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Basic Results in the Theory of Hybrid Casual Nonlinear Differential Equations

Bapurao Chandrabhan Dhage
Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
e-mail : bcdhage@gmail.com

Abstract. In this paper, some basic results concerning the existence, strict and nonstrict inequalities and existence of the maximal and minimal solutions are proved for a hybrid causal differential equation. Our results generalize some basic results of Leela and Lakshmikantham [13] and Dhage and Lakshmikantham [10] respectively for the nonlinear first order classical and hybrid differential equations.

1. Introduction

Let $\mathbb{R}$ be the real line and let, unless otherwise mentioned, $J = [t_0, t_0 + a)$ be a bounded interval in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$. Let $C(J, \mathbb{R})$ be the class of continuous real-valued functions defined on J. An operator $Q : C(J, \mathbb{R}) = E \to E$ is said to be causal or nonanticipative if for any $x, y \in E$ with $x(s) = y(s)$, $t_0 \leq s \leq t$, we have that $(Qx)(s) = (Qy)(s)$ for $t_0 \leq s \leq t$, $t < t_0 + a$. Note that the sum and product of two causal operators is again a causal operator. Again, if $\{Q_n\}$ is a sequence of causal operators in E such that

$$\lim_{n\to\infty} (Q_n x)(t) = (Qx)(t)$$

for $(t, x) \in J \times E$, then Q is again a causal operator on E into itself.

A differential equation which involves the causal operator is called a causal differential equation. Thus, the initial value problems of causal differential equations are represented by

$$\left.\begin{aligned} x'(t) &= (Qx)(t), \quad t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.1}$$

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For the study of causal differential equations, see Corduneunu [1], a recent paper by Drici *et al.* [11] and the references herein. The importance of the investigations of causal differential equations lies in the fact that they include several dynamic systems. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [3] and extensively treated in the several papers on perturbed differential equations. See Burton [4], Dhage [5] and the references therein. This class of differential equations includes the perturbations of original differential equations in different ways. A sharp classification of different types of perturbations of differential equations appears in Dhage and Lakshmikantham [10] which can be treated with hybrid fixed point theory (see Dhage [5]-[6]). In this paper, we initiate the basic theory of hybrid causal differential equations.

The causal differential equations can be perturbed in several ways to obtain different types of hybrid causal differential equations. Here, we consider the hybrid causal differential equation involving the mixed type of perturbations of second type. In such perturbations, the unknown function under derivative is perturbed in a linear and quadratic way.

Given a continuous operator $Q : E = C(J,\mathbb{R}) \to E$, consider the initial value problems for hybrid causal differential equation (in short HCDE) given by

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &= (Qx)(t), \quad t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.2}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $k : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The most simple special case of the HCDE (1.2) is the following IVP of HDE,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &= g(t,x(t)), \quad t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.3}$$

Similarly, another example of a hybrid causal integro-differential equation is the following IVP of hybrid integro-differential equations,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &= g\left(t, x(t), \int_{t_0}^{t} k(t,s,x(s))\,ds\right), \quad t \in J, \\ x_{t_0} &= x_0 \end{aligned}\right\} \tag{1.4}$$

where, all the functions involved in (1.4) belong to the appropriate function spaces.

The HDE (1.3) has recently been studied by Dhage [5] via hybrid fixed point theory in Banach algebras. However, the hybrid integro-differential equations of the type (1.4) are not discussed yet, but can be treated similarly. See Dhage and Lakshmikantham [10].

2. Strict and Nonstrict Inequalities

We list the following hypotheses.

(A_0) The function $x \to \dfrac{x - k(t,x)}{f(t,x)}$ is increasing in $\mathbb{R}$ for all $t \in J$.

(B_0) The causal operator Q is quasi-nondecreasing, that is,

$$x(t_1) = y(t_1),\ \ x(t) < y(t),\ \ t_0 \le t < t_1,$$

implies

$$(Qx)(t_1) = (Qy)(t_1),\ \ (Qx)(t) \le (Qy)(t)$$

for $t_0 \le t < t_1 < t_0 + a$.

We begin by proving the basic results dealing with hybrid causal differential inequalities.

Theorem 2.1. *Assume that hypotheses* (A_0) *and* (B_0) *hold. Suppose that there exist* $y, z \in C(J, \mathbb{R})$ *such that*

$$\frac{d}{dt}\left[\frac{y(t) - k(t, y(t))}{f(t, y(t))}\right] \le (Qy)(t) \tag{1.1}$$

and

$$\frac{d}{dt}\left[\frac{z(t) - k(t, z(t))}{f(t, z(t))}\right] \ge (Qz)(t) \tag{1.2}$$

for $t \in J$. *If one of the inequalities* (*1.1*) *and* (*1.2*) *is strict and*

$$y(t_0) < z(t_0), \tag{1.3}$$

then

$$y(t) < z(t) \tag{1.4}$$

for all $t \in J$.

