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Subalgebras and Ideals of BCK/BCI-Algebras in the Framework of the Hesitant Intersection

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Abstract. Using the hesitant intersection ($\Cap$), the notions of $\Cap$-hesitant fuzzy subalgebras, $\Cap$-hesitant fuzzy ideals and $\Cap$-hesitant fuzzy p-ideals are introduced,and their relations and related properties are investigated. Conditions for a $\Cap$-hesitant fuzzy ideal to be a $\Cap$-hesitant fuzzy p-ideal are provided. The extension property for $\Cap$-hesitant fuzzy p-ideals is established.

1. Introduction

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. The concept of hesitant fuzzy sets, which is introduced by Torra [6, 7], is another generalization of fuzzy sets. The hesitant fuzzy set is very useful to express peoples hesitancy in daily life, and it is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [11] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xu and Xia [12] defined the distance and correlation measures for hesitant fuzzy information and then discussed their properties in detail. Also, hesitant fuzzy set theory is used in decision making problem etc.(see [5, 8, 9, 10, 12]), and is applied to residuated lattices and MTL-algebras (see [2, 4]).

In this paper, we introduce the notions of hesitant fuzzy subalgebras, hesitant fuzzy ideals and hesitant fuzzy p-ideals based on the hesitant intersection ($\Cap$),

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briefly, ⋒-hesitant fuzzy subalgebras, ⋒-hesitant fuzzy ideals and ⋒-hesitant fuzzy p-ideals, in BCK/BCI-algebras. We investigate their relations and related properties. We provide conditions for a ⋒-hesitant fuzzy ideal to be a ⋒-hesitant fuzzy p-ideal. We finally establish the extension property for ⋒-hesitant fuzzy p-ideals.

2. Preliminaries

An algebra $(L; *, 0)$ of type $(2, 0)$ is calleda *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in L)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in L)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in L)\ (x * x = 0)$,

(IV) $(\forall x, y \in L)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra L satisfies the following identity:

(V) $(\forall x \in L)\ (0 * x = 0)$,

then L is called a *BCK-algebra*.

Any BCK/BCI-algebra L satisfies the following conditions:

(2.1) $$(\forall x \in L)\,(x * 0 = x)\,,$$

(2.2) $$(\forall x, y, z \in L)\,(x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,,$$

(2.3) $$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * y)\,,$$

(2.4) $$(\forall x, y, z \in L)\,((x * z) * (y * z) \le x * y)$$

where $x \le y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

(2.5) $$(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,,$$

(2.6) $$(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,,$$

(2.7) $$(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,.$$

A BCI-algebra L is said to be *p-semisimple*(see [1]) if $0 * (0 * x) = x$ for all $x \in L$.

Every p-semisimple BCI-algebra L satisfies:

(2.8) $$(\forall x, y, z \in L)\,((x * z) * (y * z) = x * y)\,.$$

A nonempty subset S of a BCK/BCI-algebra L is called a *subalgebra* of L if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra L is called an *ideal* of L if it satisfies:

(2.9) $$0 \in A,$$

(2.10) $$(\forall x \in L)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,.$$

A subset A of a BCI-algebra L is called a *p-ideal* of L (see [13]) if it satisfies (2.9) and

$$(\forall x, y, z \in L)\,((x * z) * (y * z) \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.11}$$

Note that an ideal A of a BCI-algebra L is a p-ideal of L if and only if the following assertion is valid:

$$(\forall x, y, z \in L)\,((x * z) * (y * z) \in A \ \Rightarrow\ x * y \in A)\,. \tag{2.12}$$

We refer the reader to the books [1, 3] for further information regarding BCK/BCI-algebras.

3. Subalgebras and Ideals of BCK/BCI-Algebras Based on the Hesitant Intersection

Let L be a set. A *hesitant fuzzy set* on L (see [6]) is defined in terms of a function $\mathcal{H}$ that when applied to L returns a subset of $[0, 1]$, that is, $\mathcal{H} : L \to \mathscr{P}([0, 1])$.

Given a hesitant fuzzy set $\mathcal{H}$ on L, we define $\mathrm{Inf}\mathcal{H}$ and $\mathrm{Sup}\mathcal{H}$, respectively, as follows:

$$\mathrm{Inf}\mathcal{H}(x) = \begin{cases} \text{minimum of } \mathcal{H}(x) & \text{if } \mathcal{H}(x) \text{ is finite,} \\ \text{infimum of } \mathcal{H}(x) & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$\mathrm{Sup}\mathcal{H}(x) = \begin{cases} \text{maximum of } \mathcal{H}(x) & \text{if } \mathcal{H}(x) \text{ is finite,} \\ \text{supremum of } \mathcal{H}(x) & \text{otherwise} \end{cases} \tag{3.2}$$

for all $x \in L$. It is obvious that $\mathrm{Inf}\mathcal{H}$ and $\mathrm{Sup}\mathcal{H}$ are fuzzy sets in L.

