfrac{\tau(k)}{\tau(+\infty)+\tau(\tau_0)}(Tx)\left(t\left(\tau(\tau_0)\right)\right)\\&\geq \int_{-\infty}^{k} \tfrac{1}{\rho(s)}ds\frac{1}{2\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds}(Tx)(\tau_0)\\&\geq \int_{-\infty}^{-k} \tfrac{1}{\rho(s)}ds\frac{1}{2\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds}(Tx)(\tau_0) = \mu\sup_{t\in R}(Tx)(t).\end{aligned}$$

Similarly, if $\min\{(Tx)(-k),\ (Tx)(k)\} = (Tx)(-k) = (Tx)(t(\tau(-k)))$, note $\tau(-k) <$

1, for $t \in [-k, k]$, one has

$$\begin{aligned}
&(Tx)(t) \geq (Tx)(t(\tau(-k))) \\
&= (Tx)\left(t\left(\tfrac{\tau(+\infty)+\tau(\tau_0)-\tau(-k)}{\tau(+\infty)+\tau(\tau_0)-\tau(-k)}\tfrac{\tau(+\infty)\tau(-k)}{\tau(+\infty)+\tau(\tau_0)} + \tfrac{\tau(-k)}{\tau(+\infty)+\tau(\tau_0)}\tau(\tau_0)\right)\right) \\
&\geq \tfrac{\tau(+\infty)+\tau(\tau_0)-\tau(-k)}{\tau(+\infty)+\tau(\tau_0)}(Tx)\left(t\left(\tfrac{\tau(-k)\tau(+\infty)}{\tau(+\infty)+\tau(\tau_0)-\tau(-k)}\right)\right) \\
&\qquad + \tfrac{\tau(-k)}{\tau(+\infty)+\tau(\tau_0)}(Tx)\left(t\left(\tau(\tau_0)\right)\right) \\
&\geq \int_{-\infty}^{-k} \tfrac{1}{\rho(s)}ds \frac{1}{2\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds}(Tx)(\tau_0) > \mu \sup\nolimits_{t\in R}(Tx)(t).
\end{aligned}$$

Hence (2.10) holds.

Case 2. $\sup\limits_{t\in R}(Tx)(t) = \lim\limits_{t\to-\infty}(Tx)(t)$ or $\lim\limits_{t\to+\infty}(Tx)(t))$. Choose $\tau_0 \in R$. By the same methods used in Case 1, we get

$$\min_{t\in[-k,k]}(Tx)(t) \geq \mu(Tx)(\tau_0).$$

Let $\tau_0 \to -\infty$. We get

$$\min_{t\in[-k,k]}(Tx)(t) \geq \mu \lim_{t\to-\infty}(Tx)(t) = \mu \sup_{t\in R}(Tx)(t) \text{ or } \mu \lim_{t\to+\infty}(Tx)(t)).$$

So (2.10) holds. From (2.8) and (2.10), we see $Tx \in P$. Hence $T : P \to P$ is well defined.

Now we prove that T is completely continuous. The following five steps are needed (Steps 1-2 imply that $T : X \to X$ is continuous and Steps 3-5 imply that T maps bounded sets into relatively compact sets).

Step 1. we prove that the function $A_x : X \to R$ is continuous in x.

Let $\{x_n\} \in X$ with $x_n \to x_0$ as $n \to \infty$. Let $\{A_{x_n}\}(n = 1, 2, \ldots, m)$ be constants decided by equation

$$\begin{aligned}
&\alpha\delta\Phi^{-1}(A_{x_n}) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_{x_n} + \int_s^{+\infty} f(r, x_n(r), x_n'(r))dr\right) ds \\
&+\beta\gamma\Phi^{-1}\left(A_{x_n} + \int_{-\infty}^{+\infty} f(r, x_n(r), x_n'(r))dr\right) \\
&+\gamma \int_{-\infty}^{+\infty} g(r, x_n(r), x_n'(t))dr - \alpha \int_{-\infty}^{+\infty} h(r, x_n(r), x_n'(t))dr = 0,
\end{aligned}$$

corresponding to $x_n(n = 0, 1, 2, \dots)$. Since $x_n \to x_0$ as $n \to \infty$, there exists an $M > 0$ such that $\|x_n\| \le M(n = 0, 1, 2, \cdots)$. The fact f, g, h are Carathédory functions means there exists $\phi_M \in L^1(R)$ such that

$$\begin{aligned} &f(t, x_n(t), x_n'(t)) = f\left(t, x_n(t), \tfrac{1}{\rho(t)}\rho(t)x_n'(t)\right) \le \phi_M(t), t \in R, \\ &g(t, x_n(t), x_n'(t)) \le \phi_M(t), t \in R, \\ &h(t, x_n(t), x_n'(t)) \le \phi_M(t), t \in R. \end{aligned} \tag{2.11}$$

Then

$$\begin{aligned} &\int_{-\infty}^{\infty} f(r, x_n(r), x_n'(r))dr \le \int_{-\infty}^{\infty} \phi_M(r)dr < \infty, \\ &\int_{-\infty}^{\infty} g(r, x_n(r), x_n'(r))dr \le \int_{-\infty}^{\infty} \phi_M(r)dr < \infty, \\ &\int_{-\infty}^{\infty} h(r, x_n(r), x_n'(r))dr \le \int_{-\infty}^{\infty} \phi_M(r)dr < \infty. \end{aligned} \tag{2.12}$$

Let

$$A_{0,n} = \frac{\gamma \int_{-\infty}^{+\infty} g(r, x_n(r), x_n'(r))dr - \alpha \int_{-\infty}^{+\infty} h(r, x_n(r), x_n'(r))dr}{\sigma}.$$

So

$$\begin{aligned} A_{x_n} &\in \left[-\int_{-\infty}^{\infty} f(r, x_n(r), x_n'(r))dr + \Phi(-A_{0,n}), \Phi(-A_{0,n})\right] \\ &\subseteq \left[-\int_{-\infty}^{\infty} \phi_M(r)dr - \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right),\ \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right)\right], \end{aligned}$$

which means that $\{A_{x_n}\}$ is uniformly bounded. It follows that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\left|\Phi^{-1}\left(A_{x_n} + \int_s^{+\infty} f(r, x_n(r), {x_n}'(r))dr\right)\right| ds \\ \le\ &\int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds \Phi^{-1}\left(2\int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right)\right). \end{aligned}$$

Suppose that $\{A_{x_n}\}$ does not converge to A_{x_0}. Then there exist two subsequences $\{A_{x_{n_k}{}^{(1)}}\}$ and $\{A_{x_{n_k}{}^{(2)}}\}$ of $\{A_{x_n}\}$ with $A_{x_{n_k}{}^{(1)}} \to c_1$ and $A_{x_{n_k}{}^{(2)}} \to c_2$ as $k \to \infty$, but $c_1 \neq c_2$. By the construction of A_{x_n}, $(n = 1, 2, \dots)$, we have

$$\alpha\delta\Phi^{-1}(A_{x_{n_k}{}^{(1)}}) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_{x_{n_k}{}^{(1)}} + \int_s^{+\infty} f(r, x_{n_k}{}^{(1)}(r), x_{n_k}{}^{(1)\prime}(r))dr\right)ds$$
$$+\beta\gamma\Phi^{-1}\left(A_{x_{n_k}{}^{(1)}} + \int_{-\infty}^{+\infty} f(r, x_{n_k}{}^{(1)}(r), x_{n_k}{}^{(1)\prime}(r))dr\right)$$
$$+\gamma\int_{-\infty}^{+\infty} g(r, x_{n_k}{}^{(1)}(r), x_{n_k}{}^{(1)\prime}(t))dr - \alpha\int_{-\infty}^{+\infty} h(r, x_{n_k}{}^{(1)}(r), x_{n_k}{}^{(1)\prime}(t))dr = 0.$$