Proof. Assume that the conclusion (1.4) is false and that

$$\frac{d}{dt}\left[\frac{z(t) - k(t, z(t))}{f(t, z(t))}\right] > (Qz)(t)$$

for $t \in J$. Denote

$$Y(t) = \frac{y(t) - k(t, y(t))}{f(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t) - k(t, z(t))}{f(t, z(t))}$$

for $t \in J$.

Now continuity of y and z together with the inequality (1.3) implies that there exists a $t_1 > t_0$ such that

$$y(t_1) = z(t_1) \quad \text{and} \quad y(t) < z(t) \tag{1.5}$$

for all $t_0 \leq t < t_1$. Since Q is quasi-nondecreasing, the expressions in (1.5) imply that

$$(Qy)(t_1) = (Qz)(t_1) \quad \text{and} \quad (Qy)(t) \leq (Qz)(t) \tag{1.6}$$

for all $t_0 \leq t < t_1$.

As hypothesis (A_0) holds, it follows from (1.5) that

$$Y(t_1) = Z(t_1) \quad \text{and} \quad Y(t) < Z(t) \tag{1.7}$$

for all $t_0 \leq t < t_1$. The above relation (1.7) further yields

$$\frac{Y(t_1+h) - Y(t_1)}{h} > \frac{Z(t_1+h) - Z(t_1)}{h}$$

for small $h < 0$. Taking the limit as $h \to 0$, we obtain

$$Y'(t_1) \geq Z'(t_1). \tag{1.8}$$

Hence from (1.6) and (1.8), we get

$$(Qy)(t_1) \geq Y'(t_1) \geq Z'(t_1) > (Qz)(t_1).$$

This is a contradiction and the proof is complete. □

The next result is about the nonstrict inequality for the HCDE (1.2) on J which requires a one-sided Lipschitz condition.

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Suppose that there exists a real number $L > 0$ such that*

$$(Qy)(t) - (Qz)(t) \leq L \sup_{t_0 \leq s \leq t} \left[\frac{y(s) - k(s, y(s))}{f(s, y(s))} - \frac{z(s) - k(s, z(s))}{f(s, z(s))} \right] \tag{1.9}$$

whenever $y(s) \geq z(s)$, $t_0 \leq s \leq t$. Then,

$$y(t_0) \leq z(t_0) \tag{1.10}$$

implies

$$y(t) \leq z(t) \tag{1.11}$$

for all $t \in J$.

Proof. Let $\epsilon > 0$ and let a real number $L > 0$ be given. Set

$$\frac{z_\epsilon(t) - k(t, z_\epsilon(t))}{f(t, z_\epsilon(t))} = \frac{z(t) - k(t, x(t))}{f(t, z(t))} + \epsilon e^{2L(t-t_0)} \tag{1.12}$$

so that

$$\frac{z_\epsilon(t) - k(t, z_\epsilon(t))}{f(t, z_\epsilon(t))} > \frac{z(t) - k(t, x(t))}{f(t, z(t))}.$$

Define

$$Z_\epsilon(t) = \frac{z(t) - k(t, z(t))}{f(t, z(t))} \quad \text{and} \quad Z(t) = \frac{z(t) - k(t, z(t))}{f(t, z(t))}$$

for $t \in J$.

Now using the one-sided Lipschitz condition (1.9), we obtain

$$(Qz_\epsilon)(t) - (Qz)(t) \leq L \sup_{t_0 \leq s \leq t} [Z_\epsilon(s) - Z(s)] = L\epsilon e^{2L(t-t_0)}.$$

Now,

$$\begin{aligned} Z_\epsilon'(t) &= Z'(t) + 2L\epsilon e^{2L(t-t_0)} \\ &\geq (Qz)(t) + 2L\epsilon e^{2L(t-t_0)} \\ &\geq (Qz_\epsilon)(t) + 2L\epsilon e^{2L(t-t_0)} - L\epsilon e^{2L(t-t_0)} \\ &= (Qz_\epsilon)(t) + L\epsilon e^{2L(t-t_0)} \\ &> (Qz_\epsilon)(t) \end{aligned}$$

for all $t \in J$. Also, we have

$$Z_\epsilon(t_0) > Z(t_0) \geq Y(t_0).$$

Now we apply Theorem 2.1 with $z = z_\epsilon$ to yield

$$Y(t) < Z_\epsilon(t)$$

for all $t \in J$. On taking $\epsilon \to 0$ in the above inequality, we get

$$Y(t) \leq Z(t)$$

which further in view of hypothesis (A_0) implies that (1.11) holds on J. This completes the proof. □

Remark 2.1. The conclusion of Theorems 2.1 and 2.2 also remains true if we replace the derivative in the inequalities (1.1) and (1.2) by Dini-derivative D_- of the function $\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}$ on the bounded interval J.