For a hesitant fuzzy set $\mathcal{H}$ on L and $x, y \in L$, we define

$$\mathcal{H}(x) \Cup \mathcal{H}(y) := \{t \in \mathcal{H}(x) \cup \mathcal{H}(y) \mid t \geq \max\{\mathrm{Inf}\mathcal{H}(x), \mathrm{Inf}\mathcal{H}(y)\}\} \tag{3.3}$$

and

$$\mathcal{H}(x) \Cap \mathcal{H}(y) := \{t \in \mathcal{H}(x) \cup \mathcal{H}(y) \mid t \leq \min\{\mathrm{Sup}\mathcal{H}(x), \mathrm{Sup}\mathcal{H}(y)\}\}. \tag{3.4}$$

We say that $\mathcal{H}(x) \Cup \mathcal{H}(y)$ (resp., $\mathcal{H}(x) \Cap \mathcal{H}(y)$) is the hesitant union (resp., hesitant intersection) of $\mathcal{H}(x)$ and $\mathcal{H}(y)$.

Proposition 3.1 *For any hesitant fuzzy set $\mathcal{H}$ on L, we have*

(1) $(\forall x \in L)\,(\mathcal{H}(x) \Cup \mathcal{H}(x) = \mathcal{H}(x))\,.$

(2) $(\forall x \in L)\,(\mathcal{H}(x) \Cap \mathcal{H}(x) = \mathcal{H}(x))\,.$

(3) $(\forall a, b, x, y \in L)\,(\mathcal{H}(a) \subseteq \mathcal{H}(x),\ \mathcal{H}(b) \subseteq \mathcal{H}(y)\ \Rightarrow \mathcal{H}(a) \Cap \mathcal{H}(b) \subseteq \mathcal{H}(x) \Cap \mathcal{H}(y))\,.$

(4) $(\forall a,b,x,y \in L)\,(\mathcal{H}(a) \subseteq \mathcal{H}(x),\ \mathcal{H}(b) \subseteq \mathcal{H}(y) \ \Rightarrow\ \mathcal{H}(a) \Cup \mathcal{H}(b) \subseteq \mathcal{H}(x) \Cup \mathcal{H}(y))\,.$

Proof. (1) and (2) are straightforward.

(3) Let $a,b,x,y \in L$ be such that $\mathcal{H}(a) \subseteq \mathcal{H}(x)$ and $\mathcal{H}(b) \subseteq \mathcal{H}(y)$. Then $\mathrm{Sup}\mathcal{H}(a) \leq \mathrm{Sup}\mathcal{H}(x)$ and $\mathrm{Sup}\mathcal{H}(b) \leq \mathrm{Sup}\mathcal{H}(y)$. If $t \in \mathcal{H}(a) \Cap \mathcal{H}(b)$, then

$$t \in \mathcal{H}(a) \cup \mathcal{H}(b) \subseteq \mathcal{H}(x) \cup \mathcal{H}(y)$$

and $t \leq \min\{\mathrm{Sup}\mathcal{H}(a), \mathrm{Sup}\mathcal{H}(b)\} \leq \min\{\mathrm{Sup}\mathcal{H}(x), \mathrm{Sup}\mathcal{H}(y)\}$.
Hence $t \in \mathcal{H}(x) \Cap \mathcal{H}(y)$, and so $\mathcal{H}(a) \Cap \mathcal{H}(b) \subseteq \mathcal{H}(x) \Cap \mathcal{H}(y)$.

(4) Let $a,b,x,y \in L$ be such that $\mathcal{H}(a) \subseteq \mathcal{H}(x)$ and $\mathcal{H}(b) \subseteq \mathcal{H}(y)$. Then $\mathrm{Inf}\mathcal{H}(a) \geq \mathrm{Inf}\mathcal{H}(x)$ and $\mathrm{Inf}\mathcal{H}(b) \geq \mathrm{Inf}\mathcal{H}(y)$. If $t \in \mathcal{H}(a) \Cup \mathcal{H}(b)$, then

$$t \in \mathcal{H}(a) \cup \mathcal{H}(b) \subseteq \mathcal{H}(x) \cup \mathcal{H}(y)$$

and $t \geq \max\{\mathrm{Inf}\mathcal{H}(a), \mathrm{Inf}\mathcal{H}(b)\} \geq \max\{\mathrm{Inf}\mathcal{H}(x), \mathrm{Inf}\mathcal{H}(y)\}$.
Hence $t \in \mathcal{H}(x) \Cup \mathcal{H}(y)$, and so $\mathcal{H}(a) \Cup \mathcal{H}(b) \subseteq \mathcal{H}(x) \Cup \mathcal{H}(y)$.

□

Definition 3.2 A hesitant fuzzy set on a BCK/BCI-algebra L is called a *hesitant fuzzy subalgebra* of L based on the intersection ($\cap$) (briefly, $\cap$-*hesitant fuzzy subalgebra* of L) if it satisfies:

$$(\forall x,y \in L)\,(\mathcal{H}(x*y) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y))\,. \tag{3.5}$$

Definition 3.3 A hesitant fuzzy set on a BCK/BCI-algebra L is called a *hesitant fuzzy subalgebra* of L based on the hesitant intersection ($\Cap$) (briefly, $\Cap$-*hesitant fuzzy subalgebra* of L) if it satisfies:

$$(\forall x,y \in L)\,(\mathcal{H}(x*y) \supseteq \mathcal{H}(x) \Cap \mathcal{H}(y))\,. \tag{3.6}$$

Example 3.4 Let $L = \{0,1,2,3\}$ be a BCK-algebra (see [3]) with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

(1) Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.8] & \text{if } x = 0, \\ [0.3, 0.7] & \text{if } x = 1, \\ [0.3, 0.5] & \text{if } x \in \{2, 3\}. \end{cases}$$

It is easy to check that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L.