Let $k \to \infty$, using Lebesgue's dominated convergence theorem, the above equality implies

$$\alpha\delta\Phi^{-1}(c_1) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(c_1 + \int_s^{+\infty} f(r, x_0(r), x_0'(r))dr\right)ds$$
$$+\beta\gamma\Phi^{-1}\left(c_1 + \int_{-\infty}^{+\infty} f(r, x_0(r), x_0'(r))dr\right)$$
$$+\gamma\int_{-\infty}^{+\infty} g(r, x_0(r), x_0'(t))dr - \alpha\int_{-\infty}^{+\infty} h(r, x_0(r), x_0'(t))dr = 0.$$

Since $\{A_{x_0}\}$ is unique with respect to x_0, we get $c_1 = A_{x_0}$. Similarly, $c_2 = A_{x_0}$. Thus $c_1 = c_2$, a contradiction. So, for any $x_n \to x_0$, one has $A_{x_n} \to A_{x_0}$, which means $A_x : X \to X$ is continuous.

Step 2. we show that T is continuous on X. Since A_x is continuous, then B_x is continuous too. From the continuity of A_x and B_x, f is a Caratheodory function, the result follows.

Step 3. we show that T is maps bounded subsets into bounded sets. Given a bounded set $D \subseteq X$. Then, there exists $M > 0$ such that $D \subseteq \{x \in X : \|x\| \leq M\}$. Then there exists $\phi_M \in L^1(R)$ such that (2.19) and (2.20) hold by replacing x_n by

x. Similarly we have

$$|A_x| \le \int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right) < \infty,$$

$$|B_x| = \left|\frac{\int_{-\infty}^{+\infty} g(r,x(r),x'(t))dr + \beta\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right)}{\alpha}\right|$$

$$\le \frac{\int_{-\infty}^{+\infty}\phi_M(r)dr + \beta\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right)}{\alpha}.$$

Therefore,

$$\begin{aligned}|(Tx)(t)| &= \left|B_x + \int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r))dr\right)ds\right| \\ &\le \frac{\int_{-\infty}^{+\infty}\phi_M(r)dr + \beta\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right)}{\alpha} \\ &\quad + \int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) \\ &=: M_1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\rho(t)|(Tx)'(t)| &= \left|\Phi^{-1}\left(A_x + \int_{t}^{\infty} f(u,x(u),x'(u))du\right)\right| \\ &\le \Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) =: r_1.\end{aligned}$$

Then

$$\|(Tx)\| = \max\left\{\sup_{t\in R}|(Tx)(t)|, \sup_{t\in R}\rho(t)|(Tx)'(t)|\right\} < \infty.$$

So, $\{TD\}$ is bounded.

Step 4. we prove that both $\{Tx : x \in D\}$ and $\{\rho(Tx)' : x \in D\}$ are equi-continuous on each finite subinterval on R.

Let $D \subset \{x \in X : ||x|| \leq M\}$. For any $K > 0$, $t_1, t_2 \in [-K, K]$with $t_1 \leq t_2$ and $x \in X$, since f, g, h are Caratheodory functions, then there exists $\phi_M \in L^1(R)$ such that (2.11) and (2.12) hold by replacing x_n by x. Then

$$\left| A_x + \int_t^{+\infty} f(r, x(r), x'(r))dr \right| \leq 2 \int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma + \alpha}{\sigma} \int_{-\infty}^{+\infty} \phi_M(s)ds \right) =: r.$$

Since $\Phi^{-1}(s)$ is uniformly continuous on $[-r, r]$, then for each $\epsilon > 0$ there exists $\mu > 0$ such that $|s_1 - s_2| < \mu$ with $s_1, s_2 \in [-r, r]$ implies that $|\Phi^{-1}(s_1) - \Phi^{-1}(s_2)| < \epsilon$.

Since

$$|\Phi(\rho(t_1)(Tx)'(t_1)) - \Phi(\rho(t_2)(Tx)'(t_2))| = \left| \int_{t_2}^{t_1} f(r, x(r), x'(r))dr \right|$$

$$\leq \int_{t_1}^{t_2} \phi_M(r)dr \to 0 \text{ uniformly as } t_1 \to t_2,$$

Then there exists $\sigma > 0$ such that $|t_2 - t_1| < \sigma$ implies that $|\Phi(\rho(t_1)(Tx)'(t_1)) - \Phi(\rho(t_2)(Tx)'(t_2))| < \mu$. Thus $|t_1 - t_2| < \sigma$ implies that

$$(2.13) \qquad |\rho(t_1)(Tx)'(t_1) - \rho(t_2)(Tx)'(t_2)| =$$

$$|\Phi^{-1}(\Phi(\rho(t_1)(Tx)'(t_1))) - \Phi^{-1}(\Phi(\rho(t_2)(Tx)'(t_2)))| < \epsilon.$$

On the other hand, we have

$$|(Tx)(t_1) - (Tx)(t_2)| \leq \int_{t_1}^{t_2} \frac{1}{\rho(s)} ds \Phi^{-1}$$

$$\left(2 \int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma + \alpha}{\sigma} \int_{-\infty}^{+\infty} \phi_M(s)ds \right) \right) \to 0 \text{ uniformly as } t_1 \to t_2.$$

Then there exists $\sigma_2 > 0$ such that $|t_1 - t_2| < \sigma_2$ implies

$$(2.14) \qquad |(Tx)(t_1) - (Tx)(t_2)| < \epsilon.$$

Then (2.11) and (2.12) imply both $\{Tx : x \in D\}$ and $\{\rho(Tx)' : x \in D\}$ are equi-continuous on $[-K, K]$. So both $\{Tx : x \in D\}$ and $\{\rho(Tx)' : x \in D\}$ are equi-continuous on each finite subinterval on R.

Step 5. we show that both $\{Tx : x \in D\}$ and $\{\rho(Tx)' : x \in D\}$ are equiconvergent at $+\infty$ and $-\infty$ respectively.