As an application of Theorem 2.1, consider a IVP of nonlinear hybrid integro-differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &= q(t)+\int_{t_0}^{t} g(t,s,x(s))\,ds,\;\; t\in J,\\ x(t_0) &= x_0\in\mathbb{R}, \end{aligned}\right\} \tag{1.13}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, $k : J \times \mathbb{R} \to \mathbb{R}$, $g : J \times J \times \mathbb{R} \to \mathbb{R}$ and $q : J \to \mathbb{R}$ are continuous.

Assume that $g(t, s, x)$ is monotone increasing in x for each (t, s) and

$$\frac{d}{dt}\left[\frac{y(t)-k(t,y(t))}{f(t,y(t))}\right] \le q(t)+\int_{t_0}^{t} g(t,s,y(s))\,ds = (Qy)(t)$$

and

$$\frac{d}{dt}\left[\frac{z(t)-k(t,z(t))}{f(t,z(t))}\right] \ge q(t)+\int_{t_0}^{t} g(t,s,z(s))\,ds = (Qz)(t)$$

for all $t \in J$ and one of the above inequalities is strict. Further, if the hypothesis (A_0) holds, then $y(t_0) < z(t_0)$ implies $y(t) < z(t)$ for all $t \in J$.

3. Existence Result

In this section, we prove an existence result for the HCDE (1.2) on a closed and bounded interval $J = [t_0, t_0 + a]$ under mixed Lipschitz and compactness conditions on the nonlinearities involved in it. We place the HCDE (1.2) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a supremum norm $\|\cdot\|$ in $C(J, \mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|$$

and a multiplication “ $\cdot$ ” in $C(J, \mathbb{R})$ by

$$(x \cdot y)(t) = (xy)(t) = x(t)y(t)$$

for $x, y \in C(J, \mathbb{R})$. Clearly $C(J, \mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it. By $L^1(J, \mathbb{R})$ we denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_{t_0}^{t_0+a} |x(s)|\,ds.$$

We prove the existence of solution for the HCDE (1.2) via a hybrid fixed point theorem in Banach algebras due to Dhage [5, 7].

Theorem 3.1. *Let S be a closed convex and bounded subset of the Banach algebra E and let $A, C : E \to E$ and $B : S \to E$ be three operators such that*

(a) *A and C are Lipschitz with the Lipschitz constants α and β respectively,*

(b) *B is compact and continuous,*

(c) *$x = Ax\,By + Cx$ for all $y \in S \Longrightarrow x \in S$, and*

(d) *$\alpha M + \beta < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.*

Then the operator equation $Ax\,Bx + Cx = x$ has a solution in S.

We consider the following hypotheses in what follows.

(A_1) There exist constants $L_1 > 0$ and $L_2 > 0$ such that

$$|f(t,x) - f(t,y)| \le L_1|x-y|$$

and

$$|k(t,x) - k(t,y)| \le L_2|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) There exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|(Qx)(t))| \le h(t),\ \ t \in J,$$

for all $x \in E$.

The following lemma is useful in the sequel.

Lemma 3.1. *Assume that hypothesis (A_0) holds. Then for any $h \in L^1(J,\mathbb{R}_+)$, a function $x \in C(J,\mathbb{R})$ such that $t \mapsto \dfrac{x(t) - k(t,x(t))}{f(t,x(t))}$ is differentiable is a solution of the HCDE*

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t,x(t))}{f(t,x(t))}\right] &= h(t),\ \ t \in J,\\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.1}$$

if and only if x satisfies the hybrid causal integral equation (HCIE)

$$x(t) = k(t,x(t)) + \big[f(t,x(t))\big]\left(\frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} + \int_{t_0}^{t} h(s)\,ds\right),\ \ t \in J. \tag{3.2}$$

Proof. Let $h \in L^1(J,\mathbb{R}_+)$. Assume first that x is a solution of the HCDE (3.1). By definition, $t \mapsto \dfrac{x(t) - k(t,x(t))}{f(t,x(t))}$ is differentiable, whence $\dfrac{d}{dt}\left[\dfrac{x(t) - k(t,x(t))}{f(t,x(t))}\right]$ is integrable on J. Applying integration to (3.1) from t_0 to t, we obtain the HCIE (3.2) on J.

Conversely, assume that x satisfies the HCIE (3.2). Then by direct differentiation we obtain the first equation in (3.1). Again, substituting $t = t_0$ in (3.2) yields

$$\frac{x(t_0) - k(t_0, x(t_0))}{f(t_0, x(t_0))} = \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)}.$$

Since the mapping $x \mapsto \dfrac{x - k(t, x)}{f(t, x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$, the mapping $x \mapsto \dfrac{x - k(t_0, x)}{f(t_0, x)}$ is injective in $\mathbb{R}$, whence $x(t_0) = x_0$. Hence the proof of the lemma is complete. □

Now we are in a position to prove the following existence theorem for HCDE (1.2).

Theorem 3.2. *Assume that the hypotheses* (A_0)-(A_2) *hold. Further, if*

$$L_1 \left(\left| \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) + L_2 < 1 \tag{3.3}$$

then the HCDE (*1.2*) *has a solution defined on* J.