(2) Define a hesitant fuzzy set $\mathcal{G}$ on L as follows:

$$\mathcal{G} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.2, 0.8] & \text{if } x = 0, \\ [0.2, 0.7] & \text{if } x = 1, \\ [0.2, 0.4] & \text{if } x = 2, \\ [0.2, 0.6] & \text{if } x = 3. \end{cases}$$

Then $\mathcal{G}$ is not a $\Cap$-hesitant fuzzy subalgebra of L since

$$\begin{aligned} \mathcal{G}(3) \Cap \mathcal{G}(1) &= \{t \in \mathcal{G}(3) \cup \mathcal{G}(1) \mid t \leq \min\{\text{Sup}\mathcal{G}(3), \text{Sup}\mathcal{G}(1)\} \\ &= \{t \in [0.2, 0.7] \mid t \leq \min\{0.6, 0.7\}\} \\ &= [0.2, 0.6] \nsubseteq [0.2, 0.4] = \mathcal{G}(2) = \mathcal{G}(3 * 1). \end{aligned}$$

It is clear that every $\Cap$-hesitant fuzzy subalgebra is a $\cap$-hesitant fuzzy subalgebra, but the converse is not true in general as seen in the following example.

Example 3.5 Let $L = \{0, a, b, c, d\}$ be a BCI-algebra (see [1]) with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	b	c	d
a	a	0	b	c	d
b	b	b	0	d	c
c	c	c	d	0	b
d	d	d	c	b	0

Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0, 0.9] & \text{if } x = 0, \\ [0.2, 0.7] & \text{if } x = a, \\ (0.2, 0.3] & \text{if } x = b, \\ \{0.4, 0.5, 0.6\} & \text{if } x = c, \\ [0.6, 0.7] & \text{if } x = d. \end{cases}$$

It is routine to check that $\mathcal{H}$ is a $\cap$-hesitant fuzzy subalgebra of L. Note that

$$\begin{aligned} \mathcal{H}(b) \Cap \mathcal{H}(d) &= \{x \in \mathcal{H}(b) \cup \mathcal{H}(d) \mid x \leq \min\{\text{Sup}\mathcal{H}(b), \text{Sup}\mathcal{H}(d)\} \\ &= \{x \in (0.2, 0.3] \cup [0.6, 0.7] \mid x \leq \min\{0.3, 0.7\}\} \\ &= (0.2, 0.3], \end{aligned}$$

and so $\mathcal{H}(b * d) = \mathcal{H}(c) = \{0.4, 0.5, 0.6\} \not\supseteq (0.2, 0.3] = \mathcal{H}(b) \Cap \mathcal{H}(d)$. Therefore $\mathcal{H}$ is not a $\Cap$-hesitant fuzzy subalgebra of L.

For any hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra L and $\varepsilon \in \mathscr{P}([0,1])$, we consider the set

$$\mathcal{H}_\varepsilon := \{x \in L \mid \varepsilon \subseteq \mathcal{H}(x)\}$$

which is called the *hesitant ε-level set* on L.

Theorem 3.6 *If $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of a BCK/BCI-algebra L, then the hesitant ε-level set $\mathcal{H}_\varepsilon$ on L is a subalgebra of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$.*

Proof. Assume that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of a BCK/BCI-algebra L and let $\varepsilon \in \mathscr{P}([0,1])$ be such that $\mathcal{H}_\varepsilon \neq \emptyset$. If $x, y \in \mathcal{H}_\varepsilon$, then $\varepsilon \subseteq \mathcal{H}(x)$ and $\varepsilon \subseteq \mathcal{H}(y)$. It follows from (3.6) and Proposition 3.1(3) that

$$\mathcal{H}(x * y) \supseteq \mathcal{H}(x) \Cap \mathcal{H}(y) \supseteq \varepsilon \tag{3.7}$$

and that $x * y \in \mathcal{H}_\varepsilon$. Therefore $\mathcal{H}_\varepsilon$ is a subalgebra of L. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7 Let $L = \{0, 1, 2, a, b\}$ be a BCI-algebra (see [1]) with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.8) & \text{if } x = 0, \\ (0.3, 0.5] & \text{if } x = 1, \\ [0.4, 0.7] & \text{if } x = 2, \\ (0.4, 0.6) & \text{if } x = a, \\ (0.4, 0.5] & \text{if } x = b. \end{cases}$$

Then we have

$$\mathcal{H}_\varepsilon = \begin{cases} \{0\} & \text{if } \varepsilon \subseteq [0.3, 0.8),\ \varepsilon \nsubseteq (0.3, 0.5) \text{ and } \varepsilon \nsubseteq [0.4, 0.7], \\ \{0, 2\} & \text{if } \varepsilon \subseteq [0.4, 0.7] \text{ and } \varepsilon \nsubseteq (0.4, 0.6), \\ \{0, 2, a\} & \text{if } \varepsilon \subseteq (0.4, 0.6) \text{ and } \varepsilon \nsubseteq (0.4, 0.5], \\ \{0, 1\} & \text{if } \varepsilon \subseteq (0.3, 0.5)\ \varepsilon \nsubseteq (0.4, 0.5], \\ L & \text{if } \varepsilon \subseteq (0.4, 0.5], \\ \emptyset & \text{otherwise,} \end{cases}$$

and so $\mathcal{H}_\varepsilon$ is a subalgebra of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$. Since

$$\begin{aligned}\mathcal{H}(2) \Cap \mathcal{H}(b) &= \{t \in \mathcal{H}(2) \cup \mathcal{H}(b) \mid t \leq \min\{\text{Sup}\mathcal{H}(2), \text{Sup}\mathcal{H}(b)\}\\ &= \{t \in [0.4, 0.7] \mid t \leq \min\{0.7, 0.5\}\}\\ &= [0.4, 0.5] \nsubseteq (0.4, 0.6) = \mathcal{H}(a) = \mathcal{H}(2 * b),\end{aligned}$$

$\mathcal{H}$ is not a $\Cap$-hesitant fuzzy subalgebra of L.

