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쌍곡선에서의 재킷 행렬

Jacket Matrix in Hyperbola

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요 약 Jacket 행렬[2][8]은 1984년 이문호 교수에 의해 소개되어 신호처리 및 코딩이론에 사용되는 $J^{\dagger}=\left[J_{ik}^{-1}\right]^{T}$ 인 행렬로서, Galois field F 에서 $J^{\dagger}$ 가 J 의 원소별 역행렬일 때 $JJ^{\dagger}=mI_{m}$ 의 특성을 갖는 $J=[j_{ik}]$ 인 $m\times m$ 정방행렬이다. 본 논문에서는 Jacket 행렬에 의해 고유 값으로 분해될 수 있는 정방행렬 A_{2^n} 을 제안하였다. 특히 A_2 와 그의 확장인 A_3 행렬을 가지고 쌍곡선과 쌍곡면의 성질을 수정하는데 각각 적용할 수 있음을 보였다. 특히 쌍곡선이 n배의 정보량을 갖게 되면 A_2 행렬의 EVD[7]를 이용하여 최종 행렬 A_2^n 을 쉽게 계산할 수 있다. 또한 여기서 제안한 알고리즘을 가지고 컴퓨터 그래픽에서의 응용 프로그램과 수치해석에서도 이용될 수 있음을 보였다.

Abstract Jacket matrices[2][8] which are defined to be $m\times m$ matrices $J^{\dagger}=\left[J_{ik}^{-1}\right]^{T}$ over a Galois field F with the property $JJ^{\dagger}=mI_{m}$, $J^{\dagger}$ is the transpose matrix of element-wise inverse of J, i.e., $J^{\dagger}=\left[J_{ik}^{-1}\right]^{T}$, were introduced by Lee in 1984 and are used for Digital Signal Processing and Coding theory. This paper presents some square matrices A_{2^n} which can be eigenvalue decomposed by Jacket matrices. Specially, A_2 and its extension A_3 can be used for modifying the properties of hyperbola and hyperboloid, respectively. Specially, when the hyperbola has n times transformation, the final matrices A_2^n can be easily calculated by employing the EVD[7] of matrices A_2. The ideas that we will develop here have applications in computer graphics and used in many important numerical algorithms.

Key Words : Eigenvalue decomposition, Diagonalization, Jacket matrix, Center Weighed Hadamard.

Ⅰ. Introduction

The Hadamard transform is an orthogonal matrix with highly practical values for signal sequence transforms and data processing[1]. Jacket matrices[2][8], which are motivated by the center weighted Hadamard matrices[3], is a class of matrices with its inverse being determined by the element-wise inverse of the matrix. Let $J=[j_{ik}]$ be a $m\times m$ matrix whose elements are from multiplicative group of a field *F*. Denote the

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transpose matrix of its element-wise inverses by $J^{\dagger}$ i.e., $J^{\dagger}=\left[J_{ik}^{-1}\right]^{T}$. The matrix J is called Jacket matrix if $JJ^{\dagger}=J^{\dagger}J=mI_m$ where I_m is the identity matrix over *F*. Since the inverse of the Jacket matrix can be calculated easily, it is very helpful to employ this kind of matrix in the signal processing, mobile communication, cryptography and simple Eigenvalue decomposition[7]. In addition, the Jacket matrices are associated with many kinds of matrices, such as unitary matrices and Hermitian matrices which are very important in communication, mathematics and physics.

The conventional eigenvalue decomposition (EVD) is described in [4]-[7]: With the eigenvalues of A_n on the diagonal of a diagonal matrix Λ_n and the corresponding eigenvectors forming the columns of a matrix V_n, if V_n is nonsingular, then the eigenvalue decomposition can be written as $A_n=V_n\Lambda_nV_n^{-1}$. In this paper we present a kind of 2-by-2 matrices A_2, whose EVD based on Jacket matrices can be written as $A_2=J_2\Lambda_2J_2^{-1}$. Employing a recursive relation, the EVD of higher order matrices A_{2^n} based on J_{2^n} can be obtained. The application of A_2 and its extension A_3 in geometry are also investigated.

The paper is organized as follows. Section 2 describes conventional Center Weighted Hadamard matrices. Section 3 and Section 4 discuss eigenvalue decomposition of matrix A_2 and the higher order matrices A_{2^n} based on Jacket matrices, respectively. Section 5 presents the application of the matrix A_2 in geometry. Section 6 extends A_2 to A_3, and discusses the application of the matrix A_3 in three-dimension. Finally, some conclusions are drawn in Section 7.

