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Posner's First Theorem for $*$-ideals in Prime Rings with Involution

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Abstract. Posner's first theorem states that if R is a prime ring of characteristic different from two, d_1 and d_2 are derivations on R such that the iterate d_1d_2 is also a derivation of R, then at least one of d_1, d_2 is zero. In the present paper we extend this result to $*$-prime rings of characteristic different from two.

1. Introduction

Throughout the paper, R will represent an associative ring. R is called a prime ring if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xRx = \{0\}$ implies $x = 0$. Given an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ of R into itself is called an involution on R if it satisfies the conditions; (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution '$*$' is called a ring with involution or a $*$-ring. A ring R with involution '$*$' is said to be $*$-prime if $aRb = aRb^* = \{0\}$, where $a, b \in R$ (equivalently $aRb = a^*Rb = \{0\}$, where $a, b \in R$) implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution '$*$' is $*$-prime but the converse is not true in general. Of course, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. An ideal I of R is called a $*$-ideal of R if $I^* = I$. Let R be a $*$-prime ring, $a \in R$ and $aRa = \{0\}$. This implies that $aRaRa^* = \{0\}$ also. Now $*$-primeness of R insures that $a = 0$ or $aRa^* = \{0\}$. $aRa^* = \{0\}$ together with $aRa = \{0\}$ gives us $a = 0$. Thus we conclude that every $*$-prime ring is a semiprime ring.

An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let I be a nonzero ideal of R. Then an additive mapping $d : I \longrightarrow R$ is called a derivation from I to R if $d(xy) = d(x)y +$

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$xd(y)$ holds for all $x, y \in I$. In the year 1957, E. C. Posner initiated the study of derivations in rings and proved two very striking theorems. These results have been generalized by several authors in different directions see [1,4,5] for reference where further references can be found. Posner's first theorem [8, Theorem 1] states that if R is a prime ring of characteristic not 2 and the iterate of two derivations is also a derivation, then at least one of them is zero. In this paper we extend this result to $*$-prime rings of characteristic different from 2.

2. Preliminary Results

We begin with the following lemmas which are essential for developing the proof of our main result.

Lemma 2.1 *If R is a $*$-prime ring of characteristic different from 2, then R is 2-torsion free.*

Proof. Suppose that $x \in R$ such that $2x = 0$. This implies that $2xrs = 0$ for all $r, s \in R$ i.e., $xR(2s) = \{0\}$ for all $s \in R$. Since characteristic of R is different from 2 and $R \neq \{0\}$, this provides us a nonzero element $l \in R$ such that $2l \neq 0$. Now we conclude that $xR(2l) = \{0\} = xR(2l)^*$. Finally $*$-primeness of R provides us $x = 0$ and hence R is 2-torsion free. □

Lemma 2.2 *Let R be a $*$-prime ring and I a nonzero $*$-ideal of R. If $d : I \longrightarrow R$ is a derivation such that d commutes with '$*$'. If a is an element of R and $ad(x) = 0$ (resp., $d(x)a = 0$) for all $x \in I$, then either $a = 0$ or $d = 0$.*

Proof. Replacing x by xy, where $y \in I$ in the relation $ad(x) = 0$, we obtain that $ad(x)y + axd(y) = 0$, i.e., $axd(y) = 0$ for all $x, y \in I$. Replacing x by xs where $s \in R$ in the latter relation, we arrive at $axsd(y) = 0$, i.e., $axRd(y) = \{0\}$ for all $x, y \in I$. Since d commutes with '$*$' and I is a $*$-ideal, we obtain that $axRd(y) = \{0\} = axR\{d(y)\}^*$ for all $x, y \in I$. Now $*$-primeness of R provides us $d = 0$ or $ax = 0$ for all $x \in I$. Putting tx where $t \in R$ for x in the latter relation, we arrive at $atx = 0$, i.e., $aRx = \{0\}$ for all $x \in I$. Since I is a $*$-ideal of R, we also have $aRx = aRx^* = \{0\}$. Now $*$-primeness of R and $I \neq \{0\}$ imply that $a = 0$. Similarly we can also show that $d(x)a = 0$ for all $x \in I$ implies that $a = 0$ or $d = 0$. □

