

A Further Result Related to a Conjecture of R. Brück

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ABSTRACT. In this paper, we investigate the uniqueness problem of a meromorphic function sharing one small function with its differential polynomial, and give a result which is related to a conjecture of R. Brück.

1. Introduction

In this paper, meromorphic function means meromorphic in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [12]. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Let $f(z)$ be a meromorphic function. It is known that the hyper order of $f(z)$,

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denoted by $\sigma_2(f)$, is defined by

$$\sigma_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

In 1996, R. Brück [1] posed the following conjecture.

Brück Conjecture. Let f be a non-constant entire function such that the hyper order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < +\infty$. If f and f' share a finite value a CM, then

$$\frac{f' - a}{f - a} = c,$$

where c is a nonzero constant.

In 1998, Gundersen and Yang [3] verified that the Conjecture is true when f is of finite order. In 1999, Yang [10] confirmed that the Conjecture is also true when f' is replaced by $f^{(k)}$ ($k \geq 2$) and f is of finite order. In recent years, many results have been published concerning the above conjecture, see [2, 5, 7, 8, 14, 15, 16, 17, 18] etc., and Zhang [17] was the first author who considers the case when f is a meromorphic function. We need the following definition.

Definition 1.1. Let l be a non-negative integer or infinite. Denote by $E_l(a, f)$ the set of all a -points of f where an a -point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f and g share (a, l) .

Remark. It is easy to see that f and g share (a, l) implies that f and g share (a, p) for $0 \leq p \leq l$. Also we note that f and g share the value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) , respectively. We also use $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f - a$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$.

Lahiri [5] improved the results of Zhang [17] by using the above definition and obtained the following Theorem:

Theorem A. Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1, 2)$ and

$$2\bar{N}(r, f) + N_2\left(r, \frac{1}{f^{(k)}}\right) + N_2\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)} - a}{f - a} = c$ for $c \in \mathbf{C} \setminus \{0\}$.

Let p be a positive integer and $a \in \mathbf{C} \cup \{\infty\}$. We use $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f - a$, whose multiplicities are not greater

than p , $N_{(p+1)}(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f - a$ whose multiplicities are not less than $p + 1$, and we use $\overline{N}_p(r, \frac{1}{f-a})$ and $\overline{N}_{(p+1)}(r, \frac{1}{f-a})$ to denote their corresponding reduced counting functions (ignoring multiplicities) respectively. Define

$$\delta_p(a, f) = 1 - \limsup_{r \rightarrow +\infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)}.$$

It follows that $\delta_p(a, f) \geq \delta(a, f)$.

Let $L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0f$. Zhang and Yang [16] obtained the following result which improves the results of [5, 8, 14, 18].

Theorem B. *Let f be a non-constant meromorphic function, $k \geq 1$ and $l \geq 0$ be integers. Let $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $L(f) - a$ share $(0, l)$. Then $f \equiv L(f)$ if one of the following assumptions holds,*

(1) $l \geq 2$ and

$$3\theta(\infty, f) + \delta_{2+k}(0, f) + \delta_2(0, f) + \delta(a, f) > 4,$$

(2) $l = 1$ and

$$\frac{7+k}{2}\theta(\infty, f) + \frac{1}{2}\delta_{1+k}(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) + \delta(a, f) > \frac{k}{2} + 5,$$

(3) $l = 0$ and

$$(2k + 6)\theta(\infty, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) + \delta_{2+k}(0, f) + \theta(0, f) + \delta(a, f) > 2k + 10.$$

Definition 1.2. Let p_0, p_1, \dots, p_k be non-negative integers. We call

$$M[f] = f^{p_0}(f')^{p_1} \dots (f^{(k)})^{p_k}$$

a differential monomial in f with degree $d_M = p_0 + p_1 + \dots + p_k$ and weight $\Gamma_M = p_0 + 2p_1 + \dots + (k + 1)p_k$, and

$$H[f] = \sum_{j=1}^n a_j M_j[f],$$

where a_j are small functions of f , is called a differential polynomial in f of degree $d = \max\{d_{M_j}, 1 \leq j \leq n\}$ and weight $\Gamma = \max\{\Gamma_{M_j}, 1 \leq j \leq n\}$, furthermore if $\deg(M_j) = d(j = 1, 2, \dots, n)$, then $H[f]$ is a homogeneous differential polynomial in f of degree d .