Theorem 3.8 *Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra L such that*

$$(\forall x, y \in L)\,(\mathcal{H}(x) \Cap \mathcal{H}(y) = \mathcal{H}(x) \cap \mathcal{H}(y))\,. \tag{3.8}$$

If the hesitant ε-level set $\mathcal{H}_\varepsilon$ on L is a subalgebra of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$, then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L.

Proof. Assume that the set $\mathcal{H}_\varepsilon := \{x \in L \mid \varepsilon \subseteq \mathcal{H}(x)\}$ is a subalgebra of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$. For any $x, y \in L$, let $\mathcal{H}(x) = \varepsilon_x$ and $\mathcal{H}(y) = \varepsilon_y$. Take $\varepsilon = \varepsilon_x \cap \varepsilon_y$. Then $x, y \in \mathcal{H}_\varepsilon$, and so $x * y \in \mathcal{H}_\varepsilon$. It follows from (3.8) that

$$\mathcal{H}(x * y) \supseteq \varepsilon = \varepsilon_x \cap \varepsilon_y = \varepsilon_x \Cap \varepsilon_y = \mathcal{H}(x) \Cap \mathcal{H}(y).$$

Therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L. □

Definition 3.9 A hesitant fuzzy set on a BCK/BCI-algebra L is called a *hesitant fuzzy ideal* of L based on the intersection ($\cap$) (briefly, $\cap$-*hesitant fuzzy ideal* of L) if it satisfies:

$$(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{H}(0))\,, \tag{3.9}$$

$$(\forall x, y \in L)\,(\mathcal{H}(x * y) \cap \mathcal{H}(y) \subseteq \mathcal{H}(x))\,. \tag{3.10}$$

Definition 3.10 A hesitant fuzzy set on a BCK/BCI-algebra L is called a *hesitant fuzzy ideal* of L based on the hesitant intersection ($\Cap$) (briefly, $\Cap$-*hesitant fuzzy ideal* of L) if it satisfies the condition (3.9) and

$$(\forall x, y \in L)\,(\mathcal{H}(x * y) \Cap \mathcal{H}(y) \subseteq \mathcal{H}(x))\,. \tag{3.11}$$

Example 3.11 Let $(Z, +, 0)$ be an additive group of integers. Note that $(Z, -, 0)$ is the adjoint BCI-algebra of $(Z, +, 0)$. For any BCI-algebra $(Y, *, 0)$, let $L := Y \times Z$. Then $(L, \otimes, (0, 0))$ is a BCI-algebra (see [1]) in which the operation $\otimes$ is given by

$$(\forall (x, m), (y, n) \in L)\,((x, m) \otimes (y, n) = (x * y, m - n))\,.$$

For a subset $A := Y \times N_0$ of L where N_0 is the set of nonnegative integers, let $\mathcal{H}$ be a hesitant fuzzy set on L defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x \in A, \\ [0.3, 0.6] & \text{otherwise.} \end{cases}$$

It is routine to verify that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L.

It is clear that every $\Cap$-hesitant fuzzy ideal is a $\cap$-hesitant fuzzy ideal, but the converse is not true in general as seen in the following example.

Example 3.12 Let $L = \{0, e, a, b, c\}$ be a BCI-algebra (see [1]) with the following Cayley table:

$*$	0	e	a	b	c
0	0	0	a	b	c
e	e	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0,1) & \text{if } x = 0, \\ [0.2, 0.7] & \text{if } x = e, \\ (0.2, 0.3] & \text{if } x = a, \\ \{0.4, 0.5\} & \text{if } x = b, \\ [0.6, 0.7) & \text{if } x = c. \end{cases}$$

Then $\mathcal{H}$ is a $\cap$-hesitant fuzzy subalgebra of L. Note that

$$\begin{aligned} \mathcal{H}(a * c) \Cap \mathcal{H}(c) &= \mathcal{H}(b) \Cap \mathcal{H}(c) \\ &= \{t \in \{0.4, 0.5\} \cup [0.6, 0.7] \mid t \leq \min\{0.5, 0.7\}\} \\ &= \{0.4, 0.5\} \nsubseteq (0.2, 0.3) = \mathcal{H}(a). \end{aligned}$$

Hence $\mathcal{H}$ is not a $\Cap$-hesitant fuzzy ideal of L.