II. Conventional Center Weighted Hadamard matrices

Let the Hadamard and the center weighted Hadamard matrices of order $N=2^k$ be denoted by $[H]_N$ and $[WH]_N$, respectively. The CWHT (center weighted Hadamard transform) of an $N\times 1$ vector $[f]$ and an $N\times N$ (image) matrix $[g]$ are given by [3].

$$[F]=[WH]_N[f] \qquad (1)$$

$$[G]=[WH]_N[g][WH]_N. \qquad (2)$$

The lowest order WH matrix is size of (4×4) and is defined as follows:

$$[WH]_4 \triangleq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \qquad (3)$$

The inverse of (3) is matrix is $[WH]^{-T}$ as element-wise inverse

$$[WH]_4^{-1}=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad (4)$$

This choice of weighting was dictated, to a large extent, by the requirement of digital hardware simplicity[6]. As with the Hadamard matrix, a recursive relation governs the generation of higher order WH matrices, i.e.,

$$[WH]_N \triangleq [WH]_{N/2}\otimes[H]_2 \qquad (5)$$

where $\otimes$ is the Kronecker product and $[H]_2$ is the lowest order Hadamard matrix given by (1) to (4)[8]:

$$[H]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (6)$$

We now present a fast algorithm for the CWHT which is related to the fast HT (FHT) algorithm[2]-[4],[8]: the FHT can be derived by decomposing $[H]_N$ into a product of k sparse matrices, each having rows/columns with only two nonzero elements. In order to develop a similar algorithm for the CWHT, define a coefficient matrix $[WC]_N$ by

$$[WC]_N \triangleq [H]_N [WH]_N \quad (7)$$

Since $[H]_N^{-1} = 1/N[H]_N$, we have from (7) that

$$[WH]_N = 1/N[H]_N[WC]_N \quad (8)$$

It is demonstrated that $[WC]_N$ is a sparse matrix with at most two nonzero elements per row and column. Therefore, the fast CWHT (FCWHT) is simply the FHT followed by a sparse matrix $1/N[WC]_N$.
To show the sparseness of $[WC]_N$ we start by computing the lowest order, i.e. $[WC]_4$.

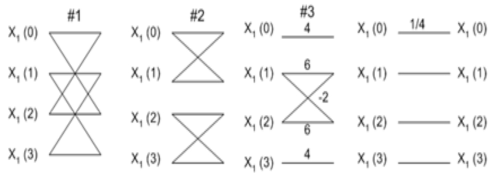


그림 1. 고속 CWHT 플로우 그래프, N=4
Fig. 1. The Fast CWHT flow graph, N=4.

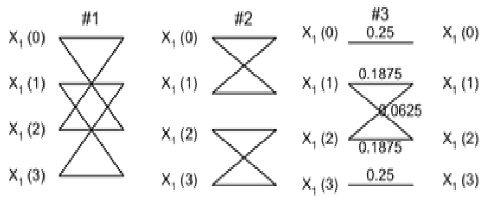


그림 2. 고속 역 CWHT 플로우 그래프, N=4
Fig. 2. Fast inverse CWHT flow graph, N=4.

From (7), we have

$$[WC]_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad (9)$$

Clearly $[WC]_4$ is sparse. Using the expansion properties of the Hadamard and weighted Hadamard matrices, (7) can be written as

$$\begin{aligned} [WC]_N &= ([H]_{N/2} \otimes [H]_2)([WH]_{N/2} \otimes [H]_2) \\ &= ([H]_{N/2}[WH]_{N/2}) \otimes ([H]_2[H]_2) \\ &= [WC]_{N/2} \otimes (2[I]_2) \end{aligned} \quad (10)$$

where $[I]_2$ is the 2×2 identity matrix. Since $[WC]_4$ is symmetric and has at most two non-zero elements in each row, from (10) it clearly follows that the same is true for $[WC]_8$, and hence, for any $[WC]_N$, $N = 2^k$, $k = 2,3,4,\cdots$. Fig.1 shows a flow graph of the 4-point FCWHT algorithm. From this figure, it is clear that the first three iterations of the algorithm are those of the FHT. These are followed by the operation of the $[WC]_4$. The $N = 2^k$ point FCWHT algorithm requires $kN + N/2$ real additions and $1.5N$ real multiplications in contrast with the N-point FHT which requires kN real additions.
The inverse FCWHT may be formulated in a similar fashion as the FCWHT. First we note that $[WC]_N^{-1} = [WC]_{N/2}^{-1} \otimes 1/2[I]_2$. Obviously,