In this paper, we improve the above Theorems and obtain the following result.

Theorem 1.3. *Let f be a non-constant meromorphic function and $H[f]$ be a non-constant homogeneous differential polynomial of degree d and weight Γ satisfying*

$\Gamma \geq (k + 2)d - 2$. Let $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $f - a$ and $H[f] - a$ share $(0, l)$. Then $\frac{H[f]-a}{f-a} = C$ for some non-zero constant C if one of the following assumptions holds,

(i) $l \geq 2$ and

$$(1.1) \quad 3\theta(\infty, f) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) > 4,$$

(ii) $l = 1$ and

$$(1.2) \quad \frac{7 + \Gamma - d}{2}\theta(\infty, f) + \frac{d}{2}\delta_{1+\Gamma-d}(0, f^d) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) > \frac{\Gamma + 9}{2},$$

(iii) $l = 0$ and

$$(1.3) \quad [2(\Gamma - d) + 6]\theta(\infty, f) + \delta_2(0, f) + d\delta_{2+\Gamma-d}(0, f^d) + 2d\delta_{1+\Gamma-d}(0, f^d) + \theta(0, f) + \delta(a, f) > 2\Gamma + 8.$$

Epecially, when $l = 0$, i.e. f and H share a IM, if (1.3) holds, then $f \equiv H[f]$.

2. Some Lemmas

Lemma 2.1.([11]) *Let f be a nonconstant meromorphic function, k be a positive integer. Then*

$$(2.1) \quad N\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f),$$

$$(2.2) \quad N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Suppose that F and G are two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p , a 1-point of G of order q . We denote by $N_L(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p > q$, by $N_E^1(r, \frac{1}{F-1})$ the counting function of those 1-points of F where $p = q = 1$, by N_E^2 the counting function of those 1 -points of F where $p = q \geq 2$; each point in these counting functions is counted only one time. Similarly, we can define $N_L(r, \frac{1}{G-1})$, $N_E^1(r, \frac{1}{G-1})$ and $N_E^2(r, \frac{1}{G-1})$.

Lemma 2.2.([13]) *Let F and G are two nonconstant meromorphic functions,*

$$(2.3) \quad \Delta = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

If F and G share 1 IM and $\Delta \neq 0$, then

$$(2.4) \quad N_E^{(1)}\left(r, \frac{1}{F-1}\right) \leq N(r, \Delta) + S(r, F) + S(r, G).$$

Lemma 2.3. *Let $H[f]$ be a non-constant differential polynomial. Let z_0 be a pole of f of order p and neither a zero nor a pole of coefficients of $H[f]$. Then z_0 is a pole of $H[f]$ with order at most $pd + (\Gamma - d)$.*

Proof. Let

$$H[f] = \sum_{j=1}^n a_j H_j[f], \quad H_j[f] = f^{p_0} (f')^{p_1} \dots (f^{(k)})^{p_k},$$

$$d_{M_j} = p_0 + p_1 + \dots + p_k, \quad \Gamma_{M_j} = p_0 + 2p_1 + \dots + (k+1)p_k.$$

Let z_0 be a pole of f of order p , then z_0 be a pole of $H_j[f]$ of order $pd_{M_j} + (\Gamma_{M_j} - d_{M_j})$. Because $d = \max\{d_{M_j}, 1 \leq j \leq n\}$, $\Gamma = \max\{\Gamma_{M_j}, 1 \leq j \leq n\}$ and z_0 neither be a zero nor be a pole of a_j , then z_0 is a pole of $H[f]$ with order at most $pd + (\Gamma - d)$. \square

Lemma 2.4. *Let f be a transcendental meromorphic function, $H[f]$ is a homogeneous differential polynomial in f of degree d and weight Γ . If $H[f] \neq 0$, we have*

$$(2.5) \quad N\left(r, \frac{1}{H}\right) \leq T(r, H) - dT(r, f) + dN\left(r, \frac{1}{f}\right) + S(r, f).$$

$$(2.6) \quad N\left(r, \frac{1}{H}\right) \leq (\Gamma - d)\bar{N}(r, f) + dN\left(r, \frac{1}{f}\right) + S(r, f).$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we have

$$\begin{aligned} N\left(r, \frac{1}{H}\right) &= T(r, H) - m\left(r, \frac{1}{H}\right) + O(1) \\ &\leq T(r, H) - \left(m\left(r, \frac{1}{f^d}\right) - m\left(r, \frac{H}{f^d}\right)\right) + O(1) \\ &= T(r, H) - \left(T\left(r, \frac{1}{f^d}\right) - N\left(r, \frac{1}{f^d}\right)\right) + S(r, f) \\ (2.7) \quad &= T(r, H) - dT(r, f) + dN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