Since

$$|\Phi(\rho(t)(Tx)'(t)) - A_x| = \left|\int_t^\infty f(r, x(r), x'(r))dr\right| \leq \int_t^\infty \phi_M(r)dr \to 0$$

uniformly as $t \to \infty$, we get similarly that

$$|\rho(t)(Tx)'(t) - \Phi^{-1}(A_x)| \to 0 \text{ uniformly as } t \to \infty.$$

In fact, for any $\epsilon > 0$, there exists $\sigma_1 > 0$ such that $|s_1 - s_2| < \sigma_1$ implies that $|\Phi^{-1}(s_1) - \Phi^{-1}(s_2)| < \frac{\epsilon}{2}$. So there exists $T_{1,\epsilon} > 0$ such that $t > T_{1,\epsilon}$ implies that $|\Phi(\rho(t)(Tx)'(t)) - A_x| < \sigma_1$. Hence

$$|\rho(t)(Tx)'(t) - \Phi^{-1}(A_x)| = |\Phi^{-1}\Big(\Phi(\rho(t)(Tx)'(t))\Big) - \Phi^{-1}(A_x)| < \frac{\epsilon}{2}, t > T_{1,\epsilon}.$$

Then

$$|\rho(t_1)(Tx)'(t_1) - \rho(t_2)(Tx)'(t_2)| < \epsilon, \ \ t_1, t_2 > T_{1,\epsilon}.$$

On the other hand, we have

$$\begin{aligned} &\left|(Tx)(t) - B_x - \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right)ds\right| \\ \leq &\int_t^{+\infty} \frac{1}{\rho(s)}ds\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) \\ \to\ & 0 \text{ uniformly as } t \to +\infty. \end{aligned}$$

Then there exists $T_{2,\epsilon} > 0$ such that

$$|(Tx)(t_1) - (Tx)(t_2)| < \epsilon, \quad t > T_{2,\epsilon}.$$

Hence $\{\rho(Tx)' : x \in D\}$ and $\{Tx : x \in D\}$ are equiconvergent at $+\infty$.

Similarly we can prove that both $\{Tx : x \in D\}$ and $\{\rho(Tx)' : x \in D\}$ are equiconvergent at $-\infty$. The details are omitted.

From Steps 3-5, we see tht T maps bounded sets into relatively compact sets.

Therefore, the operator $T : P \to P$ is completely continuous. The proof of (iii) is complete.

(iv) It is easy to see that $x \in X$ is a positive solution of BVP(1.1) if and only if x is a fixed point of T in P. The proof of (iv) is complete. Thus the proof of Lemma 2.3 is ended. □

3. Existence of Positive Solutions

In this section we establish existence result on positive solutions of BVP(1.1).

For ease expression, for nonnegative function $\phi \in L^1(R)$ and nonnegative constants L_1, L_2 and a, b, denote

$$M_0 = \max\left\{ \left(\nu\left(\frac{\int_{-k}^{0} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^0 \phi(r)dr\right)ds}{a}\right)\right)^{-1}, \left(\nu\left(\frac{\int_0^k \frac{1}{\rho(s)}\Phi^{-1}\left(\int_0^s \phi(r)dr\right)ds}{a}\right)\right)^{-1} \right\},$$

$$W_0 = \max\left\{ \left(\nu\left(\frac{\Phi^{-1}\left(\int_{-\infty}^{0} \phi(r)dr\right)}{L_1}\right)\right)^{-1}, \left(\nu\left(\frac{\Phi^{-1}\left(\int_0^{+\infty} \phi(r)dr\right)}{L_1}\right)\right)^{-1} \right\},$$

$$E_0 = \min\left\{ \begin{array}{c} \nu\left(\dfrac{b}{\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds\right)\Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)\right)+\frac{1}{\alpha}\int_{-\infty}^{+\infty}\phi_g(r)dr}\right), \\ \nu\left(\dfrac{L_2}{\Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)\right)}\right) \end{array} \right\}$$

if $\alpha > 0$ and

$$E_0 = \min\left\{ \begin{array}{c} \nu\left(\dfrac{b}{\left(\frac{\delta}{\gamma}+2\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds\right)\Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr\right)\right)+\frac{1}{\gamma}\int_{-\infty}^{+\infty}\phi_h(r)dr}\right), \\ \nu\left(\dfrac{L_2}{\Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr\right)\right)}\right) \end{array} \right\}$$

if $\alpha = 0$.

Theorem 3.1. *Suppose that* (a)-(d) *hold and there exist nonnegative function* $\phi, \phi_g, \phi_h \in L^1(R)$ *and nonnegative constants* L_1, L_2 *and* a, b *such that* $L_1 < L_2$ *and* $a < b$ *and*

$$f\left(t, u, \frac{v}{\rho(t)}\right) \geq M_0\phi(t),\ t \in [-k, k], u \in [\mu a, a], v \in [-L_1, L_1];$$

$$f\left(t, u, \frac{v}{\rho(t)}\right) \geq W_0\phi(t),\ t \in R, u \in [0, a], v \in [-L_1, L_1],$$

$$f\left(t, u, \frac{v}{\rho(t)}\right) \leq E_0\phi(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2],$$

$$g\left(t, u, \frac{v}{\rho(t)}\right) \leq \frac{1}{\nu^{-1}(1/E_0)}\phi_g(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2],$$

$$h\left(t, u, \frac{v}{\rho(t)}\right) \leq \frac{1}{\nu^{-1}(1/E_0)}\phi_h(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2].$$

If $E_0 > \max\{M_0, W_0\}$, then BVP(1.1) has at least one positive solution x satisfying

$$a \le \sup_{t\in R} x(t) \le b,\ 0 < \sup_{t\in R} \rho(t)|x'(t)| \le L_2 \tag{3.1}$$

or

$$0 < \sup_{t\in R} x(t) \le b,\ L_1 \le \sup_{t\in R} \rho(t)|x'(t)| \le L_2. \tag{3.2}$$

Proof. Let X, P and the operator T be defined in section 2. By (2.3) (the definition of T), (2.1) and (2.8), then (2.10) holds. By Lemma 2.3, $T : P \to P$ is completely continuous. x is a positive solution of BVP(1.1) if and only if x is a fixed point the T in P. By (d) and Remark 1.1, we have

$$\Phi^{-1}(uv) \le \nu^{-1}(u)\Phi^{-1}(v),\ \ \Phi^{-1}(uv) \ge \frac{\Phi^{-1}(v)}{\nu^{-1}(1/u)},\ u, v > 0.$$

Define

$$\begin{aligned}
&\xi(x) = \sup_{t\in R} x(t),\ t \in R,\ \ \eta(x) = \sup_{t\in R} \rho(t)|x'(t)|,\ t \in R,\\
&\Omega_1 = \{x \in P : \xi(x) < a,\ \eta(x) < L_1\},\ \ \Omega_2 = \{x \in P : \xi(x) < b,\ \eta(x) < L_2\}.
\end{aligned}$$

It is easy to see that ξ and η are continuous functionals and Ω_1 and Ω_2 are bounded nonempty open subsets of P and

$$\xi(x), \eta(x) \le ||x|| = \max\{\xi(x), \eta(x)\}.$$

To apply (E1) in Lemma 2.1, we do the following two steps:

Step 1. We prove that

$$Tx \ne \lambda x \text{ for all } \lambda \in [0, 1) \text{ and } x \in \partial\Omega_1. \tag{3.3}$$

Let

$$\begin{aligned}
C_1 &= \{x \in P : \xi(x) = a,\ \eta(x) \le L_1\},\\
D_1 &= \{x \in P : \xi(x) \le a,\ \eta(x) = L_1\},\\
C_2 &= \{x \in P : \xi(x) = b,\ \eta(x) \le L_2\},\\
D_2 &= \{x \in P : \xi(x) \le b,\ \eta(x) = L_2\}.
\end{aligned}$$