Proof. Set $E = C(J, \mathbb{R})$ and define a subset S of E defined by

$$S = \{x \in E \mid \|x\| \leq N\} \tag{3.4}$$

where,

$$N = \frac{F_0 \left(\left| \dfrac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) + K_0}{1 - L_1 \left(\left| \dfrac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} \right| + \|h\|_{L^1} \right) - L_2},$$

and

$$F_0 = \sup_{t \in J} |f(t, 0)| \quad \text{and} \quad K_0 = \sup_{t \in J} |k(t, 0)|.$$

Clearly S is a closed, convex and bounded subset of the Banach algebra E. Now, using the hypotheses (A_0) and (A_2) it can be shown by an application of Lemma 3.1 that the HCDE (1.2) is equivalent to the nonlinear HCIE

$$x(t) = k(t, x(t)) + \big[f(t, x(t))\big] \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx)(s)\, ds \right) \tag{3.5}$$

for $t \in J$.

Define three operators $A, C : E \to E$ and $B : S \to E$ by

$$Ax(t) = f(t, x(t)), \; t \in J, \tag{3.6}$$

$$Bx(t) = \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx)(s)\, ds,\ t \in J, \tag{3.7}$$

and

$$Cx(t) = k(t, x(t)),\ t \in J. \tag{3.8}$$

Then, the HCIE (3.5) is transformed into an operator equation as

$$Ax(t)\, Bx(t) + Cx(t) = x(t),\ t \in J. \tag{3.9}$$

We shall show that the operators A, B and C satisfy all the conditions of Theorem 3.1.

First, we show that A is a Lipschitz operator on E with the Lipschitz constant L_1. Let $x, y \in E$. Then, by hypothesis (A_1),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq L_1 |x(t) - y(t)| \leq L_1 \|x - y\|$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \leq L_1 \|x - y\|$$

for all $x, y \in E$. This shows that A is a Lipschitz operator on E with the Lipschitz constant L_1. Similarly, it can be shown that C is also a Lipschitz operator on E with the Lipschitz constant L_2.

Next, we show that B is a compact and continuous operator on S into E. First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx_n)(s)\, ds \right) \\
&= \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \lim_{n\to\infty} \int_{t_0}^{t} (Qx_n)(s)\, ds \\
&= \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} \left[\lim_{n\to\infty} (Qx_n)(s) \right] ds \\
&= \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx)(s)\, ds \\
&= Bx(t)
\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas *et al.* [2], it is proved that B is a a continuous operator on S.

Next, we show that B is compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equi-continuous set in E. Let $x \in S$ be arbitrary. Then by hypothesis (A$_2$),

$$\begin{aligned}
|Bx(t)| &\leq \Big|\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)}\Big| + \int_{t_0}^{t} |(Qx)(s)|\, ds \\
&\leq \Big|\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)}\Big| + \int_{t_0}^{t} h(s)\, ds \\
&\leq \Big|\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)}\Big| + \|h\|_{L^1}
\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|Bx\| \leq \Big|\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)}\Big| + \|h\|_{L^1}$$

for all $x \in S$. This shows that B is uniformly bounded on S.

Again, let $t_1, t_2 \in J$. Then for any $x \in S$, one has

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left|\int_{t_0}^{t_1} (Qx)(s)\, ds - \int_{t_0}^{t_2} (Qx)(s)\, ds\right| \\
&\leq \left|\int_{t_2}^{t_1} |(Qx)(s)|\, ds\right| \\
&\leq |p(t_1) - p(t_2)|
\end{aligned}$$

where $p(t) = \displaystyle\int_{t_0}^{t} h(s)\, ds$. Since the function p is continuous on compact J, it is uniformly continuous. Hence, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Longrightarrow |Bx(t_1) - Bx(t_2)| < \epsilon$$

for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $B(S)$ is an equi-continuous set in E. Now the set $B(S)$ is uniformly bounded and equicontinuous set in E, so it is compact by Arzelá-Ascoli theorem. As a result, B is a continuous and compact operator on S.

Next, we show that hypothesis (c) of Theorem 3.1 is satisfied. Let $x \in E$ and $y \in S$ be arbitrary such that $x = Ax\,By + Cx$. Then, by assumption (A$_1$), we have

$$\begin{aligned}
|x(t)| &\leq |Ax(t)|\,|By(t)| + |Cx(t)| \\
&\leq \big|f(t, x(t))\big|\,\Big|\left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qy)(s)\, ds\right)\Big| + \big|k(t, x(t))\big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \big[|f(t,x(t)) - f(t,0)| + |f(t,0)|\big] \left| \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \int_{t_0}^{t} |(Qy)(s)|\,ds \right) \right| \\
&\quad + |k(t,x(t)) - k(t,0)| + |k(t,0)| \\
&\leq [L_1|x(t)| + F_0] \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \int_{t_0}^{t} h(s)\,ds \right) \\
&\quad + L_2|x(t)| + K_0 \\
&\leq \frac{F_0 \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1} \right) + K_0}{1 - L_1 \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1} \right) - L_2}.
\end{aligned}
$$