This proves (2.5). From Lemma 3, we have

$$\begin{aligned} T(r, H) &= m(r, H) + N(r, H) \\ &\leq m\left(r, \frac{H}{f^d}\right) + m(r, f^d) + N(r, H) \\ &\leq dm(r, f) + dN(r, f) + (\Gamma - d)\bar{N}(r, f) + S(r, f) \\ &= dT(r, f) + (\Gamma - d)\bar{N}(r, f) + S(r, f). \end{aligned}$$

Combining with (2.7), we obtain (2.6). \square

Lemma 2.5. *Let f be a non-constant meromorphic function, $H[f]$ is a homogeneous differential polynomial in f of degree d and weight Γ , and let p be a positive integer. If $H[f] \not\equiv 0$ and $\Gamma \geq (k+2)d - (p+1)$, we have*

$$(2.8) \quad N_p\left(r, \frac{1}{H}\right) \leq T(r, H) - dT(r, f) + N_{p+\Gamma-d}\left(r, \frac{1}{f^d}\right) + S(r, f),$$

$$(2.9) \quad N_p\left(r, \frac{1}{H}\right) \leq (\Gamma - d)\bar{N}(r, f) + N_{p+\Gamma-d}\left(r, \frac{1}{f^d}\right) + S(r, f).$$

Proof. From (2.6), we have

$$\begin{aligned} N_p\left(r, \frac{1}{H}\right) + \sum_{j=p+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{H}\right) &\leq (\Gamma - d)\bar{N}(r, f) + N_{p+\Gamma-d}\left(r, \frac{1}{f^d}\right) \\ &\quad + \sum_{j=p+\Gamma-d+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{f^d}\right) + S(r, f). \end{aligned}$$

Since $\Gamma \geq (k+2)d - (p+1)$, then we have

$$\begin{aligned} N_p\left(r, \frac{1}{H}\right) &\leq (\Gamma - d)\bar{N}(r, f) + N_{p+\Gamma-d}\left(r, \frac{1}{f^d}\right) + \sum_{j=p+\Gamma-d+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{f^d}\right) \\ &\quad - \sum_{j=p+1}^{\infty} \bar{N}_{(j)}\left(r, \frac{1}{H}\right) + S(r, f) \\ &\leq (\Gamma - d)\bar{N}(r, f) + N_{p+\Gamma-d}\left(r, \frac{1}{f^d}\right) + S(r, f). \end{aligned}$$

Thus (2.9) holds. By the same arguments as above, we obtain (2.8) from (2.5).□

3. Proof of Theorem 1.3

Let $F = \frac{H[f]}{a}, G = \frac{f}{a}$. From the conditions of Theorem 1.3, we know that F and G share $(1, l)$ except the zeros and poles of $a(z)$. From the proof of Lemma 2.4, we have

$$(3.1) \quad T(r, F) = O(T(r, f)) + S(r, f), \quad T(r, G) = T(r, f) + S(r, f).$$

It is obvious that f is a transcendental meromorphic function. Let Δ be defined by (2.3). We distinguish two cases.

Case 1. $\Delta \equiv 0$. Integrating (2.3), yields

$$(3.2) \quad \frac{1}{G-1} = \frac{C}{F-1} + D,$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or pole of a , then z_0 is a pole of F with multiplicity $pd + (\Gamma - d)$, a pole of G with multiplicity p . This contradicts (3.2) as H contains at least one derivative. Therefore, we have

$$(3.3) \quad \bar{N}(r, F) = \bar{N}(r, G) = \bar{N}(r, f) = S(r, f).$$

(3.2) also shows that F and G share the value 1 CM.

Next, we will prove $D = 0$.

