

Existence and Exponential Stability for a Thermoviscoelastic Equation with Boundary Output Feedback Control

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ABSTRACT. In this paper, we consider a thermoviscoelastic equation which has one end fixed and output feedback control at the other end. We prove the existence of solutions using the Galerkin method and then investigate the exponential stability of solutions by using multiplier technique.

1. Introduction

In this paper, we consider the following thermoviscoelastic equation which has one end fixed and output feedback control at the other end :

$$(1.1) \quad u_{tt} - M\left(\int_0^L |u_x|^2 dx\right)u_{xx} + \alpha\theta_x - \beta u_{xxt} + g(u_t) = 0 \text{ in } (0, L) \times (0, \infty),$$

$$(1.2) \quad \theta_t - k\theta_{xx} + \alpha u_{xt} = 0 \text{ in } (0, L) \times (0, \infty),$$

$$(1.3) \quad u(0, t) = \theta(0, t) = \theta(L, t) = 0 \text{ for } t > 0,$$

$$(1.4) \quad -M\left(\int_0^L |u_x|^2 dx\right)u_x(L, t) - \beta u_{xt}(L, t) = v(t) \text{ for } t \geq 0,$$

$$(1.5) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \text{ for } x \in [0, L],$$

$$(1.6) \quad u_{out}(t) = u_t(L, t),$$

where $u = u(x, t)$ and $\theta = \theta(x, t)$ denote the displacement and the temperature, respectively, $\alpha > 0$, $\beta > 0$, $k > 0$, M is a function satisfying some conditions, $v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the boundary control force applied at the free end of thermoviscoelastic body and $u_{out}(t)$ stands for the measured signal of the system at time t . System (1.1)-(1.6) describes the transverse vibration of an extensible Timoshenko clamped at $x = 0$ and supported at $x = L$ by a control force. The advantage of

Received December 26, 2015; accepted May 28, 2016.

2010 Mathematics Subject Classification: 35B40, 35L75, 93D20, 93B52, 35Q79.

Key words and phrases: thermoviscoelasticity, decay estimates, output feedback control.

the adaptive stabilization is that stabilization and good control performance can be automatically achieved even in the presence of various types of uncertainty. The boundary stabilization and boundary controllability for beams were considered by many authors [5, 3, 7]. The nonlinear boundary stabilization can be found in [1, 4]. Guos [1] considered the adaptive stabilization for a Kirchhoff-type nonlinear beam under boundary output feedback control. In [2], the exponential stability of a semi-linear wave equation with variable coefficients under the nonlinear boundary feedback was investigated. Park et al. [4] studied the existence and exponential stability for a Euler-Bernoulli beam equation with memory and boundary output feedback control term. Moreover, Nakao [6] studied the contact problem in thermoviscoelastic materials. Motivated above papers, we prove the existence and exponential stability for a thermoviscoelastic equation with boundary output feedback control term. To this end, we design the following adaptive output feedback controller:

$$(1.7) \quad v(t) = h(t)u_t(L, t),$$

$$(1.8) \quad h_t(t) = ru_t^2(L, t), \quad h(0) = h_0 > 0, \quad r > 0.$$

Then the closed-loop system of (1.1)-(1.6) is given by

$$(1.9) \quad u_{tt} - M\left(\int_0^L |u_x|^2 dx\right)u_{xx} + \alpha\theta_x - \beta u_{xxt} + g(u_t) = 0 \text{ in } (0, L) \times (0, \infty),$$

$$(1.10) \quad \theta_t - k\theta_{xx} + \alpha u_{xt} = 0 \text{ in } (0, L) \times (0, \infty),$$

$$(1.11) \quad u(0, t) = \theta(0, t) = \theta(L, t) = 0 \text{ for } t > 0,$$

$$(1.12) \quad -M\left(\int_0^L |u_x|^2 dx\right)u_x(L, t) - \beta u_{xt}(L, t) = h(t)u_t(L, t) \text{ for } t \geq 0,$$

$$(1.13) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \text{ for } x \in [0, L],$$

$$(1.14) \quad h_t(t) = ru_t^2(L, t) \text{ for } t \geq 0, \quad h(0) = h_0 > 0, \quad r > 0.$$

The energy of the system (1.9)-(1.14) is given by

$$(1.15) \quad E(t) = \frac{1}{2} \int_0^L |u_t|^2 dx + \frac{1}{2} \int_0^L |\theta|^2 dx + \frac{1}{2} \hat{M} \left(\int_0^L |u_x|^2 dx \right),$$

where $\hat{M}(s) = \int_0^s M(\sigma) d\sigma$.

