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The Existence of Fixed Points for Generalized Weak Contractions

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Abstract. In this paper, we study the existence and uniqueness of fixed points for generalized weak contractions under some proper assumptions. Our theorems include the known results of [1]-[6].

1. Introduction

Let (X, d) be a metric space and T be a self-map of X. T is said to be contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \tag{1.1}$$

for all $x, y \in X$. T is called φ-weak contraction if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.2}$$

for all $x, y \in X$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(t) = 0$ iff $t = 0$.

The weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who established a fixed point theorem for such map in Hilbert spaces. Later, Rhoades [2], in 2001, extended the result of [1] to complete metric spaces. The result is as follows.

Theorem 1.1. *Let (X, d) be a complete metric space, and let T be a φ-weak contraction on X, then T has a unique fixed point.*

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However, Boyd and Wong [3], as early as 1969, introduced the notion of Φ-contraction, i.e., there exists an upper semi-continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \le \Phi(d(x, y) \tag{1.3}$$

for all $x, y \in X$. Further, they also showed that if $\Phi(t) < t$ for all $t > 0$ and $\Phi(0) = 0$, then T has a unique fixed point u, and $T^n x \to u$ for each $x \in X$. In fact, it is easy to find from (1.2)

$$d(Tx, Ty) \le (I - \varphi)(d(x, y)), \tag{1.4}$$

where I is identity map. Denote $\Phi = I - \varphi$, then

$$d(Tx, Ty) \le \Phi(d(x, y)), \tag{1.5}$$

here Φ is continuous. But Φ of (1.3) is upper semi-continuous. Therefore Φ-contraction is weaker than φ-weak contraction above.

In 2008, Dutta and Choudhury [4] gave the following the existence theorem of fixed points for φ-weak contractions.

Theorem 1.2.([4, Theorem 2.1]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality:*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \varphi(d(x, y)), x, y \in X, \tag{1.6}$$

where $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2009, Dorić [6] generalized above Theorem 1.2 as follows.

Theorem 1.3.([6, Theorem 2.2]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)), x, y \in X \tag{1.7}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$, $\psi : [0, +\infty) \to [0, +\infty)$ is continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has unique fixed point.

If one takes $\psi(t) = t$ for $t \in [0, +\infty)$ in Theorem 1.3, then it reduces to Corollary 2.2 of Zhang et al.[5].

Remark 1.4. For (1.6) and (1.7), we can write them again in the following

$$\psi(d(Tx, Ty)) \le (\psi - \varphi)(d(x, y)) = \Phi(d(x, y)), x, y \in X, \tag{1.8}$$

and

$$\psi(d(Tx,Ty)) \le (\psi-\varphi)(M(x,y)) = \Phi(M(x,y)), x,y\in X, \tag{1.9}$$

respectively.

Inspired and motivated by these facts, we establish more general definitions as follows.

Definition 1.5. T is said to be (ψ,φ)-weak contraction if there exist $\psi,\varphi : [0,+\infty)\to[0,+\infty)$ with $\psi(0)=\varphi(0)=0$ and $\psi(t),\varphi(t)>0$ for all $t>0$ such that

$$\psi(d(Tx,Ty)) \le \varphi(d(x,y)), x,y\in X. \tag{1.10}$$

Definition 1.6. T is said to be generalized (ψ,φ)-weak contraction if there exist $\psi,\varphi : [0,+\infty)\to[0,+\infty)$ with $\psi(0)=\varphi(0)=0$ and $\psi(t),\varphi(t)>0$ for all $t>0$ such that

$$\psi(d(Tx,Ty)) \le \varphi(M(x,y)), x,y\in X, \tag{1.11}$$

where $M(x,y)=\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\}$.

The main aim of this paper is to study the existence and uniqueness of fixed points for generalized (ψ,φ)-weak contractions in complete metric spaces.