Taking supremum over t,

$$
\|x\| \leq \frac{F_0 \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1} \right) + K_0}{1 - L_1 \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1} \right) - L_2} = N.
$$

This shows that hypothesis (c) of Theorem 3.1 is satisfied. Finally, we have

$$
M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq \left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1}
$$

and so,

$$
L_1 M + L_2 \leq L_1 \left(\left| \frac{x_0 - k(t_0,x_0)}{f(t_0,x_0)} \right| + \|h\|_{L^1} \right) + L_2 < 1.
$$

Thus, all the conditions of Theorem 3.1 are satisfied and hence the operator equation $Ax\,Bx + Cx = x$ has a solution in S. As a result, the HCDE (1.2) has a solution defined on J. This completes the proof. □

4. Existence of Maximal and Minimal Solutions

In this section, we shall prove the existence of maximal and minimal solutions for the HCDE (1.2) on $J = [t_0, t_0 + a]$. We need the following definition in what follows.

Definition 4.1. A solution r of the HCDE (1.2) is said to be maximal if for any other solution x to the HCDE (1.2) one has $x(t) \leq r(t)$ for all $t \in J$. Again, a solution ρ of the HCDE (1.2) is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any solution of the HCDE (1.2) existing on J.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the similar arguments with appropriate modifications. Given an arbitrary small real number $\epsilon > 0$, consider the following initial value problem of HCDE,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &\leq (Qx)(t)+\epsilon, \;\; t \in J, \\ x(t_0) &= x_0+\epsilon, \end{aligned}\right\} \tag{4.1}$$

where, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $k \in C(J \times \mathbb{R}, \mathbb{R})$ and the operator $Q : E \to E$ is continuous.

An existence theorem for the HCDE (4.1) can be stated as follows:

Theorem 4.1. *Assume that the hypotheses* (A_0)-(A_2) *hold. Suppose also that the inequality* (*3.3*) *holds. Then for every small number* $\epsilon > 0$, *the HCDE* (*4.1*) *has a solution defined on* J.

Proof. By hypothesis, since

$$L_1\left(\left|\frac{x_0-k(t_0,x_0)}{f(t_0,x_0)}\right| + \|h\|_{L^1}\right) + L_2 < 1,$$

there exists an $\epsilon_0 > 0$ such that

$$L_1\left(\left|\frac{x_0+\epsilon-k(t_0,x_0+\epsilon)}{f(t_0,x_0+\epsilon)}\right| + \|h\|_{L^1} + \epsilon a\right) + L_2 < 1 \tag{4.2}$$

for all $0 < \epsilon \leq \epsilon_0$. Now the rest of the proof is similar to Theorem 3.2. □

Our main existence theorem for maximal solution for the HCDE (1.2) is

Theorem 4.2. *Assume that the hypotheses* (A_0)-(A_2) *hold. Further, if the condition (3.3) holds, then the HCDE (1.2) has a maximal solution defined on* J.

Proof. Let $\{\epsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim\limits_{n\to\infty} \epsilon_n = 0$, where ϵ_0 is a positive real number satisfying the inequality

$$L_1\left(\left|\frac{x_0+\epsilon_0-k(t_0,x_0+\epsilon_0)}{f(t_0,x_0+\epsilon_0)}\right| + \|h\|_{L^1} + \epsilon_0 a\right) + L_2 < 1. \tag{4.3}$$

The number ϵ_0 exists in view of the inequality (3.3). Then for any solution u of the HCDE (1.2), by Theorem 2.1, one has

$$u(t) < r(t,\epsilon_n) \tag{4.4}$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$, where $r(t,\epsilon_n)$ is a solution of the HCDE,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] &= (Qx)(t)+\epsilon_n, \;\; t \in J, \\ x(t_0) &= x_0+\epsilon_n \in \mathbb{R}, \end{aligned}\right\} \tag{4.5}$$

defined on J.

Since, by Theorems 3.1 and 3.2, $\{r(t,\epsilon_n)\}$ is a decreasing sequence of positive real numbers, the limit