Sub-step 1.1. For $x \in C_1$, we prove that $\xi(Tx) \ge a$.
In fact, $x \in C_1$ implies that

$$\mu a \le x(t) \le a,\ \ t \in [-k, k],\ -L_1 \le \rho(t)x'(t) \le L_1.$$

Then we get

$$f(t,x(t),x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \geq M_0\phi(t), t \in [-k,k].$$

We consider three cases:

Case 1. $\rho(t)(Tx)'(t) > 0$ for all $t \in R$. In this case, we have from (2.8) $(Tx)(t) \geq 0$ for all $t \in R$. Then $B_x \geq 0$ and $A_x \geq 0$. So

$$\begin{aligned}
\xi(Tx) &= B_x + \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr\right) ds \\
&\geq \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty} f(r,x(r),x'(r))dr\right) ds \\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} f(r,x(r),x'(r))dr\right) ds \\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} \phi(r)M_0 dr\right) ds \\
&\geq \int_{-k}^{0} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{0} \phi(r)M_0 dr\right) ds \\
&\geq \int_{-k}^{0} \frac{1}{\rho(s)} \frac{\Phi^{-1}\left(\int_s^0 \phi(r)dr\right) ds}{\nu^{-1}(1/M_0)} \\
&\geq a.
\end{aligned}$$

Case 2. $\rho(t)(Tx)'(t) < 0$ for all $t \in R$. In this case, we have from (2.8) $(Tx)(t) \geq 0$ for all $t \in R$. Furthermore we have $\lim_{t\to+\infty}(Tx)(t) \geq 0$ and $\lim_{t\to-\infty}\rho(t)(Tx)'(t) \leq 0$.

Then

$$\begin{aligned}
\xi(Tx) &= \sup_{t\in R}(Tx)(t)\\
&= \sup_{t\in R}\Bigg[\lim_{t\to+\infty}(Tx)(t)\\
&\quad -\int_t^{+\infty}\frac{1}{\rho(s)}\Phi^{-1}\left(\lim_{t\to-\infty}\rho(t)(Tx)'(t)-\int_{-\infty}^{s} f(r,x(r),x'(r))dr\right)ds\Bigg]\\
&\ge \sup_{t\in R}\left[-\int_t^{+\infty}\frac{1}{\rho(s)}\Phi^{-1}\left(-\int_{-\infty}^{s} f(r,x(r),x'(r))dr\right)ds\right]\\
&\ge \int_{-k}^{k}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_{-k}^{s} f(r,x(r),x'(r))dr\right)ds\\
&\ge \int_{0}^{k}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_{0}^{s} f(r,x(r),x'(r))dr\right)ds\\
&\ge \int_{0}^{k}\frac{1}{\rho(s)}\frac{\Phi^{-1}\left(\int_0^s\phi(r)dr\right)ds}{\nu^{-1}(1/M_0)} \ge a.
\end{aligned}$$

Case 3. there exists $\tau_0\in R$ such that $\rho(\tau_0)(Tx)'(\tau_0)=0$. In this case we have $\lim_{t\to-\infty}(Tx)(t)\ge 0$ and $\lim_{t\to+\infty}(Tx)(t)\ge 0$. Then

$$\rho(t)(Tx)'(t)=\begin{cases}\Phi^{-1}\left(-\int_{\tau_0}^{t} f(r,x(r),x'(r))dr\right),\ t\ge\tau_0,\\ \Phi^{-1}\left(\int_{t}^{\tau_0} f(r,x(r),x'(r))dr\right),\ t\le\tau_0.\end{cases}$$

So

$$\begin{aligned}
(Tx)(t) &= \begin{cases}\lim_{t\to+\infty}(Tx)(t)-\int_t^{+\infty}\frac{1}{\rho(s)}\Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right)ds,\ t\ge\tau_0,\\ \lim_{t\to-\infty}(Tx)(t)+\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right)ds,\ t\le\tau_0.\end{cases}\\
&\ge \begin{cases}-\int_t^{+\infty}\frac{1}{\rho(s)}\Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right)ds,\ t\ge\tau_0,\\ \int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right)ds,\ t\le\tau_0.\end{cases}
\end{aligned}$$

If $\tau_0 \geq 0$, then

$$\begin{aligned}\xi(Tx) &\geq \int_{-\infty}^{0} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right) ds \\ &\geq \int_{-k}^{0} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{s}^{0} f(r,x(r),x'(r))dr\right) ds \\ &\geq \int_{-k}^{0} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{s}^{0} \phi(r)M_0 dr\right) ds \geq a,\end{aligned}$$

If $\tau_0 \leq 0$, then

$$\begin{aligned}\xi(Tx) &\geq -\int_{t}^{+\infty} \frac{1}{\rho(s)} \Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right) ds \\ &\geq \int_{0}^{k} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{0}^{s} f(r,x(r),x'(r))dr\right) ds \\ &\geq \int_{0}^{k} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{0}^{s} \phi(r)M_0 dr\right) ds \geq a.\end{aligned}$$

Sub-step 1.2. For $x \in D_1$, we prove that $\eta(Tx) \geq L_1$.
In fact, $x \in D_1$ implies that

$$0 \leq x(t) \leq a,\ -L_1 \leq \rho(t)x'(t) \leq L_1,\ t \in R.$$

Then we get

$$f(t,x(t),x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \geq W_0\phi(t), t \in R.$$

We consider three cases:

Case 1. $\rho(t)(Tx)'(t) > 0$ for all $t \in R$. In this case, we have $\lim_{t\to+\infty} \Phi(\rho(t)(Tx)'(t)) \geq 0$. Then

$$\rho(t)(Tx)'(t) = \Phi^{-1}\left(\lim_{t\to+\infty} \rho(t)(Tx)'(t) + \int_{t}^{+\infty} f(r,x(r),x'(r))dr\right).$$

So

$$\begin{aligned}
\eta(Tx) &= \sup_{t\in R}\rho(t)|(Tx)'(t)| \\
&\geq \sup_{t\in R}\Phi^{-1}\left(\int_t^{+\infty} f(r,x(r),x'(r))dr\right) \\
&= \Phi^{-1}\left(\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right) \geq \Phi^{-1}\left(\int_{-\infty}^{+\infty}\phi(r)W_0dr\right) \\
&\geq \frac{\Phi^{-1}\left(\int_{-\infty}^{+\infty}\phi(r)dr\right)}{\nu^{-1}(1/W_0)} \geq \frac{\Phi^{-1}\left(\int_{-\infty}^{0}\phi(r)dr\right)}{\nu^{-1}(1/W_0)} \\
&\geq L_1.
\end{aligned}$$