2. Main Results

Theorem 2.1. *Let $T : X\to X$ be a generalized (ψ,φ)-weak contraction with $\psi(t)>\varphi(t)$ and $\lim_{\tau\to t}\inf\psi(\tau) > \lim_{\tau\to t}\sup\varphi(\tau)$ for all $t>0$. Then T has a unique fixed point.*

Proof. Let $x_0\in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by $x_{n+1}=Tx_n$. Without loss of generality, we assume that $x_{n+1}\ne x_n$ for all $n\ge 0$. Then it follows from (1.9) with $x:=x_n, y:=x_{n-1}$ that

$$\begin{aligned}\psi(d(x_{n+1},x_n)) &= \psi(d(Tx_n,Tx_{n-1}))\\ &\le \varphi(M(x_n,x_{n-1})),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}M(x_n,x_{n-1}) &= \max\{d(x_n,x_{n-1}), d(x_{n+1},x_n), \tfrac{1}{2}d(x_{n+1},x_{n-1})\}\\ &= \max\{d(x_n,x_{n-1}), d(x_{n+1},x_n)\}.\end{aligned} \tag{2.2}$$

If $d(x_{n+1},x_n)>d(x_n,x_{n-1})$ for some n, we get from (2.1) and (2.2)

$$0<\psi(d(x_{n+1},x_n)) \le \varphi(d(x_{n+1},x_n)), \tag{2.3}$$

which is a contradiction and so $d(x_{n+1},x_n)\le d(x_n,x_{n-1})$ for each $n\ge 1$. Thus there exists $r\ge 0$ such that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = r. \tag{2.4}$$

And we also obtain from (2.1) that

$$\psi(d(x_{n+1}, x_n)) \le \varphi(d(x_n, x_{n-1})), \tag{2.5}$$

which implies that

$$\begin{aligned} \inf_{i\ge n} \psi(d(x_{i+1}, x_i)) &\le \psi(d(x_{n+1}, x_n)) \\ &\le \varphi(d(x_n, x_{n-1})) \le \sup_{j\ge n} \varphi(d(x_j, x_{j-1})). \end{aligned} \tag{2.6}$$

If $r > 0$, then letting $n \to \infty$ in the inequality (2.6) we get

$$\lim_{n\to\infty} \inf_{i\ge n} \psi(d(x_{i+1}, x_i)) \le \lim_{n\to\infty} \sup_{j\ge n} \varphi(d(x_j, x_{j-1})), \tag{2.7}$$

that means, $\lim_{\tau\to r} \inf \psi(\tau) \le \lim_{\tau\to r} \sup \varphi(\tau)$, which implies that $r = 0$, contradicting our assumption. So $r = 0$, i.e., $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence. If it is not true, there exist $\epsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \ge \epsilon$. This implies that $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$ for all $k \ge 1$. By the triangle inequality, we obtain that

$$\begin{aligned} \epsilon \le d(x_{m(k)}, x_{n(k)}) &\le d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\le \epsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{2.8}$$

Letting $k \to \infty$ in (2.8) we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.9}$$

Since

$$d(x_{m(k)}, x_{n(k)}) - d(x_{n(k)-1}, x_{n(k)}) \le d(x_{m(k)}, x_{n(k)-1}) < \epsilon,$$

then

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{2.10}$$

Similarly, we also obtain that

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.11}$$

Again using Definition 1.2, then

$$\begin{aligned} &\psi(d(x_{m(k)}, x_{n(k)})) \\ &= \psi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\le \varphi(M(x_{m(k)-1}, x_{n(k)-1})), \end{aligned} \tag{2.12}$$

which deduces

$$\inf_{i\ge k} \psi(d(x_{m(i)}, x_{n(i)})) \le \sup_{j\ge k} \varphi(M(x_{m(j)-1}, x_{n(j)-1})), \tag{2.13}$$

where

$$
\begin{aligned}
&M(x_{m(k)-1}, x_{n(k)-1}) \\
&= \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}), \\
&\tfrac{1}{2}[d(x_{m(k)-1}, x_{n(k)}) + d(x_{m(k)}, x_{n(k)-1})]\} \\
&\leq \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}), \\
&\tfrac{1}{2}[2d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)}, x_{m(k)-1})]\} \\
&\leq d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)}).
\end{aligned} \tag{2.14}
$$

It implies that

$$
\begin{aligned}
&d(x_{m(k)-1}, x_{n(k)-1}) \leq M(x_{m(k)-1}, x_{n(k)-1}) \\
&\leq d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)}).
\end{aligned} \tag{2.15}
$$

Letting $k \to \infty$ in (2.15), we have

$$
\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \tag{2.16}
$$

Taking the limit as $k \to \infty$ in (2.13), we have

$$
\lim_{k\to\infty} \inf_{i\geq k} \psi(d(x_{m(i)}, x_{n(i)})) \leq \lim_{k\to\infty} \sup_{j\geq k} \varphi(M(x_{m(j)-1}, x_{n(j)-1})),
$$

which contradicts with the condition of Theorem 2.1 and therefore $\{x_n\}$ is a Cauchy sequence and hence it is convergent. Let $\lim_{n\to\infty} x_n = q$.