$$\begin{aligned} [WC]_N[WC]_N^{-1} &= ([WC]_{N/2} \otimes 2[I]_2)([WC]_{N/2}^{-1} \otimes 1/2[I]_2) \\ &= ([WC]_{N/2}[WC]_{N/2}^{-1}) \otimes ([I]_2[I]_2) = [I]_N \end{aligned} \quad (11)$$

Equation (11) can be verified by multiplying $[WC]_N^{-1}$ in (11) by the expression for $[WC]_N$ given in (10). From (11) and the sparseness and symmetry of $[WC]_N^{-1}$, it follows that $[WC]_N^{-1}$ is also symmetric and sparse. Furthermore, using (8), we have

$$[WH]_N^{-1} = N[WC]_N^{-1}[H]_N^{-1} \qquad (12)$$

But $[WC]_N$ and $[H]_N^{-1} = 1/N[H]_N$ are both symmetric with a symmetric products. Thus

$$[WH]_N^{-1} = [H]_N[WC]_N^{-1} \qquad (13)$$

Equation (13), with the exception of scale factor 1/*N*, is of the same form as (18). This signifies a fast algorithm for the inverse of $[WH]_N$ composed of FHT followed by $[WC]_N$. Fig.2 and Fig.3 show a flow graph of the inverse FCWHT for N = 4 and a signal flowchart of the FCWHT, respectively.

Examples: The simple recursive relationship in (8) and (13) can now be used to formulate a sparse matrix decomposition of $[WH]_N$ and $[WC]_N^{-1}$. As an example for N = 8, $[WC]_8$ can be represented as

$$\begin{aligned}[WC]_8 &= \frac{1}{8}([H]_4 \otimes [H]_2)([WC]_4 \otimes 2[I]_2) \\ &= \frac{1}{8}([H]_4[WC]_4 \otimes 2[H]_2 \\ &= \frac{1}{8}([WC]_4[H]_4 \otimes 2[H]_2, \end{aligned} \qquad (14)$$

i.e., $[WC]_8$,

$$[WC]_8 = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes 2\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -2 & -2 & 2 & 2 & -1 & -1 \\ 1 & -1 & -2 & 2 & 2 & -2 & -1 & 1 \\ 1 & 1 & 2 & 2 & -2 & -2 & -1 & -1 \\ 1 & -1 & 2 & -2 & -2 & 2 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \qquad (15)$$

In a similar manner as the FCWHT, we note that

$$\begin{aligned}[WH]_8^{-1} &= ([H]_4 \otimes [H]_2)([WC]_4^{-1} \otimes 1/2[I]_2) \\ &= ([H]_4[WC]_4^{-1}) \otimes 1/2[H]_2 \end{aligned} \qquad (16)$$

Therefore, (16) becomes

$$\begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.1875 & 0.0625 & 0 \\ 0 & 0.0625 & 0.1875 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 & -1 \\ 1 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1 & -1 \\ 1 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \qquad (17)$$

The symmetrical matrix decomposition of the $[WH]_N$ and $[WC]_N^{-1}$ often assume the form of the Kronecker products of lowest order Hadamard matrices and successively lower order weighted Hadamard matrices. Using the algebra of Kronecker products, (8) and (10), it is evident that

$$\begin{aligned}[WH]_N[WH]_N^{-1} &= 1/N(\{[H]_{N/2}[WC]_{N/2} \otimes 2[H]_2\}.\{[H]_{N/2}[WC]_{N/2}^{-1} \otimes 1/2[H]_2\}) \\ &= ([WH]_{N/2} \otimes [H]_2)([WH]_{N/2}^{-1} \otimes 1/2[H]_2) \\ &= ([WH]_{N/2}[WH]_{N/2}^{-1}) \otimes 1/2[H]_2[H]_2 \\ &= [I]_N \end{aligned} \qquad (18)$$

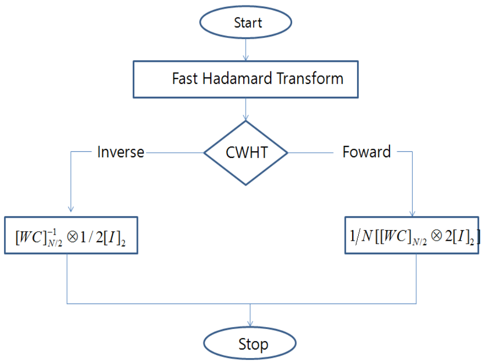


그림 3. 고속 CWHT 플로우 차트
Fig. 3. The Fast CWHT flow chart.