Case 2. $\rho(t)(Tx)'(t) < 0$ for all $t \in R$. In this case, we have $\lim_{t\to-\infty}\Phi(\rho(t)(Tx)'(t)) \leq 0$. Furthermore we have

$$\rho(t)(Tx)'(t) = \Phi^{-1}\left(\lim_{t\to-\infty}\Phi(\rho(t)(Tx)'(t)) - \int_{-\infty}^{t} f(r,x(r),x'(r))dr\right).$$

$$\begin{aligned}
\eta(Tx) &= \sup_{t\in R}\rho(t)|(Tx)'(t)| \\
&= \sup_{t\in R}\left|\Phi^{-1}\left(\lim_{t\to-\infty}\Phi(\rho(t)(Tx)'(t)) - \int_{-\infty}^{t} f(r,x(r),x'(r))dr\right)\right| \\
&\geq \Phi^{-1}\left(\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right) \\
&\geq \Phi^{-1}\left(\int_{-\infty}^{+\infty}\phi(r)W_0dr\right) \\
&\geq \Phi^{-1}\left(\int_{0}^{+\infty}\phi(r)W_0dr\right) \geq \frac{\Phi^{-1}\left(\int_{0}^{+\infty}\phi(r)dr\right)}{\nu^{-1}(1/W_0)} \\
&\geq L_1.
\end{aligned}$$

Case 3. there exists $\tau_0 \in R$ such that $\rho(\tau_0)(Tx)'(\tau_0) = 0$. In this case we have

$\lim_{t\to-\infty}(Tx)(t)\geq 0$ and $\lim_{t\to+\infty}(Tx)(t)\geq 0$. Then

$$\rho(t)(Tx)'(t)=\begin{cases}\Phi^{-1}\left(-\int_{\tau_0}^{t}f(r,x(r),x'(r))dr\right),\ t\geq\tau_0,\\ \Phi^{-1}\left(\int_{t}^{\tau_0}f(r,x(r),x'(r))dr\right),\ t\leq\tau_0.\end{cases}$$

If $\tau_0\geq 0$, then

$$\begin{aligned}\eta(Tx)&=\sup_{t\in R}\rho(t)|(Tx)'(t)|\geq\sup_{t\in R}\Phi^{-1}\left(\int_{t}^{\tau_0}f(r,x(r),x'(r))dr\right)\\ &\geq\Phi^{-1}\left(\int_{-\infty}^{0}f(r,x(r),x'(r))dr\right)\\ &\geq\Phi^{-1}\left(\int_{-\infty}^{0}\phi(r)W_0dr\right)\geq L_1,\end{aligned}$$

If $\tau_0\leq 0$, then

$$\begin{aligned}\eta(Tx)&=\sup_{t\in R}\rho(t)|(Tx)'(t)|\\ &\geq\sup_{t\in R}\left|\Phi^{-1}\left(-\int_{\tau_0}^{t}f(r,x(r),x'(r))dr\right)\right|\\ &\geq\Phi^{-1}\left(\int_{0}^{+\infty}f(r,x(r),x'(r))dr\right)\\ &\geq\Phi^{-1}\left(\int_{0}^{+\infty}\phi(r)W_0dr\right)\geq L_1,\end{aligned}$$

Now we prove (3.25). It is easy to see that

$$\partial\Omega_1\subseteq C_1\cup D_1,\ \ \partial\Omega_2\subseteq C_2\cup D_2.$$

If $Tx=\lambda x$ for some $\lambda\in[0,1)$ and $x\in\partial\Omega_1$, then either $x\in C_1$ or $x\in D_1$.

If $x\in C_1$, we get from Sub-step 1.1 that $\xi(Tx)\geq a$. On the other hand, we have $\xi(Tx)=\lambda\xi(x)<\xi(x)=a$, a contradiction.

If $x\in D_1$, from Sub-step 1.2, we have $\eta(Tx)\geq L_1$. On the other hand, we have $\eta(Tx)=\lambda\eta(x)<\eta(x)=L_1$, a contradiction too.

From above discussion, (3.3) holds.

Step 2. We prove that

$$Tx \neq \lambda x \text{ for all } \lambda \in (1, +\infty) \text{ and } x \in \partial\Omega_2. \tag{3.4}$$

By (d), we have $\Phi(uv) \leq \frac{\Phi(v)}{\nu(1/u)}$ for $u > 0, v \geq 0$ and $\frac{1}{\nu^{-1}(1/v)} \leq \nu^{-1}(v)$ for $v > 0$. For $x \in C_2$, one has

$$0 \leq x(t) \leq b, \ \ -L_2 \leq \rho(t)x'(t) \leq L_2, \ t \in R.$$

For $\alpha > 0$, then we get

$$f(t, x(t), x'(t)) = f\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \leq E_0\phi(t), t \in R,$$
$$g(t, x(t), x'(t)) = g\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \leq \frac{1}{\nu^{-1}(1/E_0)}\phi_g(t), t \in R,$$
$$h(t, x(t), x'(t)) = h\left(t, x(t), \frac{\rho(t)x'(t)}{\rho(t)}\right) \leq \frac{1}{\nu^{-1}(1/E_0)}\phi_h(t), t \in R.$$

Then, we have

$$\begin{aligned}
\xi(Tx) &= \sup_{t\in R}(Tx)(t) \\
&= \sup_{t\in R}\left[B_x + \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(r, x(r), x'(r))dr\right)ds\right] \\
&\leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr\right) \\
&\quad + \frac{1}{\alpha}\int_{-\infty}^{+\infty} g(r, x(r), x'(r))dr \\
&\leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)\Phi^{-1}\left(2\int_{-\infty}^{+\infty} f(r, x(r), x'(r))dr\right. \\
&\quad \left. + \Phi\left(\frac{\gamma\int_{-\infty}^{+\infty} g(r, x(r), x'(r))dr + \alpha\int_{-\infty}^{+\infty} h(r, x(r), x'(r))dr}{\sigma}\right)\right) \\
&\quad + \frac{1}{\alpha}\int_{-\infty}^{+\infty} g(r, x(r), x'(r))dr \\
&\leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}ds\right)\Phi^{-1}
\end{aligned}$$

$$\left(2\int_{-\infty}^{+\infty}\phi(r)E_0dr+\Phi\left(\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)\frac{1}{\nu^{-1}(1/E_0)}\right)\right)$$
$$+\frac{1}{\alpha}\int_{-\infty}^{+\infty}\phi_g(r)\frac{1}{\nu^{-1}(1/E_0)}dr$$

$$\begin{aligned}
\leq\ & \left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds\right)\Phi^{-1}\\
& \left(2\int_{-\infty}^{+\infty}\phi(r)E_0dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)E_0\right)\\
& +\frac{1}{\alpha}\int_{-\infty}^{+\infty}\phi_g(r)dr\nu^{-1}(E_0)\\
\leq\ & \left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\rho(s)}ds\right)\Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)\right)\\
& \nu^{-1}(E_0)+\frac{1}{\alpha}\int_{-\infty}^{+\infty}\phi_g(r)dr\nu^{-1}(E_0)\leq b.
\end{aligned}$$

For $x\in D_2$, we have

$$\begin{aligned}
\eta(Tx) &= \sup_{t\in R}\rho(t)|(Tx)'(t)|\\
&= \sup_{t\in R}\left|\Phi^{-1}\left(A_x+\int_t^{+\infty}f(r,x(r),x'(r))dr\right)\right|\\
&\leq \Phi^{-1}\left(2\int_{-\infty}^{+\infty}f(r,x(r),x'(r))dr\right.\\
&\quad\left.+\Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}g(r,x(r),x'(r))dr+\alpha\int_{-\infty}^{+\infty}h(r,x(r),x'(r))dr}{\sigma}\right)\right)\\
&\leq \Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(r)dr+\Phi\left(\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}\phi_g(r)dr+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_h(r)dr\right)\right)\nu^{-1}(E_0)\leq L_2.
\end{aligned}$$

Similarly we can prove that $\xi(Tx) \le b$ and $\eta(Tx) \le L_2$ for all $x \in D_2$ if $\alpha = 0$.