3. Main Results

The study of derivation in ring was initiated by E. C. Posner [8] in the year 1957, who proved two very striking theorems. Posner' first theorem deals with the composition of two derivations on a ring and states that if R is a prime ring of characteristic different from 2 and d_1, d_2 are derivations of R such that the iterate d_1d_2 is also a derivation, then at least one of d_1 and d_2 is zero. We extend Posner's first theorem in the setting of $*$-prime rings having characteristic different from 2 and establish the following:

Theorem 3.1 *Let R be a $*$-prime ring of characteristic not 2, I a nonzero $*$-ideal and $d_1, d_2 : I \longrightarrow R$ are derivations such that the iterate $d_1d_2 : I \longrightarrow R$ is also a derivation. If at least one of d_1 and d_2 commutes with '$*$', then $d_1 = 0$ or $d_2 = 0$.*

Proof. We divide the proof in following two cases:

Case I : Let us suppose that d_1 commutes with '$*$'. Since the map $d_1d_2 : I \longrightarrow R$ is a derivation, it is obvious that $d_2(I) \subseteq I$ and $d_1d_2(xy) = d_1d_2(x)y + xd_1d_2(y)$ for all $x, y \in I$. As $d_1, d_2 : I \longrightarrow R$ are derivations, we obtain that

$$\begin{aligned} d_1d_2(xy) &= d_1(d_2(xy)) \\ &= d_1(d_2(x)y + xd_2(y)) \\ &= d_1d_2(x)y + d_2(x)d_1(y) + d_1(x)d_2(y) + xd_1d_2(y). \end{aligned}$$

By above relations we conclude that

$$d_2(x)d_1(y) + d_1(x)d_2(y) = 0 \text{ for all } x, y \in I. \tag{3.1}$$

Now replacing x by $xd_2(z)$, where $z \in I$ in the relation (3.1) we obtain that $d_2(xd_2(z))d_1(y) + d_1(xd_2(z))d_2(y) = 0$ for all $x, y, z \in I$. This gives us $d_2(x)d_2(z)d_1(y) + xd_2^2(z)d_1(y) + d_1(x)d_2(z)d_2(y) + xd_1d_2(z)d_2(y) = 0$. In view of equation (3.1) and using the fact that $d_2(I) \subseteq I$, we find that $(d_2(d_2(z))d_1(y) + d_1(d_2(z))d_2(y)) = 0$. Hence we arrive at

$$d_2(x)d_2(z)d_1(y) + d_1(x)d_2(z)d_2(y) = 0 \text{ for all } x, y, z \in I. \tag{3.2}$$

Using the relation (3.1) and Lemma 2.1, the relation (3.2) reduces to $d_1(x)d_2(z)d_2(y) = 0$ for all $x, y, z \in I$. Now Lemma 2.2 provides us either $d_1 = 0$ or $d_2(z)d_2(y) = 0$ for all $y, z \in I$. If the first case holds then nothing to do, if not we have $d_2(z)d_2(y) = 0$ for all $y, z \in I$. Replacing y by yz in the latter relation and using the same argument as above again we arrive at $d_2(z)yd_2(z) = 0$ for all $y, z \in I$. Replacing y by sy where $s \in R$ in the latter relation we arrive at $d_2(z)Ryd_2(z) = \{0\}$ i.e., $yd_2(z)Ryd_2(z) = \{0\}$ for all $y, z \in I$. Since R is a $*$-prime ring, it is semiprime also and hence we obtain that $yd_2(z) = 0$ for all $y, z \in I$. Replacing y by yt where $t \in R$ in the latter relation we arrive at $ytd_2(z) = 0$ i.e., $yRd_2(z) = \{0\}$ for all $y, z \in I$. But we know that I is a $*$-ideal of R. Therefore we also get $y^*Rd_2(z) = \{0\}$ for all $y, z \in I$. Finally $*$-primeness of R and $I \neq \{0\}$ imply that $d_2 = 0$.