Ⅲ. The EVD of matrix A_2 based on Jacket matrices

Diagonal matrices play an important role in many applications, because, in many respects, they represent the simplest kinds of linear operators[5][6]. Here, the primary purpose of eigenvalue decomposing the matrix A_2 is just to find its similar diagonal matrix.

1. Diagonalizing matrix A_2 based on asymmetrical Jacket matrices

From (3), we know W_2 is an asymmetrical Jacket matrix, and then we give $J_2=\begin{bmatrix}1 & -w\\ 1 & w\end{bmatrix}$. Suppose one kind of 2-by-2 square matrices with the pattern $A_2=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ which can be diagonalized by asymmetrical Jacket matrices, then we have

$$J_2A_2J_2^{-1}=\frac{1}{2}\begin{bmatrix}1 & -w\\ 1 & w\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 1\\ -\frac{1}{w} & \frac{1}{w}\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}a+d-cw-\frac{b}{w} & a-d-cw+\frac{b}{w}\\ a-d+cw-\frac{b}{w} & a+d+cw+\frac{b}{w}\end{bmatrix} \quad (19)$$

The result must be a diagonal matrix, then the entries should satisfy

$$\begin{cases} a+d-cw-\frac{b}{w}\neq 0\\ a+d+cw+\frac{b}{w}\neq 0\\ a-d-cw+\frac{b}{w}=0\\ a-d+cw-\frac{b}{w}=0\end{cases} \quad (20)$$

finally, we have

$$\begin{cases} a=d\\ b=cw^2\\ \det(A)=|a^2-c^2w^2|\neq 0\end{cases} \quad (21)$$

Based on the analysis above, we can fixed the form of the 2-by-2 nonsingular basic matrix as $A_2=\begin{bmatrix}a & cw^2\\ c & a\end{bmatrix}$, where a, c can be any complex number on the premise of $\det(A)\neq 0$.
Simplifying (19) as

$$J_2A_2J_2^{-1}=\frac{1}{2}\begin{bmatrix}1 & -w\\ 1 & w\end{bmatrix}\begin{bmatrix}a & cw^2\\ c & a\end{bmatrix}\begin{bmatrix}1 & 1\\ -\frac{1}{w} & \frac{1}{w}\end{bmatrix}$$

$$=\begin{bmatrix}a-cw & 0\\ 0 & a+cw\end{bmatrix} \quad (22)$$

then the EVD of A_2 can be written as

$$A_2=J_2^{-1}\Lambda_2J_2$$

$$=\frac{1}{2}\begin{bmatrix}1 & 1\\ -\frac{1}{w} & \frac{1}{w}\end{bmatrix}\begin{bmatrix}a-cw & 0\\ 0 & a+cw\end{bmatrix}\begin{bmatrix}1 & -w\\ 1 & w\end{bmatrix} \quad (23)$$

The entries on the diagonal of the diagonal matrix Λ_2 are the eigenvalues of A_2, and the columns of the matrix J_2^{-1} are the corresponding eigenvectors.

Example 1 : Suppose a nonsingular real matrix $A_2 = \begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix}$, Obviously, $a = 3,\ c = 2,\ w^2 = 4$. Taking $w = 2$, we construct the Jacket matrix as , and easily get the inverse matrix $J_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then the EVD is

$$A_2 = J_2 A_2 J_2^{-1}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$$

Example 2 : Given a nonsingular complex matrix $A_2 = \begin{bmatrix} 2i & -i \\ i & 2i \end{bmatrix}$, take $w = \sqrt{\frac{-i}{i}} = i$, then $J_2 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$. The EVD can be written as

$$A_2 = J_2^{-1}\Lambda J_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} 2i+1 & 0 \\ 0 & 2i-1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