In fact, if $Tx = \lambda x$ for some $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$, then either $x \in C_2$ or $x \in D_2$.

If $x \in C_2$, we get from above discussion that $\xi(Tx) \le b$. On the other hand, we have $\xi(Tx) = \lambda\xi(x) > \xi(x) = b$, a contradiction.

If $x \in D_2$, from above discussion, we have $\eta(Tx) \le L_2$. On the other hand, we have $\eta(Tx) = \lambda\eta(x) > \eta(x) = L_2$, a contradiction too.

From above discussion, (3.4) holds.

It follows from (3.3), (3.4) and (E1) in Lemma 2.1 that T has at least one fixed point $x \in \overline{\Omega_2} \setminus \Omega_1$. So BVP(1.1) has at least one positive solution x such that $x \in \overline{\Omega_2} \setminus \Omega_1$ and that x satisfies (3.1) and (3.2). The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Suppose that* (a)-(d) *hold and there exist nonnegative function* $\phi, \phi_g, \phi_h \in L^1(R)$ *and* $L_1 > L_2 > 0$ *and* $a > b > 0$ *such that*

$$
\begin{aligned}
&f\left(t, u, \frac{v}{\rho(t)}\right) \ge M_0\phi(t),\ t \in [-k, k], u \in [\mu a, a], v \in [-L_1, L_1];\\
&f\left(t, u, \frac{v}{\rho(t)}\right) \ge W_0\phi(t),\ t \in R, u \in [0, a], v \in [-L_1, L_1],\\
&f\left(t, u, \frac{v}{\rho(t)}\right) \le E_0\phi(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2],\\
&g\left(t, u, \frac{v}{\rho(t)}\right) \le \frac{1}{\nu^{-1}(1/E_0)}\phi_g(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2],\\
&h\left(t, u, \frac{v}{\rho(t)}\right) \le \frac{1}{\nu^{-1}(1/E_0)}\phi_h(t),\ t \in R, u \in [0, b], v \in [-L_2, L_2].
\end{aligned}
$$

If $E_)$ $> \max\{M_0, W_0\}$, *then BVP(1.8) has at least one positive solution* x *satisfying*

$$b \le \sup_{t\in R} x(t) \le a,\ 0 < \sup_{t\in R} \rho(t)|x'(t)| \le L_1 \tag{3.5}$$

or

$$0 < \sup_{t\in R} x(t) \le a,\ L_2 \le \sup_{t\in R} \rho(t)|x'(t)| \le L_1. \tag{3.6}$$

Proof. Let X, P and the nonlinear operator T be defined in Section 2. Using (E2) in Lemma 2.1. The proof is similar to the proof of Theorem 3.1 and is omitted. □

4. Non-Existence of Positive Solutions

Now we establish non-existence results on positive solutions of BVP(1.1).

Theorem 4.1. *Let* $\Omega = R \times [0, +\infty) \times R$. *Suppose that* (a)-*(d) hold and there* *exists a function* $\phi \in L^1(R)$ *such that*

$$\sup_{(t,u,v)\in\Omega} \frac{f\left(t,u,\frac{v}{\rho(t)}\right)}{\phi(t)\Phi(|v|)} \leq 1,$$

(4.1)

$$\sup_{(t,u,v)\in\Omega} \frac{g\left(t,u,\frac{v}{\rho(t)}\right)}{\phi(t)|v|} \leq 1 \sup_{(t,u,v)\in\Omega} \frac{h\left(t,u,\frac{\phi(t)v}{\rho(t)}\right)}{\phi(t)|v|} \leq 1$$

If

$$2\int_{-\infty}^{+\infty} \phi(s)dsds + \Phi\left(\frac{\gamma \int_{-\infty}^{+\infty} \phi(r)dr + \alpha \int_{-\infty}^{+\infty} \phi(r)dr}{\sigma}\right) < 1, \tag{4.2}$$

then $BVP(1.1)$ *does not admit positive solutions.*

Proof. From (4.1), we have

$$f\left(t,u,\frac{v}{\rho(t)}\right) \leq \phi(t)\Phi(|v|),\ (t,u,v) \in R\times[0,+\infty)\times R,$$

$$g\left(t,u,\frac{v}{\rho(t)}\right) \leq \phi(t)|v|,\ (t,u,v) \in R\times[0,+\infty)\times R,$$

$$h\left(t,u,\frac{v}{\rho(t)}\right) \leq \phi(t)|v|, (t,u,v) \in R\times[0,+\infty)\times R.$$

Assume that x is a positive fixed point of T. We have

$$f\left(t,x(t),x'(t)\right) \leq \phi(t)\Phi(\rho(t)x'(t)) \leq \phi(t)\Phi(||\rho x'||_0),\ t \in R,$$

$$g\left(t,x(t),x'(t)\right) \leq \phi(t)\rho(t)|x'(t)| \leq \phi(t)||\rho x'||_0,\ t \in R,$$

$$h\left(t,x(t),x'(t)\right) \leq \phi(t)\rho(t)|x'(t)| \leq \phi(t)||\rho x'||_0, t \in R.$$

Then (2.2) implies that

$$\begin{aligned} |A_x| &\leq \int_{-\infty}^{+\infty} f(s,x(s),x'(s))ds + \Phi\left(\frac{\gamma \int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr + \alpha \int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr}{\sigma}\right) \\ &\leq \Phi(||\rho x'||_0) \int_{-\infty}^{+\infty} \phi(s)dsds + \Phi\left(\frac{\gamma \int_{-\infty}^{+\infty} \phi(r)dr + \alpha \int_{-\infty}^{+\infty} \phi(r)dr}{\sigma}\right) \Phi\left(||\rho x'||_0\right). \end{aligned}$$

It follows that

$$\begin{aligned}||\rho x'||_0 &= ||\rho(Tx)'||_0 = \sup_{t\in R}\left[\Phi^{-1}\left(A_x + \int_t^{+\infty} f(r,x(r),x'(r))dr\right)\right] \\ &\le ||\rho x'||_0 \Phi^{-1}\left(2\int_{-\infty}^{+\infty}\phi(s)dsds + \Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}\phi(r)dr + \alpha\int_{-\infty}^{+\infty}\phi(r)dr}{\sigma}\right)\right) \\ &< ||\rho x'||_0\end{aligned}$$

which is a contradiction. The proof is complete. □

Theorem 4.2. *Let* $\Omega = R\times[0,+\infty)\times R$. *Suppose that* (a)-(d) *hold and there exists a function* $\phi \in L^1(R)$ *such that*

$$\inf_{(t,u,v)\in\Omega}\frac{f\left(t,u,\frac{v}{\rho(t)}\right)}{\phi(t)\Phi(u)} \ge 1. \tag{4.3}$$