$$r(t) = \lim_{n\to\infty} r(t,\epsilon_n) \tag{4.6}$$

exists. We show that the convergence in (4.6) is uniform on J. To finish, it is enough to prove that the sequence $\{r(t,\epsilon_n)\}$ is equi-continuous in $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
&|r(t_1,\epsilon_n) - r(t_2,\epsilon_n)| \\
&\quad = \big|k(t_1, r(t_1,\epsilon_n)) - k(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \Bigg|\big[f(t_1, r(t_1,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_1}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_1}\epsilon_n\,ds\Big) \\
&\qquad\quad - \big[f(t_2, r(t_2,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_2}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_2}\epsilon_n\,ds\Big)\Bigg| \\
&\quad \le \big|k(t_1, r(t_1,\epsilon_n)) - k(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \Bigg|\big[f(t_1, r(t_1,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_1}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_1}\epsilon_n\,ds\Big) \\
&\qquad\quad - \big[f(t_2, r(t_2,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_1}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_1}\epsilon_n\,ds\Big)\Bigg| \\
&\qquad + \Bigg|\big[f(t_2, r(t_2,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_1}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_2}\epsilon_n\,ds\Big) \\
&\qquad\quad - \big[f(t_2, r(t_2,\epsilon_n))\big]\Big(\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)} + \int_{t_0}^{t_2}(Qr)(s,\epsilon_n)\,ds + \int_{t_0}^{t_2}\epsilon_n\,ds\Big)\Bigg| \\
&\quad \le \big|k(t_1, r(t_1,\epsilon_n)) - k(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big|\,\Big(\Big|\frac{x_0+\epsilon_n}{f(t_0,x_0+\epsilon_n)}\Big| + \|h\|_{L^1} + \epsilon_n a\Big) \\
&\qquad + F\,\big[\big|p(t_1)-p(t_2)\big| + |t_1-t_2|\epsilon_n\big]
\end{aligned} \tag{4.7}$$

where, $F = \sup_{(t,x)\in J\times[-N,N]} |f(t,x)|$ and $p(t) = \int_{t_0}^{t} h(s)\,ds$.

Since f and k are continuous on compact set $J\times[-N,N]$, they are uniformly continuous there. Hence,

$$\big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

and

$$\big|k(t_1, r(t_1, \epsilon_n)) - k(t_2, r(t_2, \epsilon_n))\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Similarly, since the function p is continuous on compact set J, it is uniformly continuous and hence

$$\big|p(t_1) - p(t_2)\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for $t_1, t_2 \in J$.

Therefore, from the above inequality (4.7), it follows that

$$|r(t_1, \epsilon_n) - r(t_1, \epsilon_n)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t, \epsilon_n) \to r(t) \quad \text{as} \quad n \to \infty$$

for all $t \in J$. Next, we show that the function $r(t)$ is a solution of the HCDE (3.1) defined on J. Now, since $r(t, \epsilon_n)$ is a solution of the HCDE (4.5), we have

$$\begin{aligned} r(t, \epsilon_n) = {} & \big[f(t, r(t, \epsilon_n))\big] \left(\frac{x_0 + \epsilon_n}{f(t_0, x_0 + \epsilon_n)} + \int_{t_0}^{t} (Qr)(s, \epsilon_n)\, ds + \int_{t_0}^{t} \epsilon_n\, ds \right) \\ & + k(t, r(t, \epsilon_n)) \end{aligned} \tag{4.8}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above equation (4.8) yields

$$r(t) = k(t, r(t)) + \big[f(t, r(t))\big] \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qr)(s)\, ds \right)$$

for $t \in J$. Thus, the function r is a solution of the HCDE (1.2) on J. Finally, form the inequality (4.4) it follows that

$$u(t) \le r(t)$$

for all $t \in J$. Hence the HCDE (1.2) has a maximal solution on J. This completes the proof. □

5. Comparison Theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the HCDE (1.2). In this section we prove that the maximal and minimal solutions serve the bounds for the solutions of the related differential inequality to HCDE (1.2) on $J = [t_0, t_0 + a]$.

Theorem 5.1. *Assume that the hypotheses* (A_0)-(A_2) *hold. Suppose that the condition* (*3.3*) *holds. Further, if there exists a function* $u \in C(J, \mathbb{R})$ *such that*

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{u(t) - k(t, u(t))}{f(t, u(t))}\right] &\le (Qu)(t), \;\; t \in J, \\ u(t_0) &\le x_0, \end{aligned}\right\} \tag{5.1}$$

then,

$$u(t) \leq r(t) \tag{5.2}$$

for all $t \in J$, where r is a maximal solution of the HCDE (1.2) on J.

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 4.3, $r(t, \epsilon)$ is a maximal solution of the HCDE (4.1) and that the limit

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \tag{5.3}$$

is uniform on J and the function r is a maximal solution of the HCDE (1.2) on J. Hence, we obtain

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{r(t,\epsilon) - k(t, r(t,\epsilon))}{f(t, r(t,\epsilon))}\right] &= (Qr)(t,\epsilon) + \epsilon, \ \ t \in J, \\ r(t_0, \epsilon) &= x_0 + \epsilon. \end{aligned}\right\} \tag{5.4}$$

From above inequality it follows that

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{r(t,\epsilon) - k(t, r(t,\epsilon))}{f(t, r(t,\epsilon))}\right] &> (Qr)(t,\epsilon), \ \ t \in J, \\ r(t_0, \epsilon) &> x_0. \end{aligned}\right\} \tag{5.5}$$