Suppose $D \neq 0$, then we have

$$(3.4) \quad \frac{1}{G-1} = \frac{D(F-1 + \frac{C}{D})}{F-1}.$$

So, we have

$$(3.5) \quad \bar{N}\left(r, \frac{1}{D(F-1 + \frac{C}{D})}\right) = \bar{N}\left(r, \frac{G-1}{F-1}\right) = S(r, f).$$

Subcase 1.1. If $\frac{C}{D} \neq 1$, then by using (3.3), (3.5) and the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-1 + \frac{C}{D}}\right) + S(r, F) \\ &\leq \bar{N}\left(r, \frac{1}{F}\right) + S(r, F) \leq (1 + o(1))T(r, F). \end{aligned}$$

This gives that

$$T(r, F) = \bar{N}\left(r, \frac{1}{F}\right) + S(r, F) = N_1\left(r, \frac{1}{F}\right) + S(r, F).$$

So we have

$$T(r, H) = N\left(r, \frac{1}{H}\right) + S(r, f) = N_1\left(r, \frac{1}{H}\right) + S(r, f).$$

Let $p = 1$, then from assumption we have

$$\Gamma \geq (k+2)d - 2 = (k+2)d - (p+1).$$

Thus from (2.8) in Lemma 2.5, we get

$$T(r, H) = N_1\left(r, \frac{1}{H}\right) + S(r, f) \leq T(r, H) - dT(r, f) + N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right) + S(r, f).$$

So we have

$$dT(r, f) \leq N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right) + S(r, f).$$

This gives that

$$dT(r, f) = N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right) + S(r, f).$$

So we have

$$\delta_{2+\Gamma-d}(0, f^d) = \delta_{1+\Gamma-d}(0, f^d) = 0.$$

Since (3.3), we get

$$(3.6) \quad \theta(\infty, f) = 1.$$

Subcase 1.1.1. $l \geq 2$.

From $\delta_2(0, f) + \delta(a, f) > 1$ and the definition of deficiency, we have

$$(3.7) \quad T(r, f) > N_2\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-a}\right).$$

Using the second fundamental theorem of Nevanlinna and (3.3), we have

$$(3.8) \quad \begin{aligned} T(r, f) &\leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}(r, f) + S(r, f) \\ &= \bar{N}(r, 1f) + \bar{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned}$$

Combining (3.7) with (3.8), we have

$$N_2\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-a}\right) < T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f).$$

So we have

$$N_2\left(r, \frac{1}{f}\right) = S(r, f), N\left(r, \frac{1}{f-a}\right) = S(r, f).$$

This gives that

$$\overline{N}\left(r, \frac{1}{f}\right) = S(r, f), \overline{N}\left(r, \frac{1}{f-a}\right) = S(r, f).$$

From (3.8), we get a contradiction.

Subcase 1.1.2. $l = 1$.

When $d \geq 2$, by using (1.2) and the definition of deficiency, we get a contradiction.

When $d = 1$, using the similar method in subcase 1.1.1, we get a contradiction.

Subcase 1.1.3. $l = 0$.

By using (1.3) and the definition of deficiency, we get a contradiction.

Subcase 1.2. If $\frac{C}{D} = 1$, then from (3.4), we have

$$\frac{1}{G-1} \equiv C \frac{F}{F-1}.$$

This gives us that

$$\left(G - 1 - \frac{1}{C}\right)F \equiv -\frac{1}{C}.$$

Using that $F = \frac{H}{a}$ and $G = \frac{f}{a}$, we get

$$(3.9) \quad f - a \left(1 + \frac{1}{C}\right) \equiv -\frac{a^2}{C} \cdot \frac{1}{H}.$$

Using (3.3) (3.9), Lemma 2.3 and the first fundamental theorem, we get

$$\begin{aligned} (d+1)T(r, f) &= T\left(r, \frac{1}{f^d(f - (1 + \frac{1}{C})a)}\right) + O(1) \\ &= T\left(r, -\frac{CH}{f^d a^2}\right) + O(1) \\ &= N\left(r, \frac{H}{f^d}\right) + S(r, f) \\ &\leq dN\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq (d + o(1))T(r, f), \end{aligned}$$

which is a contradiction, hence $D=0$. This gives from (3.2) that

$$\frac{F-1}{G-1} \equiv C.$$

So we get $\frac{H[f]-a}{f-a} = C (C \neq 0)$.