If

$$\mu\min\left\{\int_0^k \frac{1}{\rho(s)}\Phi^{-1}\left(\int_0^s \phi(r)dr\right)ds, \int_{-k}^0 \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^0 \phi(r)dr\right)ds\right\} > 1, \tag{4.4}$$

then $BVP(1.1)$ *does not admit positive solutions.*

Proof. From (4.3), we have

$$f\left(t,u,\frac{v}{\rho(t)}\right) \ge \phi(t)\Phi(u),\ (t,u,v)\in R\times[0,+\infty)\times R.$$

Suppose that x is a positive solution of BVP(1.1). Then $x(t) = (Tx)(t)$ for all $t \in R$. We have

$$f(t,x(t),x'(t)) \ge \phi(t)\Phi(x(t)),\ t\in R.$$

We consider three cases:

Case 1. $\rho(t)x'(t) > 0$ for all $t \in R$. In this case, we have $x(t) \ge 0$ for all $t \in R$.

Then $B_x \geq 0$ and $A_x \geq 0$. Using (4.4). So

$$\begin{aligned}
||x||_0 &= \alpha(x) = \alpha(Tx)\\
&= B_x + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r, x(r), x'(r))dr\right) ds\\
&\geq \int_{-\infty}^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty} f(r, x(r), x'(r))dr\right) ds\\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} f(r, x(r), x'(r))dr\right) ds\\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} \phi(r)\Phi(x(r))dr\right) ds\\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} \phi(r)\Phi(\mu||x||_0)dr\right) ds\\
&\geq \mu\int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{k} \phi(r)dr\right) ds||x||_0 > ||x||_0,
\end{aligned}$$

which is a contradiction.

Case 2. $\rho(t)x'(t) < 0$ for all $t \in R$. In this case, we have v $\lim_{t\to+\infty} x(t) \geq 0$ and $\lim_{t\to-\infty} \rho(t)x'(t) \leq 0$. Using (4.32). Then
$||x||_0 = \alpha(x) = \alpha(Tx) = \sup_{t\in R}(Tx)(t) =$

$$\sup_{t\in R}\left[\lim_{t\to+\infty}(Tx)(t) - \int_t^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(\lim_{t\to-\infty}\rho(t)(Tx)'(t) - \int_{-\infty}^{s} f(r, x(r), x'(r))dr\right) ds\right]$$

$$\begin{aligned}
&\geq \sup_{t\in R}\left[\int_t^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{-\infty}^{s} f(r, x(r), x'(r))dr\right) ds\right]\\
&\geq \int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{-k}^{s} f(r, x(r), x'(r))dr\right) ds\\
&\geq \mu\int_{-k}^{k} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{-k}^{s} \phi(r)dr\right) ds||x||_0 > ||x||_0,
\end{aligned}$$

which is a contradiction.

Case 3. there exists $\tau_0 \in R$ such that $\rho(\tau_0)x'(\tau_0) = 0$. In this case we have $\lim_{t\to-\infty} x \geq 0$ and $\lim_{t\to+\infty} x \geq 0$. Using (4.32). Then

$$\rho(t)x'(t) = \begin{cases} \Phi^{-1}\left(-\int_{\tau_0}^{t} f(r,x(r),x'(r))dr\right), \ t \geq \tau_0, \\ \Phi^{-1}\left(\int_{t}^{\tau_0} f(r,x(r),x'(r))dr\right), \ t \leq \tau_0. \end{cases}$$

So

$$\begin{aligned} x(t) &= \\ (Tx)(t) &= \begin{cases} \lim_{t\to+\infty}(Tx)(t) - \int_t^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right)ds, \ t \geq \tau_0, \\ \lim_{t\to-\infty}(Tx)(t) + \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right)ds, \ t \leq \tau_0. \end{cases} \\ &\geq \begin{cases} -\int_t^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right)ds, \ t \geq \tau_0, \\ \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right)ds, \ t \leq \tau_0. \end{cases} \end{aligned}$$

If $\tau_0 \geq 0$, then

$$\begin{aligned} ||x||_0 = \alpha(x) &\geq \int_{-\infty}^{0} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{\tau_0} f(r,x(r),x'(r))dr\right)ds \\ &\geq \int_{-k}^{0} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{0} f(r,x(r),x'(r))dr\right)ds \\ &\geq \mu\int_{-k}^{0} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_{s}^{0} \phi(r)dr\right)ds||x||_0 \\ &> ||x||_0, \end{aligned}$$

which is a contradiction.

If $\tau_0 \leq 0$, then

$$\begin{aligned} ||x||_0 &= \alpha(x) \geq -\int_t^{+\infty} \frac{1}{\rho(s)} \Phi^{-1}\left(-\int_{\tau_0}^{s} f(r,x(r),x'(r))dr\right) ds \\ &\geq -\int_0^k \frac{1}{\rho(s)} \Phi^{-1}\left(-\int_0^s f(r,x(r),x'(r))dr\right) ds \\ &\geq \mu \int_0^k \frac{1}{\rho(s)} \Phi^{-1}\left(\int_0^s \phi(r)dr\right) ds||x||_0 \\ &> ||x||_0, \end{aligned}$$

which is a contradiction.

From above discussion, we know that BVP(1.1) has no positive solution. The proof is completed. □

5. Examples

In this section, we present two examples to illustrate the main theorems.

Example 5.1. Consider the following boundary value problem of second order differential equation on the whole line

$$\left\{ \begin{array}{l} \left[\left(\frac{e^{t^2}}{|t|}x'(t)\right)^3 + \frac{e^{t^2}}{t}x'(t)\right]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to-\infty} x(t) - \lim\limits_{t\to-\infty} \frac{e^{t^2}}{t}x'(t) = 0, \\ \lim\limits_{t\to+\infty} x(t) + \lim\limits_{t\to+\infty} \frac{e^{t^2}}{t}x'(t) = 0, \end{array} \right. \tag{5.1}$$

where

$$f\left(t,u,\frac{v}{\rho(t)}\right) = \phi(t)f_0(u,v),$$

$$\phi(t) = \left\{ \begin{array}{l} \frac{1}{t^2},\ |t| > 1, \\ \frac{1}{\sqrt{t}},\ |t| \leq 1 \end{array} \right.$$

$$f_0 : [0,+\infty) \times R \to [0,+\infty) \text{ is continuous.}$$

Then BVP(5.1) has at least one positive solutions if there exist sufficiently small $a > 0, L_1 > 0$ and sufficiently large $b > 0, L_2 > 0$ such that

$$f_0\,(u,v) \geq M_0,\ u \in [e^{-1}a,a], v \in [-L_1,L_1];$$

$$f_0\,(u,v) \geq W_0,\ u \in [0,a], v \in [-L_1,L_1],$$

$$f_0\,(u,v) \leq E_0,\ u \in [0,b], v \in [-L_2,L_2].$$

Proof. Corresponding to BVP(1.1), we have $\rho(t) = \frac{e^{t^2}}{|t|}$ (that is singular at $t = 0$), $\Phi(s) = s^3 + s$, $\alpha = \beta = \gamma = \delta = 1$. Then