Case II : Let us suppose that d_2 commutes with '$*$'. From Case I, we have $d_1(x)d_2(z)d_2(y) = 0$ for all $x, y, z \in I$. Now Lemma 2.2 provides us either $d_2 = 0$ or $d_1(x)d_2(z) = 0$ for all $x, z \in I$. If first case holds then nothing to do, if not we have $d_1(x)d_2(z) = 0$ for all $x, z \in I$. Again using Lemma 2.2 we conclude that either $d_1 = 0$ or $d_2 = 0$. □

The following example shows that the hypothesis of $*$-primeness is crucial in the above theorem.

Example 3.1 Let $R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers. Consider the map

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^*$$

of R into itself such that

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^* = \begin{pmatrix} z & 0 \\ -y & x \end{pmatrix}.$$

It is easy to verify that '$*$' is an involution of the ring R, where characteristic of R is different from 2. Further if we set $I = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y, 0 \in \mathbb{Z} \right\}$, then I is a nonzero $*$-ideal of R. Now consider the maps $d_1, d_2, : I \longrightarrow R$ defined by

$$d_1 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, d_2 \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -y & 0 \end{pmatrix}.$$

Then it is obvious to observe that d_1 and d_2 are derivations and '$*$' commutes with d_1. Further it can be also shown that the iterate $d_1 d_2 : I \longrightarrow R$ is a derivation and R is not a $*$-prime ring. However neither $d_1 = 0$ nor $d_2 = 0$.

The following example shows that the hypothesis of "characteristic different from 2" is crucial in the above theorem.

Example 3.2 Suppose that $R = \mathbb{Z}_2 < x > \times \mathbb{Z}_2 < x >$, where $\mathbb{Z}_2 < x >$ is the polynomial ring over $\mathbb{Z}_2$. Let us consider the map $(f(x), g(x)) \mapsto (f(x), g(x))^*$ of R into itself such that $(f(x), g(x))^* = (g(x), f(x))$. It is easy to check that '$*$' is an involution of R, known as the exchange involution denoted by $*_{ex}$ and R is a $*_{ex}$-prime ring. Further assume that $I =< x^2 >$ is the ideal of $\mathbb{Z}_2 < x >$ generated by $x^2 \in \mathbb{Z}_2 < x >$. Then it can be easily shown that $\mathfrak{I} = I \times I$ is a nonzero $*_{ex}$-ideal of R. Next consider $D_1, D_2 : \mathfrak{I} \longrightarrow R$ such that $D_1(f(x), g(x)) = (d(f(x)), d(g(x)))$ and $D_2(f(x), g(x)) = (d(f(x)), 0)$, where d is the usual differentiation in $\mathbb{Z}_2 < x >$. It is obvious to see that D_1, D_2 and $D_1 D_2 : \mathfrak{I} \longrightarrow R$ are derivations. Moreover, R is a ring of characteristic 2 and $D_1 *_{ex} = *_{ex} D_1$. However $D_1 \neq 0$ and $D_2 \neq 0$.

Now taking $I = R$ in the above theorem we obtain the following:

Corollary 3.1 *Let R be a $*$-prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation of R. If at least one of d_1 and d_2 commutes with '$*$', then $d_1 = 0$ or $d_2 = 0$.*

Now using the above theorem we can obtain Posner's first theorem.

Corollary 3.2 ([8], Theorem 1) *Let R be a prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation, then one at least of d_1, d_2 is zero.*

Proof. Since R is a prime ring of characteristic not 2, consider $\mathcal{R} = R \times R^o$, which is clearly a $*_{ex}$-prime ring of characteristic not 2. Set $I = \mathcal{R}$, which is a nonzero $*_{ex}$-ideal of $\mathcal{R}$. Now define $D_1, D_2 : I \longrightarrow \mathcal{R}$ by $D_1(x, y) = (d_1(x), d_1(y))$ and $D_2(x, y) = (d_2(x), d_2(y))$. Using hypothesis it can be easily seen that $D_1, D_2 : I \longrightarrow \mathcal{R}$ are derivations and the iterate $D_1D_2 : I \longrightarrow \mathcal{R}$ is also a derivation. Moreover $D_1 *_{ex} = *_{ex} D_1$. In view of the Theorem 3.1 we deduce that either $D_1 = 0$ or $D_2 = 0$, in turn we obtain that either $d_1 = 0$ or $d_2 = 0$. □

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