Proposition 3.13 *Every $\Cap$-hesitant fuzzy ideal $\mathcal{H}$ of a BCI-algebra L satisfies the following assertion:*

$$(\forall x \in L)\,(\mathcal{H}(x) \subseteq \mathcal{H}(0 * (0 * x))). \tag{3.12}$$

Proof. For every $x \in L$, we have

$$\begin{aligned} \mathcal{H}(x) = \mathcal{H}(x) \Cap \mathcal{H}(x) &\subseteq \mathcal{H}(0) \Cap \mathcal{H}(x) \\ &= \mathcal{H}((0 * (0 * x)) * x) \Cap \mathcal{H}(x) \\ &\subseteq \mathcal{H}(0 * (0 * x)) \end{aligned}$$

by Proposition 3.1, (III), (2.3) and (3.11). □

Theorem 3.14 *If $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of a BCK/BCI-algebra L, then the hesitant ε-level set $\mathcal{H}_\varepsilon$ on L is an ideal of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$.*

Proof. Suppose that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of a BCK/BCI-algebra L. Let

$x, y \in L$ and $\varepsilon \in \mathscr{P}([0,1])$ be such that $x * y \in \mathcal{H}_\varepsilon$ and $y \in \mathcal{H}_\varepsilon$. Then $\varepsilon \subseteq \mathcal{H}(x * y)$ and $\varepsilon \subseteq \mathcal{H}(y)$. It follows from (3.9), (3.11) and Proposition 3.1(3) that

$$\mathcal{H}(0) \supseteq \mathcal{H}(x) \supseteq \mathcal{H}(x * y) \Cap \mathcal{H}(y) \supseteq \varepsilon.$$

Hence $0 \in \mathcal{H}_\varepsilon$ and $x \in \mathcal{H}_\varepsilon$. Therefore $\mathcal{H}_\varepsilon$ is an ideal of L. □

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.15 Consider the BCI-algebra L in Example 3.11. For a subset $A := Y \times N_0$ of L where N_0 is the set of nonnegative integers, let $\mathcal{H}$ be a hesitant fuzzy set on L defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x \in A, \\ (0.4, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}_\varepsilon$ is an ideal of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$. For any $a \in Y$, we have

$$\begin{aligned} &\mathcal{H}((a,-3) \otimes (a,3)) \Cap \mathcal{H}(a,3) = \mathcal{H}(0,-6) \Cap \mathcal{H}(a,3) \\ &= \{t \in \mathcal{H}(0,-6) \cup \mathcal{H}(a,3) \mid t \leq \min\{\mathrm{Sup}\mathcal{H}(0,-6), \mathrm{Sup}\mathcal{H}(a,3)\}\} \\ &= \{t \in [0.3, 0.9) \mid t \leq \min\{0.6, 0.9\}\} \\ &= [0.3, 0.6] \nsubseteq (0.4, 0.6] = \mathcal{H}(a,-3). \end{aligned}$$

Hence $\mathcal{H}$ is not a $\Cap$-hesitant fuzzy ideal of L.

We provide a condition for the converse of Theorem 3.14 to be true.

Theorem 3.16 *Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra L satisfying the condition* (3.8). *If the hesitant ε-level set $\mathcal{H}_\varepsilon$ on L is an ideal of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$, then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L.*

Proof. For any $x \in L$, let $\mathcal{H}(x) = \varepsilon_x$. Then $x \in \mathcal{H}_{\varepsilon_x}$, and so $\mathcal{H}_{\varepsilon_x}$ is an ideal of L by assumption. Thus $0 \in \mathcal{H}_{\varepsilon_x}$, and hence $\mathcal{H}(0) \supseteq \varepsilon_x = \mathcal{H}(x)$. For any $x, y \in L$, let $\mathcal{H}(x * y) = \varepsilon_{x*y}$ and $\mathcal{H}(y) = \varepsilon_y$. Taking $\varepsilon = \varepsilon_{x*y} \cap \varepsilon_y$ implies that $x * y \in \mathcal{H}_\varepsilon$ and $y \in \mathcal{H}_\varepsilon$. Hence $x \in \mathcal{H}_\varepsilon$, and it follows from the condition (3.8) that

$$\mathcal{H}(x) \supseteq \varepsilon = \varepsilon_{x*y} \cap \varepsilon_y = \varepsilon_{x*y} \Cap \varepsilon_y = \mathcal{H}(x * y) \Cap \mathcal{H}(y).$$

Therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L. □

Theorem 3.17 *Let ε_1 and ε_2 be subintervals of* $[0,1]$ *such that*

(1) $\varepsilon_2 \subsetneq \varepsilon_1$, $\mathrm{Inf}\varepsilon_1 = \mathrm{Inf}\varepsilon_2$ *and* $\mathrm{Sup}\varepsilon_2 \in \varepsilon_2$,

(2) $\mathrm{Inf}\varepsilon_1 \in \varepsilon_1$ *and* $\mathrm{Inf}\varepsilon_2 \in \varepsilon_2$ *(or,* $\mathrm{Inf}\varepsilon_1 \notin \varepsilon_1$ *and* $\mathrm{Inf}\varepsilon_2 \notin \varepsilon_2$*).*

Define a hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} \varepsilon_1 & \text{if } x \in A, \\ \varepsilon_2 & \text{otherwise,} \end{cases}$$

where A is a nonempty proper subset of L. Then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal (resp., subalgebra) of L if and only if A is an ideal (resp., subalgebra) of L.

Proof. Note that

$$\mathcal{H}_\varepsilon = \begin{cases} A & \text{if } \varepsilon \subseteq \varepsilon_1 \text{ and } \varepsilon \nsubseteq \varepsilon_2, \\ L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L, then $\mathcal{H}_\varepsilon$ is an ideal of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$ by Theorem 3.14. Hence A is an ideal of L.