$$\Phi^{-1}(s) = \frac{\sqrt[3]{\frac{3}{2}\sqrt{81s^2+12}+\frac{27}{2}s} - \sqrt[3]{\frac{3}{2}\sqrt{81s^2+12}-\frac{27}{2}s}}{3},$$

$$\nu(s) = \min\{s^4, s\},$$

$$\nu^{-1}(s) = \begin{cases} s, \ s \geq 1, \\ s^{\frac{1}{4}}, \ s \leq 1, \end{cases} = \max\{s^{\frac{1}{4}}, s\}, \quad \phi \in L^1(R),$$

$$g(t, u, v) = h(t, u, v) \equiv 0,$$

$$\sigma = 2 + \int_{-\infty}^{+\infty} |s| e^{-s^2} ds = 3 > 0,$$

Choose $k = 1$ and we get

$$\mu = \int_{-\infty}^{-k} \frac{1}{\rho(s)} ds \left(2 \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds\right)^{-1} = e^{-1}.$$

By direct computation, we have

$$M_0 = \max\left\{ \left(\nu\left(\frac{\int_{-k}^{0} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{s}^{0} \phi(r) dr\right) ds}{a}\right)\right)^{-1}, \left(\nu\left(\frac{\int_{0}^{k} \frac{1}{\rho(s)} \Phi^{-1}\left(\int_{0}^{s} \phi(r) dr\right) ds}{a}\right)\right)^{-1} \right\},$$

$$W_0 = \max\left\{ \left(\nu\left(\frac{\Phi^{-1}\left(\int_{-\infty}^{0} \phi(r) dr\right)}{L_1}\right)\right)^{-1}, \left(\nu\left(\frac{\Phi^{-1}\left(\int_{0}^{+\infty} \phi(r) dr\right)}{L_1}\right)\right)^{-1} \right\},$$

$$E_0 = \min\left\{ \nu\left(\frac{b}{\left(1 + \int_{-\infty}^{+\infty} \frac{1}{\rho(s)} ds\right) \Phi^{-1}\left(2 \int_{-\infty}^{+\infty} \phi(r) dr\right)}\right), \nu\left(\frac{L_2}{\Phi^{-1}\left(2 \int_{-\infty}^{+\infty} \phi(r) dr\right)}\right) \right\}.$$

One sees that $E_0 > \max\{M_0, W_0\}$ for sufficiently small $a > 0, L_1 > 0$ and sufficiently large $b > 0, L_2 > 0$. Then the growth assumptions imposed on f_0 and Theorem 3.1 implies that BVP(5.1) has at least one positive solution x.

Example 5.2. Consider the following boundary value problem of second order differential equation on the whole line

$$\begin{cases} \left[\left|\frac{e^{t^2}}{|t|} x'(t)\right| \frac{e^{t^2}}{t} x'(t) + \frac{e^{t^2}}{t} x'(t)\right]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t \to -\infty} x(t) - \lim\limits_{t \to -\infty} \frac{e^{t^2}}{t} x'(t) = 0, \\ \lim\limits_{t \to +\infty} x(t) + \lim\limits_{t \to +\infty} \frac{e^{t^2}}{t} x'(t) = 0, \end{cases} \tag{5.2}$$

where

$$f(t,u,v)=\phi(t)\left(\frac{1}{2}+\frac{1}{\pi}\arctan u\right)\left(\frac{e^{-2t^2}}{t^2}|v|^2+\frac{e^{-t^2}}{|t|}|v|\right),\quad \phi\in L^1(R)$$

or

$$f(t,u,v)=\phi(t)\left(\frac{3}{2}+\frac{1}{\pi}\arctan v\right)(u^2+u),\quad \phi\in L^1(R).$$

Then BVP(5.2) has no positive solution if $\int\limits_{-\infty}^{+\infty}\phi(r)dr$ is sufficiently small or

$$\min\left\{\int\limits_0^{+\infty}\frac{1}{\rho(s)}\Phi^{-1}\left(\int\limits_0^s\phi(r)dr\right)ds,\int\limits_{-\infty}^0\frac{1}{\rho(s)}\Phi^{-1}\left(\int\limits_s^0\phi(r)dr\right)ds\right\}>2, \tag{5.3}$$

Proof. Corresponding to BVP(1.1), we have we have $\rho(t)=\frac{e^{t^2}}{|t|}$ (that is singular at $t=0$), $\Phi(s)=|s|s+s$, $\alpha=\beta=\gamma=\delta=1$. Then

$$\Phi^{-1}(s)=\begin{cases}\frac{1-\sqrt{1-4s}}{2}, & s\le 0,\\ \frac{-1+\sqrt{1+4s}}{2}, & s\ge 0.\end{cases}$$

$$\nu(s)=\min\{s^3,s\},$$

$$\nu^{-1}(s)=\begin{cases}s, & s\ge 1,\\ s^{\frac{1}{3}}, & s\le 1,\end{cases}=\max\{s^{\frac{1}{3}},s\},$$

$$g(t,u,v)=h(t,u,v)\equiv 0,$$

$$\sigma=2+\int\limits_{-\infty}^{+\infty}|s|e^{-s^2}ds=3>0,$$

Case 1. It is easy to see that

$$\sup_{(t,u,v)\in\Omega}\frac{f\left(t,u,\frac{v}{\rho(t)}\right)}{\phi(t)\Phi(|v|)}\le 1. \tag{5.4}$$

If

$$\frac{4}{9}\left(\int\limits_{-\infty}^{+\infty}\phi(r)dr\right)^2+\frac{8}{3}\int\limits_{-\infty}^{+\infty}\phi(s)ds<1, \tag{5.5}$$

Then Theorem 4.1 implies that BVP(5.2) does not admit positive solutions.

Case 2. It is easy to see that

$$\inf_{(t,u,v)\in\Omega} \frac{f\left(t,u,\frac{v}{\rho(t)}\right)}{\phi(t)\Phi(u)} \geq 1, \tag{5.6}$$

and

$$\mu \min\left\{\int_0^k \frac{1}{\rho(s)}\Phi^{-1}\left(\int_0^s \phi(r)dr\right)ds, \int_{-k}^0 \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^0 \phi(r)dr\right)ds\right\}$$

$$= \frac{1}{2}\int_{-\infty}^{-k} \frac{1}{\rho(s)}ds \min\left\{\int_0^k \frac{1}{\rho(s)}\Phi^{-1}\left(\int_0^s \phi(r)dr\right)ds, \int_{-k}^0 \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^0 \phi(r)dr\right)ds\right\}$$

$$\to \frac{1}{2}\min\left\{\int_0^{+\infty} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_0^s \phi(r)dr\right)ds, \int_{-\infty}^0 \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^0 \phi(r)dr\right)ds\right\} \text{ as } k \to +\infty.$$

Hence If (5.3) holds, then we can choose $k > 0$ such that (4.4) holds. Then Theorem 4.2 implies that BVP(5.2) does not admit positive solutions.

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