2. Diagonalizing matrix A_2 based on symmetrical Jacket matrices

The 2-by-2 symmetrical Jacket matrices' pattern is proposed as follow[7]:

$$J_2 = \begin{bmatrix} x_1 & \pm\sqrt{x_1 x_2} \\ \pm\sqrt{x_1 x_2} & -x_2 \end{bmatrix}$$

where the entries of this matrix are non zero, and

$$J_2^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{x_1} & \frac{1}{\pm\sqrt{x_1 x_2}} \\ \frac{1}{\pm\sqrt{x_1 x_2}} & -\frac{1}{x_2} \end{bmatrix}.$$

Employing the same analytic method as (19)-(22), the matrix $A_2 = \begin{bmatrix} a & cw^2 \\ c & a \end{bmatrix}$ can also be eigenvalue decomposed by symmetrical Jacket matrix:

$$A_2 = J_2^{-1}\Lambda J_2 = \frac{1}{2}\begin{bmatrix} \frac{1}{x_1} & \frac{1}{x_1 w} \\ \frac{1}{x_1 w} & \frac{1}{x_1 w^2} \end{bmatrix}\begin{bmatrix} a+cw & 0 \\ 0 & a-cw \end{bmatrix}\begin{bmatrix} x_1 & x_1 w \\ x_1 w & -x_1 w^2 \end{bmatrix} \quad (23)$$

Example 3 : The matrix A_2 in Example 3.1 can also be eigenvalue decomposed by symmetrical Jacket matrices. It's easy to know $w^2 = \frac{b}{c} = \frac{8}{2} = 4$. We get $w=2$ and assume $x_1 = 1$, from (22) we have:

$$A_2 = J_2^{-1}\Lambda_2 J_2 = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$$

Obviously, the nonsingular matrices A_2 with the same entries on its diagonal can be eigenvalue decomposed by symmetric and A_2 asymmetric Jacket matrices.

Ⅳ. The EVD of higher order matrices A_{2^n} based on Jacket matrices

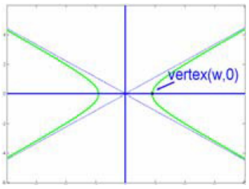
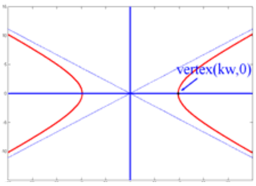
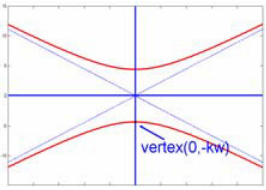
Then the EVD of A_{2^n} based on J_{2^n} is given as follow:

$$A_{2^n} = J_{2^n}^{-1}\Lambda_{2^n} J_{2^n} \quad (24)$$

where $A_{2^n} = A_{2^{n-1}} \otimes A_2 = A_2^{\otimes n}$, $J_{2^n} = J_{2^{n-1}} \otimes J_2 = J_2^{\otimes n}$ and $\Lambda_{2^n} = \Lambda_{2^{n-1}} \otimes \Lambda_2 = \Lambda_2^{\otimes n}$.

Proof : We use induction on the index n, when n=1, it is clearly true $A_{2^1} = J_{2^1}^{-1}\Lambda_{2^1} J_{2^1}$. Assume the hypothesis is true for n and then show it must therefore hold for *n+1*. Following the hypothesis we have:

표 1. Original hyperbola 의 성질과 그의 영상
Table1. The properties of the original hyperbola and its images.

	Original hyperbola	Image hyperbola(I)	Image hyperbola(II)
Variable	(x, y)	$\begin{bmatrix} x' \\ y' \end{bmatrix} = A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & cw^2 \\ c & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $(a^2 - c^2w^2 > 0)$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & cw^2 \\ c & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $(a^2 - c^2w^2 < 0)$
Equation	$\frac{x^2}{w^2} - y^2 = 1$	$\frac{x'^2}{(wk)^2} - \frac{y'^2}{k^2} = 1$	$\frac{y'^2}{k^2} - \frac{x'^2}{(wk)^2} = 1$
Semimajor x-axis	w	kw	k
Semimajor y-axis	1	k	wk
Directrix	$x = \pm \frac{w^2}{\sqrt{w^2+1}}$	$x = \pm \frac{kw^2}{\sqrt{w^2+1}}$	$y = \pm \frac{kw^2}{\sqrt{w^2+1}}$
Vertex	$(\pm w, 0)$	$(\pm kw, 0)$	$(0, \pm w)$
Focus	$F(\pm\sqrt{w^2+1}, 0)$	$F(\pm k\sqrt{w^2+1}, 0)$	$F(0, \pm k\sqrt{w^2+1})$
asymptotes	$y = \pm \frac{1}{w}x$	$y = \pm \frac{1}{w}x$	$y = \pm \frac{1}{w}x$
Eccentricity	$e = \frac{\sqrt{w^2+1}}{w}$	$e = \frac{\sqrt{w^2+1}}{w}$	$e = \sqrt{w^2+1}$
Curve			