Conversely, suppose that A is an ideal of L. Then $\mathcal{H}_\varepsilon$ is an ideal of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $\mathcal{H}_\varepsilon \neq \emptyset$. Let $x, y \in L$. If $x, y \in A$, then

$$\begin{aligned} \mathcal{H}(x) \Cap \mathcal{H}(y) &= \{t \in \mathcal{H}(x) \cup \mathcal{H}(y) \mid t \leq \min\{\text{Sup}\mathcal{H}(x), \text{Sup}\mathcal{H}(y)\}\} \\ &= \varepsilon_1 = \mathcal{H}(x) \cap \mathcal{H}(y). \end{aligned}$$

If $x, y \in L \setminus A$, then

$$\begin{aligned} \mathcal{H}(x) \Cap \mathcal{H}(y) &= \{t \in \mathcal{H}(x) \cup \mathcal{H}(y) \mid t \leq \min\{\text{Sup}\mathcal{H}(x), \text{Sup}\mathcal{H}(y)\}\} \\ &= \varepsilon_2 = \mathcal{H}(x) \cap \mathcal{H}(y). \end{aligned}$$

If $x \in A$ and $y \in L \setminus A$, then

$$\begin{aligned} \mathcal{H}(x) \Cap \mathcal{H}(y) &= \{t \in \mathcal{H}(x) \cup \mathcal{H}(y) \mid t \leq \min\{\text{Sup}\mathcal{H}(x), \text{Sup}\mathcal{H}(y)\}\} \\ &= \{t \in \varepsilon_1 \mid t \leq \min\{\text{Sup}\varepsilon_1, \text{Sup}\varepsilon_2\}\} \\ &= \{t \in \varepsilon_1 \mid t \leq \text{Sup}\varepsilon_2\} \\ &= \varepsilon_2 = \mathcal{H}(x) \cap \mathcal{H}(y). \end{aligned}$$

Similarly, if $x \in L \setminus A$ and $y \in A$, then $\mathcal{H}(x) \Cap \mathcal{H}(y) = \mathcal{H}(x) \cap \mathcal{H}(y)$. Thus $\mathcal{H}$ satisfies the condition (3.8), and therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L by Theorem 3.16. By the similar way, we can prove that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L if and only if A is a subalgebra of L. □

Proposition 3.18 *For every $\Cap$-hesitant fuzzy ideal $\mathcal{H}$ of a BCK/BCI-algebra L, the following assertions are valid.*

(1) $(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mathcal{H}(x) \supseteq \mathcal{H}(y))$,

(2) $(\forall x, y, z \in L)\,(x * y \leq z \;\Rightarrow\; \mathcal{H}(x) \supseteq \mathcal{H}(y) \Cap \mathcal{H}(z))$,

Proof. (1) Assume that $x \leq y$ for all $x, y \in L$. Then $x * y = 0$, which implies from (3.9), Proposition 3.1 and (3.11) that

$$\mathcal{H}(y) = \mathcal{H}(y) \Cap \mathcal{H}(y) \subseteq \mathcal{H}(0) \Cap \mathcal{H}(y) = \mathcal{H}(x * y) \Cap \mathcal{H}(y) \subseteq \mathcal{H}(x).$$

(2) Let $x, y, z \in L$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\mathcal{H}(z) = \mathcal{H}(z) \Cap \mathcal{H}(z) \subseteq \mathcal{H}(0) \Cap \mathcal{H}(z) = \mathcal{H}((x * y) * z) \Cap \mathcal{H}(z) \subseteq \mathcal{H}(x * y)$$

by (3.9), Proposition 3.1 and (3.11). It follows from Proposition 3.1 and (3.11) that

$$\mathcal{H}(y) \Cap \mathcal{H}(z) \subseteq \mathcal{H}(x * y) \Cap \mathcal{H}(y) \subseteq \mathcal{H}(x).$$

□

Proposition 3.19 *For every $\Cap$-hesitant fuzzy ideal $\mathcal{H}$ of a BCK/BCI-algebra L, the following assertions are equivalent.*

(1) $(\forall x, y \in L)\,(\mathcal{H}((x * y) * y) \subseteq \mathcal{H}(x * y))$,

(2) $(\forall x, y, z \in L)\,(\mathcal{H}((x * y) * z) \subseteq \mathcal{H}((x * z) * (y * z)))$.

Proof. Suppose that (1) is true and let $x, y, z \in L$. Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

by (2.3),(2.4) and (2.2). It follows from Proposition 3.18(1), (1) and (2.3) that

$$\begin{aligned} \mathcal{H}((x * y) * z) &\subseteq \mathcal{H}(((x * (y * z)) * z) * z) \\ &\subseteq \mathcal{H}((x * (y * z)) * z) \\ &= \mathcal{H}((x * z) * (y * z)), \end{aligned}$$

which shows that (2) is valid.

Now, assume that (2) holds and take $z := y$ in (2). Then

$$\mathcal{H}((x * y) * y) \subseteq \mathcal{H}((x * y) * (y * y)) = \mathcal{H}((x * y) * 0) = \mathcal{H}(x * y)$$

by using (III) and (2.1). Thus (1) is valid. □

We consider relations between a $\Cap$-hesitant fuzzy subalgebra and a $\Cap$-hesitant fuzzy ideal.