Finally we show that q is unique fixed point of T. If $q \neq Tq$, then $d(q, Tq) > 0$. By taking $x = q, y = x_n$ in (1.9), we obtain

$$
\begin{aligned}
\psi(d(Tq, x_{n+1})) &= \psi(d(Tq, Tx_n)) \\
&\leq \varphi(M(q, x_n)),
\end{aligned} \tag{2.17}
$$

which implies that

$$
\inf_{i\geq n} \psi(d(Tq, x_{i+1})) \leq \sup_{j\geq n} \varphi(M(q, x_j)), \tag{2.18}
$$

where

$$
\begin{aligned}
M(q, x_n) &= \max\{d(q, x_n), d(q, Tq), d(x_{n+1}, x_n), \\
&\quad \tfrac{1}{2}[d(q, x_{n+1}) + d(x_n, Tq)]\} \\
&\leq \max\{d(q, x_n), d(q, Tq), d(x_{n+1}, x_n), \\
&\quad \tfrac{1}{2}[2d(q, x_n) + d(x_n, x_{n+1}) + d(q, Tq)]\} \\
&\leq d(q, x_n) + d(x_n, x_{n+1}) + d(q, Tq)]\}.
\end{aligned} \tag{2.19}
$$

Since

$$
d(q, Tq) \leq M(q, x_n) \leq d(q, x_n) + d(x_n, x_{n+1}) + d(q, Tq), \tag{2.20}
$$

then we have

$$
M(q, x_n) \to d(q, Tq)
$$

as $n \to \infty$. It follows from (2.18) that

$$\lim_{n\to\infty} \inf_{i\geq n} \psi(d(Tq, x_{i+1})) \leq \lim_{n\to\infty} \sup_{j\geq n} \varphi(M(q, x_j)), \tag{2.21}$$

which is a contradiction and so $q = Tq$. For uniqueness of fixed point of T. If otherwise, there exists $p \in X$ for $Tp = p \neq q = Tq$. Observe that

$$\begin{aligned} 0 <&\psi(d(q,p)) \\ =&\psi(d(Tq,Tp)) \\ \leq&\varphi(M(q,p)) \\ =&\varphi(\max\{d(q,p), \frac{1}{2}[d(q,Tp)+d(Tq,p)]\}) \\ =&\varphi(d(q,p)), \end{aligned}$$

which is a contradiction. Hence $p = q$. □

Remark 2.2. Theorem 2.1 extends and improves Theorem 2.2 of [6] in the following sense.

1. It is unnecessary that the functions ψ and φ are continuous monotone nondecreasing and lower semi-continuous, respectively.
2. The condition of functions ψ and φ is weaked to

$$\lim_{\tau\to t} \inf \psi(\tau) > \lim_{\tau\to t} \sup \varphi(\tau)$$

for all $t > 0$. That is, functions ψ and φ neither is continuous or lower semi-continuous nor monotone nondecreasing.

Corollary 2.3. *Let $T : X \to X$ be a generalized (ψ, φ)-weak contraction. Where* (a) *φ is an upper semi-continuous function;* (b) *ψ is a lower semi-continuous function;* (c) *$\psi(t) > \varphi(t)$ for all $t > 0$. Then T has a unique fixed point.*

Proof. By (a), (b) and (c), we have

$$\lim_{\tau\to t} \inf \psi(\tau) > \lim_{\tau\to t} \sup \varphi(\tau) \tag{2.22}$$

for all $t > 0$. If (2.22) does not hold, then there exists some $t_0 > 0$ such that

$$\lim_{\tau\to t_0} \inf \psi(\tau) \leq \lim_{\tau\to t_0} \sup \varphi(\tau). \tag{2.23}$$

Using (a) and (b), we have

$$\psi(t_0) \leq \lim_{\tau\to t_0} \inf \psi(\tau) \leq \lim_{\tau\to t_0} \sup \varphi(\tau) \leq \varphi(t_0), \tag{2.24}$$

which contradicts with (c). In the view of Theorem 2.1, we obtain the conclusion of Corollary 2.3. □

Remark 2.4. In Corollary 2.3, the condition of function ψ is weaked to upper semi-continuous from the corresponding condition of Theorem 2.2 of [6].

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