$A_{2^n} = J_{2^n}^{-1}\Lambda_{2^n}J_{2^n}$, and then

$$\begin{aligned} A_{2^{n+1}} &= A_{2^n} \otimes A_{2^1} \\ &= (J_{2^n}^{-1}\Lambda_{2^n}J_{2^n}) \otimes (J_{2^1}^{-1}\Lambda_{2^1}J_{2^1}) \\ &= (J_{2^n}^{-1} \otimes J_{2^1}^{-1})(\Lambda_{2^n} \otimes \Lambda_{2^1})(J_{2^n} \otimes J_{2^1}) \\ &= (J_{2^n} \otimes J_{2^1})^{-1}(\Lambda_{2^n} \otimes \Lambda_{2^1})(J_{2^n} \otimes J_{2^1}) \\ &= J_{2^{n+1}}^{-1}\Lambda_{2^{n+1}}J_{2^{n+1}} \end{aligned}$$

Obviously, when the exponent is $n+1$, it still satisfies (24). The proof of (24) is finished. ■

Example 4 : Based on Example 3.1, from (24) the EVD of A_4 can be written as

$$A_{2^2} = J_4^{-1}\Lambda_4 J_4 = (J_2 \otimes J_2)^{-1}(\Lambda_2 \otimes \Lambda_2)(J_2 \otimes J_2)$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix}\begin{bmatrix} 1 & -2 & -2 & 4 \\ 1 & 2 & -2 & -4 \\ 1 & -2 & 2 & -4 \\ 1 & 2 & 2 & 4 \end{bmatrix}.$$

V. The application of the matrix A_2 in geometry

Let A_2 be the transformation matrix that maps the points (x, y) in 2-space into the points (x', y') in 2-space. The relationship can be expressed in the following way:

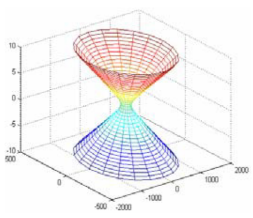
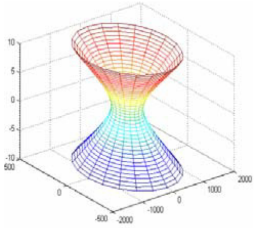
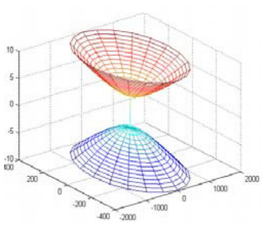
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & cw^2 \\ c & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax & cw^2y \\ cx & ay \end{bmatrix}, \qquad (25)$$

then we can have the equation as

$$\frac{x'^2}{w^2} - y'^2 = (a^2 - c^2w^2)(\frac{x^2}{w^2} - y^2) \qquad (26)$$

assume the original points (x, y) satisfy the hyperbola expression $\frac{x^2}{w^2} - y^2 = r^2$ where r is a nonzero real number, we'll take $r = 1$ for convenient discussion. After multiplying the transformation matrix

표 2. Original hyperboloids 의 성질과 영상
Table 2. The properties of the original hyperboloids and its images.

	Original hyperbola	Image hyperbola(Ⅰ)	Image hyperbola(Ⅱ)
Variable	(x, y, z)	$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & 0 & cw^2 \\ 0 & v\sqrt{a^2-c^2w^2} & 0 \\ c & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $(a^2-c^2w^2>0)$	$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & 0 & cw^2 \\ 0 & v\sqrt{a^2-c^2w^2} & 0 \\ c & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $(a^2-c^2w^2<0)$
Equation	$\frac{x^2}{w^2} + y^2 - z^2 = 1$	$\frac{x'^2}{(kw)^2} + \frac{y'^2}{k^2} - \frac{z'^2}{k^2} = 1$	$\frac{x'^2}{(kw)^2} + \frac{y'^2}{k^2} - \frac{z'^2}{k^2} = -1$
Vertex	$(\pm w,0,0),(0,\pm 1,0)$	$(\pm kw,0,0),(0,\pm k,0)$	$(0,0,\pm k)$
Curve			