Theorem 3.20 *In a BCK-algebra, every $\Cap$-hesitant fuzzy ideal is a $\Cap$-hesitant fuzzy subalgebra.*

Proof. Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy ideal of a BCK-algebra L. Using (3.11), (2.3), (III), (V), (3.9) and Proposition 3.1, we have

$$\begin{aligned} \mathcal{H}(x * y) &\supseteq \mathcal{H}((x * y) * x) \Cap \mathcal{H}(x) \\ &= \mathcal{H}((x * x) * y) \Cap \mathcal{H}(x) \\ &= \mathcal{H}(0 * y) \Cap \mathcal{H}(x) \\ &= \mathcal{H}(0) \Cap \mathcal{H}(x) \\ &\supseteq \mathcal{H}(x) \Cap \mathcal{H}(y) \end{aligned}$$

for all $x, y \in L$. Hence $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L. □

The converse of Theorem 3.20 is not true in general. In fact, consider a BCK-algebra $L = \{0, 1, 2\}$ with the following Cayley table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Let $\mathcal{H}$ be a hesitant fuzzy set on L defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} [0.3, 0.8) & \text{if } x = 0, \\ [0.3, 0.6] & \text{if } x = 1, \\ [0.3, 0.7] & \text{if } x = 2. \end{cases}$$

Then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy subalgebra of L, but it is not a $\Cap$-hesitant fuzzy ideal of L since

$$\begin{aligned} \mathcal{H}(1 * 2) \Cap \mathcal{H}(2) &= \mathcal{H}(0) \Cap \mathcal{H}(2) \\ &= \{t \in \mathcal{H}(0) \cup \mathcal{H}(2) \mid t \leq \min\{\mathrm{Sup}\mathcal{H}(0), \mathrm{Sup}\mathcal{H}(2)\} \\ &= \{t \in [0.3, 0.8) \mid t \leq \min\{0.8, 0.7\} \\ &= [0.3, 0.7] \nsubseteq [0.3, 0.6] = \mathcal{H}(1). \end{aligned}$$

In a BCI-algebra, any $\Cap$-hesitant fuzzy ideal may not be a $\Cap$-hesitant fuzzy subalgebra. In fact, the $\Cap$-hesitant fuzzy ideal $\mathcal{H}$ of L in Example 3.11 is not a $\Cap$-hesitant fuzzy subalgebra of L since

$$\begin{aligned} \mathcal{H}(a, 0) \Cap \mathcal{H}(a, 2) &= \{t \in \mathcal{H}(a, 0) \cup \mathcal{H}(a, 2) \mid t \leq \{\mathrm{Sup}\mathcal{H}(a, 0), \mathrm{Sup}\mathcal{H}(a, 2)\}\} \\ &= \{t \in [0.3, 0.9) \mid t \leq 0.9\} \\ &= [0.3, 0.9) \nsubseteq [0.3, 0.6] \\ &= \mathcal{H}((a, 0) \otimes (a, 2)) \end{aligned}$$

for all $a \in Y$.

Definition 3.21 A hesitant fuzzy set $\mathcal{H}$ on a BCI-algebra L is called a *hesitant fuzzy p-ideal* of L based on the hesitant intersection ($\Cap$) (briefly, $\Cap$-*hesitant fuzzy p-ideal* of L) if it satisfies (3.9) and

$$(3.13) \qquad (\forall x, y, z \in L)\, (\mathcal{H}((x * z) * (y * z)) \Cap \mathcal{H}(y) \subseteq \mathcal{H}(x))\,.$$

Example 3.22 Let $L = \{0, a, b, c\}$ be a BCI-algebra (see [1]) with the following Cayley table.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} (0.4, 0.7) & \text{if } x \in \{0, b\} \\ (0.4, 0.5] & \text{otherwise,} \end{cases}$$

It is routine to verify that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L.

Theorem 3.23 *Let L be a BCI-algebra. Then every $\Cap$-hesitant fuzzy p-ideal of L is a $\Cap$-hesitant fuzzy ideal of L.*

Proof. Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy p-ideal of L. Since $x * 0 = x$ for all $x \in X$, it follows from taking $z := 0$ in (3.13) that

$$\mathcal{H}(x) \supseteq \mathcal{H}((x * 0) * (y * 0)) \Cap \mathcal{H}(y) = \mathcal{H}(x * y) \Cap \mathcal{H}(y)$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L. □

The following example shows that the converse of Theorem 3.23 is not true in general.

Example 3.24 Consider a BCI-algebra $L = \{0, 1, a, b, c\}$ with the following Cayley table (see [1]).

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Define a hesitant fuzzy set $\mathcal{H}$ on L as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} (0.2, 0.9) & \text{if } x = 0, \\ (0.2, 0.7] & \text{if } x = 1, \\ (0.2, 0.5] & \text{otherwise,} \end{cases}$$

Then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy ideal of L. But it is not a $\Cap$-hesitant fuzzy p-ideal of L since

$$\begin{aligned} &\mathcal{H}((1 * a) * (0 * a)) \Cap \mathcal{H}(0) = \mathcal{H}(c * c) \Cap \mathcal{H}(0) \\ &= \mathcal{H}(0) = (0, 2, 0.9) \nsubseteq (0.2, 0.7] = \mathcal{H}(1). \end{aligned}$$

Proposition 3.25 *Every $\Cap$-hesitant fuzzy p-ideal $\mathcal{H}$ of a BCI-algebra L satisfies the following assertion:*

$$(\forall x \in L)\, (\mathcal{H}(0 * (0 * x)) \subseteq \mathcal{H}(x)) \,. \tag{3.14}$$