Now we apply Theorem 2.1 to the inequalities (5.1) and (5.5) and conclude that

$$u(t) < r(t, \epsilon) \tag{5.6}$$

for all $t \in J$. This further in view of limit (5.3) implies that inequality (5.2) holds on J. This completes the proof. □

Theorem 5.2. *Assume that the hypotheses (A_0)-(A_2) hold. Suppose that the condition (3.3) holds. Further, if there exists a function $v \in C(J, \mathbb{R})$ such that*

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{v(t) - k(t, v(t))}{f(t, v(t))}\right] &\geq (Qv)(t), \ \ t \in J, \\ v(t_0) &\geq x_0, \end{aligned}\right\} \tag{5.7}$$

then,

$$\rho(t) \leq v(t) \tag{5.8}$$

for all $t \in J$, where ρ is a minimal solution of the HCDE (1.2) on J.

Note that Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for the HCDE (1.2) on J. A result in this direction is

Theorem 5.3. *Assume that the hypotheses* (A_0)-(A_2) *hold and let the condition (3.3) be satisfied. Suppose that there exists a function* $G : J \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{aligned} &|(Qx_1)(t) - (Qx_2)(t)| \\ &\qquad \leq G\left(t, \max_{s\in[t_0,t]}\left|\frac{x_1(s) - k(s, x_1(s))}{f(s, x_1(s))} - \frac{x_2(s) - k(s, x_2(s))}{f(s, x_2(s))}\right|\right) \end{aligned} \tag{5.9}$$

for all $t \in J$ *and* $x_1, x_2 \in E$. *If identically zero function is the only solution of the differential equation*

$$m'(t) = G(t, m(t)), \ \ t \in J, \ m(t_0) = 0, \tag{5.10}$$

then the HCDE (1.2) has a unique solution on J.

Proof. By Theorem 3.2, the HCDE (1.2) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of the HCDE (1.2) existing on J. Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = \left|\frac{u_1(t) - k(t, u_1(t))}{f(t, u_1(t))} - \frac{u_2(t) - k(t, u_2(t))}{f(t, u_2(t))}\right|. \tag{5.11}$$

As $(|x(t)|)' \leq |x'(t)|$ for $t \in J$, we have that

$$\begin{aligned} m'(t) &\leq \left|\frac{d}{dt}\left[\frac{u_1(t) - k(t, u_1(t))}{f(t, u_1(t))}\right] - \frac{d}{dt}\left[\frac{u_2(t) - k(t, u_2(t))}{f(t, u_2(t))}\right]\right| \\ &\leq |(Qx_1)(t) - (Qx_2)(t)| \\ &\leq G\left(t, \left|\frac{u_1(t) - k(t, u_1(t))}{f(t, u_1(t))} - \frac{u_2(t) - k(t, u_2(t))}{f(t, u_2(t))}\right|\right) \\ &= G(t, m(t)) \end{aligned}$$

for all $t \in J$; and that $m(t_0) = 0$.

Now, we apply Theorem 6.1 to get that $m(t) = 0$ for all $t \in J$. This gives

$$\frac{u_1(t) - k(t, u_1(t))}{f(t, u_1(t))} = \frac{u_2(t) - k(t, u_2(t))}{f(t, u_2(t))}$$

for all $t \in J$. Finally, in view of hypothesis (A_0) we conclude that $u_1(t) = u_2(t)$ on J. This completes the proof. □

6. Existence of Extremal Solutions in a Vector Segment

Sometimes it is desirable to have knowledge of existence of extremal solutions for the HCDE (1.2) in a vector segment defined on J. Therefore, in this section we shall prove the existence of maximal and minimal solutions for HCDE (1.2) between the given upper and lower solutions on $J = [t_0, t_0 + a]$. We use a hybrid fixed point theorem of Dhage [7] in ordered Banach algebras for establishing our results. We need the following preliminaries in the sequel.

A non-empty closed set K in a Banach algebra E is called a **cone** with vertex 0, if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of E. A cone K is called to be **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in E. We introduce an order relation $\leq$ in E as follows. Let $x, y \in E$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\| \cdot \|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in E, then every order-bounded set in E is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [12].

Lemma 6.1. *Let K be a positive cone in a real Banach algebra E and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.*

For any $a, b \in E, a \leq b$, the order interval $[a, b]$ is a set in E given by

$$[a, b] = \{x \in E : a \leq x \leq b\}.$$

Definition 6.1. A mapping $T : [a, b] \to E$ is said to be **nondecreasing** or **monotone increasing** if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in [a, b]$.

We use the following fixed point theorems of Dhage [8] for proving the existence of extremal solutions for the HCDE (1.2) under certain monotonicity conditions.