Next, we will prove $C = 1$ when $l = 0$.

Suppose $C \neq 1$, then we have

$$G \equiv \frac{1}{C}(F-1+C)$$

and

$$(3.10) \quad N\left(r, \frac{1}{G}\right) = N\left(r, \frac{1}{F-1+C}\right).$$

By the second fundamental theorem and (3.3) (3.10), we have

$$\begin{aligned} T(r, F) &\leq \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-1+C}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right) + S(r, f) \\ &= N_1\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{G}\right). \end{aligned}$$

By Lemma 2.5 for $p = 1$, we have

$$dT(r, f) \leq N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f).$$

From the above formula and the definition of deficiency, we have

$$(3.11) \quad d\delta_{1+\Gamma-d}(0, f^d) + \theta(0, f) \leq 1.$$

So we have

$$(3.12) \quad d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) \leq 1, \quad d\delta_{1+\Gamma-d}(0, f^d) \leq 1.$$

Combining (3.11) (3.12) (3.6) with the assumptions of Theorem 1.3, we get a contradiction.

So $C = 1$ and $G \equiv F$, i.e. $f \equiv H[f]$.

This is just the conclusion of this theorem.

Case 2. $\Delta \neq 0$.

By a similar method that used in the proof of Theorem B[16], we get

$$\begin{aligned}
 T(r, F) + T(r, G) &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F-1}\right) \\
 &\quad + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G-1}\right) \\
 (3.13) \quad &\quad - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, f)
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) &\leq \bar{N}_{(2)}\left(r, \frac{1}{F}\right) + \bar{N}_{(2)}\left(r, \frac{1}{G}\right) \\
 &\quad + \bar{N}(r, G) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) + N_E^{(1)}\left(r, \frac{1}{F-1}\right) \\
 (3.14) \quad &\quad + 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f).
 \end{aligned}$$

Subcase 2.1. $l \geq 2$. It is easy to see that

$$\begin{aligned}
 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) + 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + N_E^{(1)}\left(r, \frac{1}{F-1}\right) \\
 (3.15) \quad \leq N\left(r, \frac{1}{G-1}\right) + S(r, f).
 \end{aligned}$$

From (3.13) (3.14) and (3.15), we have

$$\begin{aligned}
 T(r, F) + T(r, G) &\leq 3\bar{N}(r, G) + N_2\left(r, \frac{1}{F}\right) \\
 &\quad + N_2\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) + S(r, f).
 \end{aligned}$$

Noting that

$$N_2\left(r, \frac{1}{F}\right) = N_2\left(r, \frac{a}{H}\right) \leq N_2\left(r, \frac{1}{H}\right) + S(r, f).$$

Let $p = 2$, then from assumption we have

$$\Gamma \geq (k + 2)d - 2 > (k + 2)d - (p + 1).$$

Thus, from (2.8) in Lemma 2.5 we obtain that

$$\begin{aligned}
 T(r, H) + T(r, f) &\leq 3\bar{N}(r, f) + T(r, H) - dT(r, f) + N_{2+\Gamma-d}\left(r, \frac{1}{fd}\right) \\
 &\quad + N_2\left(r, \frac{1}{f}\right) + T(r, f) - m\left(r, \frac{1}{f-a}\right) + S(r, f).
 \end{aligned}$$

So we have

$$dT(r, f) \leq 3\bar{N}(r, f) + N_{2+\Gamma-d}\left(r, \frac{1}{fd}\right) + N_2\left(r, \frac{1}{f}\right) - m\left(r, \frac{1}{f-a}\right) + S(r, f).$$

This gives that

$$3\theta(\infty, f) + d\delta_{2+\Gamma-d}(0, f^d) + \delta_2(0, f) + \delta(a, f) \leq 4.$$

Which contradicts the assumption (1.1) of Theorem 1.3.