A_2 , based on (23), the image points (x', y') satisfy a new hyperbola equation:

$$\frac{x'^2}{w^2} - y'^2 = a^2 - c^2w^2 , \qquad (27)$$

case1: $a^2 - c^2w^2 > 0$, we have

$$\frac{x'^2}{w^2} - y'^2 = k^2(\frac{x^2}{w^2} - y^2) = k^2 , \qquad (28)$$

where $k = \sqrt{|\det(A_2)|} = \sqrt{|a^2 - c^2w^2|}$.

case2: $a^2 - c^2w^2 < 0$, then

$$\frac{x'^2}{w^2} - y'^2 = -k^2(\frac{x^2}{w^2} - y^2) = -k^2 . \qquad (29)$$

Comparing the original equation $\frac{x^2}{w^2} - y^2 = 1$ with (28) and (29), we note that after transformation, some properties of the original hyperbola are changed. More details about the properties are presented in Table 1.

Otherwise, if a hyperbola has n times transformation by A_2 , it can be expressed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A_2^n \begin{bmatrix} x \\ y \end{bmatrix} \qquad (30)$$

Based on (22), A_2^n can be easily calculated:

$$A_2^n = J_2^{-1} \Lambda_2^n J_2 \qquad (31)$$

then we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = J_2^{-1} \Lambda_2^n J_2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -\frac{1}{w} & \frac{1}{w} \end{bmatrix} \begin{bmatrix} (a-cw)^n & 0 \\ 0 & (a+cw)^n \end{bmatrix} \begin{bmatrix} 1 & -w \\ 1 & w \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (32)$$

Ⅵ. A simple extension from two-dimension A_2 to three-dimension A_3

Based on the fixed form of A_2 , the pattern of A_3 can be constructed as

$$A_3 = \begin{bmatrix} a & 0 & cw^2 \\ 0 & v\sqrt{a^2 - c^2w^2} & 0 \\ c & 0 & a \end{bmatrix},$$

the variables have the same conditions with that of A_2, and $v \neq 0$.

In three-dimension, using A_3 as the transformation matrix, we can map the points into the (x, y, z) points (x', y', z'). The relationship can be expressed in the following way:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & 0 & cw^2 \\ 0 & v\sqrt{a^2 - c^2w^2} & 0 \\ c & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad (33)$$

then we have

$$\frac{x'}{w^2} + y'^2 - z'^2 = (a^2 - c^2w^2)(\frac{x}{w^2} + v^2y^2 - z^2), \quad (34)$$

Employing the similar analysis with two-dimension (25), we have the original hyperboloid equation $\frac{x^2}{w^2} + v^2y^2 - z^2 = r^2$ where $r \neq 0$, we'll take $r = v = 1$ for convenient discussion.

Using A_3 as transformation matrix, the equation can be changed as:

$$\frac{x'^2}{w^2} + y'^2 - z'^2 = a^2 - c^2w^2 \quad (35)$$

case1: $a^2 - c^2w^2 > 0$, we have

$$\frac{x'^2}{w^2} + y'^2 - z'^2 = k^2, \quad (36)$$

where $k = \sqrt{|\det(A_2)|} = \sqrt{|a^2 - c^2w^2|}$. In this case the vertexes of the original hyperboloid can be changed.

case2: $a^2 - c^2w^2 < 0$, then

$$\frac{x'^2}{w^2} + y'^2 - z'^2 = -k^2. \quad (37)$$

In this case, the vertexes also can be changed, and if the original hyperboloid is a hyperboloid of one sheet, after the transformation we can change it to a hyperboloid of two sheets, vice versa. More details about the properties of these hyperboloids are presented in Table 2.

Ⅶ. Conclusion

This paper has presented a kind of 2-by-2 matrices A_2 which can be eigenvalue decomposed by Jacket matrices. With the aid of the recursive relation, we can extend the EVD to the high order matrices A_{2^n}. The applications of two-dimension A_2 and three-dimension A_3 in geometry have also been discussed. We can modify the properties of hyperbola and hyperboloid by multiplying the transformation matrices A_2 and A_3 respectively. Specially, when the hyperbola has n times transformation, the final matrices A_2^n can be easily calculated by employing the EVD of matrices A_2.

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