Theorem 6.1.(Dhage [8]) *Let K be a cone in a Banach algebra E and let $a, b \in E$. Suppose that $A, B : [a, b] \to K$ and $C : [a, b] \to E$ are three nondecreasing operators such that*

(a) *A and C are Lipschitz with the Lipschitz constant α, and β respectively,*

(b) *B is completely continuous, and*

(c) *$Ax\,Bx + Cx \in [a, b]$ for each $x \in [a, b]$.*

Further, if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has a least and a greatest solution in $[a, b]$, whenever $\alpha M + \beta < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone K in it defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0 \text{ for all } t \in J\}. \tag{5.1}$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$. We need the following definitions in the sequel.

Definition 6.2. A function $a \in C(J, \mathbb{R})$ is called a lower solution of the HCDE (1.2) defined on J if it satisfies

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)-k(t,a(t))}{f(t,a(t))}\right] &\leq (Qa)(t),\ t \in J, \\ a(t_0) &\leq x_0. \end{aligned}\right\}$$

Similarly, a function $b \in C(J, \mathbb{R})$ is called an upper solution of the HCDE (1.2) defined on J if it satisfies

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{b(t)-k(t,b(t))}{f(t,b(t))}\right] &\geq (Qb)(t),\ t \in J, \\ b(t_0) &\geq x_0. \end{aligned}\right\}$$

A solution to the HCDE (1.2) is a lower as well as an upper solution for the HCDE (1.2) defined on J and vice versa.

We consider the following set of assumptions:

(B_1) $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}^+$.

(B_2) The HCDE (1.2) has a lower solution a and an upper solution b defined on J with $a \leq b$.

(B_3) The function $x \mapsto \dfrac{x-k(t,x)}{f(t,x)}$ is increasing in the interval $\big[\min\limits_{t\in J} a(t), \max\limits_{t\in J} b(t)\big]$ almost everywhere for $t \in J$.

(B_4) The functions $f(t,x)$ and $k(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.

(B_5) The causal operator Q is nondecreasing on E.

(B_6) There exists a function $k \in L^1(J, \mathbb{R}_+)$ such that

$$(Qb)(t) \leq k(t)$$

for all $t \in J$.

Theorem 6.2. *Suppose that the assumptions* (A_1) *and* (B_1) *through* (B_6) *hold. Further, if*

$$L_1\left(\left|\frac{x_0-k(t_0,x_0)}{f(t_0,x_0)}\right| + \|k\|_{L^1}\right) + L_2 < 1, \tag{5.2}$$

then the HCDE (*1.2*) *has a minimal and a maximal solution in* $[a,b]$ *defined on* J*.*

Proof. Now, the HCDE (1.2) is equivalent to hybrid integral equation (3.5) defined on J. Let $E = C(J, \mathbb{R})$. Define three operators A, B and C on $[a,b]$ by (3.6), (3.7) and (3.8) respectively. Then the integral equation (3.5) is transformed into an operator equation as $Ax(t)\,Bx(t) + Cx(t) = x(t)$ in the ordered Banach algebra E. Notice that hypothesis (B$_1$) implies $A, B : [a, b] \to K$ and $C : [a, b] \to E$. Since the cone K in E is normal, $[a, b]$ is a norm-bounded set in E. Now it is shown, as in the proof of Theorem 3.2, that the operators A and C are Lipschitz with the Lipschitz constant L_1 and L_2. Similarly, B is completely continuous operator on $[a, b]$ into E. Again, the hypothesis (B$_4$) implies that A, B and C are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \leq y$. Then, by hypothesis (B$_4$),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$Cx(t) = k(t, x(t)) \leq k(t, y(t)) = Cy(t)$$

for all $t \in J$. Again,

$$\begin{aligned} Bx(t) &= \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx)(s)\,ds \right) \\ &\leq \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qy)(s)\,ds \right) \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A, B and C are nondecreasing operators on $[a, b]$. Again, Lemma 6.1 and hypothesis (B$_4$) together imply that

$$\begin{aligned} a(t) &\leq k(t, a(t)) + [f(t, a(t))] \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qa)(s)\,ds \right) \\ &\leq k(t, x(t)) + [f(t, x(t))] \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qx)(s)\,ds \right) \\ &\leq k(t, b(t)) + [f(t, b(t))] \left(\frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} + \int_{t_0}^{t} (Qb)(s)\,ds \right) \\ &\leq b(t), \end{aligned}$$

for all $t \in J$ and $x \in [a, b]$. As a result $a(t) \leq Ax(t)Bx(t) + Cx(t) \leq b(t)$ for all $t \in J$ and $x \in [a, b]$. Hence, $Ax\,Bx + Cx \in [a, b]$ for all $x \in [a, b]$. Again,

$$M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\} \leq \left| \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} \right| + \|k\|_{L^1}$$

and so,

$$L_1 M + L_2 \leq L_1 \left(\left| \frac{x_0 - k(t_0, x_0)}{f(t_0, x_0)} \right| + \|k\|_{L^1} \right) + L_2 < 1.$$

Now, we apply Theorem 6.1 to the operator equation $Ax\,Bx + Cx = x$ to yield that the HCDE (1.2) has a minimal and a maximal solution in $[a, b]$ defined on J. This completes the proof. $\square$

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