Subcase 2.2. $l = 1$. Noting that

$$\begin{aligned} 2N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) &+ 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + N_E^{(1)}\left(r, \frac{1}{F-1}\right) \\ &\leq N\left(r, \frac{1}{G-1}\right) + S(r, f) \end{aligned}$$

and

$$\begin{aligned} N_L\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r, \frac{F}{F'}\right) \leq \frac{1}{2}T\left(r, \frac{F}{F'}\right) = \frac{1}{2}T\left(r, \frac{F'}{F}\right) + O(1) \\ &\leq \frac{1}{2}N\left(r, \frac{F'}{F}\right) + S(r, f) \\ &\leq \frac{1}{2}\left(\bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F)\right) + S(r, f) \\ &\leq \frac{1}{2}\left(\bar{N}\left(r, \frac{1}{H}\right) + \bar{N}(r, f)\right) + S(r, f) \\ &\leq \frac{1}{2}\left[(\Gamma - d + 1)\bar{N}(r, f) + N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right)\right] + S(r, f). \end{aligned}$$

Using the same method as Subcase 2.1, we get

$$\begin{aligned} dT(r, f) &\leq \frac{\Gamma - d + 7}{2}\bar{N}(r, f) + N_{2+\Gamma-d}\left(r, \frac{1}{fd}\right) + \frac{1}{2}N_{1+\Gamma-d}\left(r, \frac{1}{fd}\right) \\ &\quad + N_2\left(r, \frac{1}{f}\right) - m\left(r, \frac{1}{f-a}\right) + S(r, f). \end{aligned}$$

Which contradicts with (1.2) of Theorem 1.3.

Subcase 2.3. $l = 0$. Noting that

$$\begin{aligned} (3.16) \quad N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) &+ 2N_E^{(2)}\left(r, \frac{1}{G-1}\right) + N_E^{(1)}\left(r, \frac{1}{F-1}\right) \\ &\leq N\left(r, \frac{1}{G-1}\right) + S(r, f). \end{aligned}$$

From Lemma 2.5, we have

$$\begin{aligned}
 N_L\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{F}{F'}\right) \\
 &\leq N\left(r, \frac{F'}{F}\right) + S(r, f) \\
 &\leq \left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, F)\right) + S(r, f) \\
 &\leq \left(\overline{N}\left(r, \frac{1}{H}\right) + \overline{N}(r, f)\right) + S(r, f) \\
 &\leq N_{1+\Gamma-d}\left(r, \frac{1}{f^d}\right) + (\Gamma - d + 1)\overline{N}(r, f) + S(r, f).
 \end{aligned}$$

So we have

$$\begin{aligned}
 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) &\leq 2N_{1+\Gamma-d}\left(r, \frac{1}{f^d}\right) + 2(\Gamma - d + 1)\overline{N}(r, f) \\
 (3.17) \qquad \qquad \qquad &+ \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f).
 \end{aligned}$$

Combining (3.13) (3.14) (3.16) with (3.17), we have

$$\begin{aligned}
 T(r, H) + T(r, f) &\leq N_2\left(r, \frac{1}{H}\right) + N_2\left(r, \frac{1}{f}\right) + 3\overline{N}(r, f) + N\left(r, \frac{1}{f-a}\right) \\
 &+ 2N_{1+\Gamma-d}\left(r, \frac{1}{f^d}\right) + 2(\Gamma - d + 1)\overline{N}(r, f) \\
 (3.18) \qquad \qquad &+ \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f).
 \end{aligned}$$

From (2.8), we have

$$N_2\left(r, \frac{1}{H}\right) \leq T(r, H) - dT(r, f) + N_{2+\Gamma-d}\left(r, \frac{1}{f^d}\right) + S(r, f).$$

Substituting this into (3.18), we have

$$\begin{aligned}
 dT(r, f) &\leq N_2\left(r, \frac{1}{f}\right) + 2(\Gamma - d + 3)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) \\
 &+ 2N_{1+\Gamma-d}\left(r, \frac{1}{f^d}\right) + N_{2+\Gamma-d}\left(r, \frac{1}{f^d}\right) - m\left(r, \frac{1}{f-a}\right) + S(r, f).
 \end{aligned}$$

So we have

$$\begin{aligned}
 \delta_2(0, f) + \theta(0, f) + 2(\Gamma - d + 3)\theta(\infty, f) + d\delta_{2+\Gamma-d}(0, f^d) \\
 + 2d\delta_{1+\Gamma-d}(0, f^d) + \delta(a, f) \leq 2\Gamma + 8.
 \end{aligned}$$

Which contradicts the assumption of Theorem 1.3.
Now the proof has been completed. \square

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