Proof. Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy p-ideal of L. If we put $z := x$ and $y := 0$ in (3.13), then

$$\mathcal{H}(x) \supseteq \mathcal{H}((x * x) * (0 * x)) \Cap \mathcal{H}(0) = \mathcal{H}(0 * (0 * x)) \Cap \mathcal{H}(0) \supseteq \mathcal{H}(0 * (0 * x))$$

for all $x \in L$ by (III), (3.9) and Proposition 3.1. □

Proposition 3.26 *Every $\Cap$-hesitant fuzzy p-ideal $\mathcal{H}$ of a BCI-algebra L satisfies:*

$$(\forall x, y, z \in L) \left(\mathcal{H}(x * y) \subseteq \mathcal{H}((x * z) * (y * z)) \right). \tag{3.15}$$

Proof. Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy p-ideal of L. Then it is a $\Cap$-hesitant fuzzy ideal of L by Theorem 3.23. Using (3.11), (2.4) and Proposition 3.1, we have

$$\begin{aligned} \mathcal{H}((x * z) * (y * z)) &\supseteq \mathcal{H}(((x * z) * (y * z)) * (x * y)) \Cap \mathcal{H}(x * y) \\ &= \mathcal{H}(0) \Cap \mathcal{H}(x * y) \supseteq \mathcal{H}(x * y) \end{aligned}$$

for all $x, y, z \in L$. □

We provide conditions for a $\Cap$-hesitant fuzzy ideal to be a $\Cap$-hesitant fuzzy p-ideal.

Theorem 3.27 *Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy ideal of L such that*

$$(\forall x, y, z \in L) \left(\mathcal{H}(x * y) \supseteq \mathcal{H}((x * z) * (y * z)) \right). \tag{3.16}$$

Then $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L.

Proof. If the condition (3.16) is valid, then

$$\mathcal{H}(x) \supseteq \mathcal{H}(x * y) \Cap \mathcal{H}(y) \supseteq \mathcal{H}((x * z) * (y * z)) \Cap \mathcal{H}(y)$$

for all $x, y, z \in L$ by (3.11) and Proposition 3.1. Therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L. □

Theorem 3.28 *If a $\Cap$-hesitant fuzzy ideal $\mathcal{H}$ of L satisfies the condition* (3.14)*, then it is a $\Cap$-hesitant fuzzy p-ideal of L.*

Proof. Let $x, y, z \in L$. Using Proposition 3.13, (2.5), (2.6) and (3.14), we have

$$\begin{aligned} \mathcal{H}((x * z) * (y * z)) &\subseteq \mathcal{H}(0 * (0 * ((x * z) * (y * z)))) \\ &= \mathcal{H}((0 * y) * (0 * x)) \\ &= \mathcal{H}(0 * (0 * (x * y))) \\ &\subseteq \mathcal{H}(x * y). \end{aligned}$$

It follows from Theorem 3.27 that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L. □

Theorem 3.29 *In a p-semisimple BCI-algebra, every $\Cap$-hesitant fuzzy ideal is a $\Cap$-hesitant fuzzy p-ideal.*

Proof. Let $\mathcal{H}$ be a $\Cap$-hesitant fuzzy ideal of a p-semisimple BCI-algebra L. Using (3.11) and (2.8), we have

$$\mathcal{H}(x) \supseteq \mathcal{H}(x * y) \Cap \mathcal{H}(y) = \mathcal{H}((x * z) * (y * z)) \Cap \mathcal{H}(y)$$

for all $x, y, z \in L$. Therefore $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L. □

Theorem 3.30 (Extension property for $\Cap$-hesitant fuzzy p-ideals) *Let $\mathcal{H}$ and $\mathcal{G}$ be $\Cap$-hesitant fuzzy ideals of a BCI-algebra L such that $\mathcal{H}(0) = \mathcal{G}(0)$ and $\mathcal{H}(x) \subseteq \mathcal{G}(x)$ for all $x \in L$. If $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of L, then so is $\mathcal{G}$.*

Proof. Assume that $\mathcal{H}$ is a $\Cap$-hesitant fuzzy p-ideal of X. Using (2.6), (2.7) and (III), we have $0*(0*(x*(0*(0*x)))) = 0$ for all $x \in X$. It follows from hypothesis and (3.14) that

$$\begin{aligned} \mathcal{G}(x*(0*(0*x))) &\supseteq \mathcal{H}(x*(0*(0*x))) \\ &\supseteq \mathcal{H}(0*(0*(x*(0*(0*x))))) \\ &= \mathcal{H}(0) = \mathcal{G}(0), \end{aligned}$$

and that

$$\begin{aligned} \mathcal{G}(x) &\supseteq \mathcal{G}(x*(0*(0*x))) \Cap \mathcal{G}(0*(0*x)) \\ &\supseteq \mathcal{G}(0) \Cap \mathcal{G}(0*(0*x)) \\ &\supseteq \mathcal{G}(0*(0*x)) \Cap \mathcal{G}(0*(0*x)) \\ &= \mathcal{G}(0*(0*x)) \end{aligned}$$

by (3.11), (3.9) and Proposition 3.1. Therefore $\mathcal{G}$ is a $\Cap$-hesitant fuzzy p-ideal of X by Theorem 3.28. □

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