2. Existence of Solutions

Let $L^2(0, L)$ be the usual Hilbert space with the inner product (\cdot, \cdot) and the inner product induced norm $\|\cdot\|$. Throughout this paper, we define

$$V = \{u \in H^1(0, L) : u(0) = 0\}.$$

Let $\lambda > 0$ be a constant such that $\|u\|^2 \leq \lambda \|\nabla u\|^2$ for all $u \in V$. We state the following hypotheses :

(H₁) Let $(u_0, u_1, \theta_0) \in (V \cap H^2(0, L)) \times V \times (H_0^1(0, L) \cap H^2(0, L))$ and

$$(2.1) \quad -M\left(\int_0^L |u_x(0)|^2 dx\right)u_x(L, 0) - \beta u_{xt}(L, 0) = h_0 u_1(L).$$

(H₂) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differential function and there exist positive constants μ_1 and μ_2 such that $g(s)s \geq 0$ and $\mu_1|s| \leq |g(s)| \leq \mu_2|s| \quad \forall s \in \mathbb{R}$.

(H₃) $M \in C([0, \infty)) \cap C^1(0, \infty)$, $M(s) \geq \gamma$ for some $\gamma > 0$ and $M(s) \leq M(s)s$.

(H₄) We assume that $\alpha < 4k$ and $\gamma > \frac{\alpha\lambda}{2} + \beta$.

Theorem 2.1. *Suppose that (H₁), (H₂) and (H₃) are satisfied. Then, there exists a solution to the system (1.9)-(1.14) satisfying*

$$u \in L^\infty(0, T; V), u_t \in L^\infty(0, T; V), u_{tt} \in L^2(0, T; V),$$

$$\theta \in L^2(0, T; H_0^1(0, L)), \theta_t \in L^2(0, T; H_0^1(0, L)).$$

Proof. We will use the Galerkin method. Let us denote by w_1, w_2, \dots, w_m a basis for $V \cap H^2(0, L)$ and for v_1, v_2, \dots, v_m a basis for $H_0^1(0, L) \cap H^2(0, L)$. Let us define

$$u^m(x, t) = \sum_{i=1}^m \chi_{im}(t)w_i(x), \quad \theta^m(x, t) = \sum_{i=1}^m \eta_{im}(t)v_i(x).$$

We have to find the coefficients χ_{im} and η_{im} satisfying the system

$$(2.2) \quad \begin{aligned} &(u_{tt}^m, w_i) + M(\|u_x^m\|^2)(u_x^m, w_{ix}) - \alpha(\theta^m, w_{ix}) + \beta(u_{xt}^m, w_{ix}) \\ &+ (g(u_t^m), w_i) + h^m(t)u_t^m(L, t)w_i(L) = 0, \end{aligned}$$

$$(2.3) \quad (\theta_t^m, v_i) + k(\theta_x^m, v_{ix}) - \alpha(u_t^m, v_{ix}) = 0,$$

$$(2.4) \quad h_t^m(t) = r\left[\sum_{i=1}^m \chi'_{im}(L, t)w_i(L)\right]^2 = r[u_t^m(L, t)]^2, \quad h^m(0) = h_0 > 0,$$

$$(2.5) \quad u^m(0) = u_0^m \rightarrow u_0 \text{ in } V \cap H^2(0, L), \quad u_t^m(0) = u_1^m \rightarrow u_1 \text{ in } V,$$

$$(2.6) \quad \theta^m(0) = \theta_0^m \rightarrow \theta_0 \text{ in } H_0^1(0, L) \cap H^2(0, L).$$

By standard methods in differential equations, we can prove the existence of a local solution to (2.2)-(2.6) on some interval $[0, t_m]$, where $t_m = \infty$ by using the first estimate below.

Estimate I. Multiplying Eq. (2.2) by χ'_{im} and taking summation on i and using (2.4), we have

$$(2.7) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \{ \|u_t^m\|^2 + \hat{M}(\|u_x^m\|^2) + \frac{1}{r} |h^m(t)|^2 \} + (g(u_t^m), u_t^m) \\ &+ \alpha(\theta_x^m, u_t^m) + \beta \|u_{xt}^m\|^2 = 0. \end{aligned}$$

Now, multiplying Eq. (2.3) by η'_{im} and summation on i , we see that

$$(2.8) \quad \frac{1}{2} \frac{d}{dt} \|\theta^m\|^2 + k \|\theta_x^m\|^2 - \alpha(\theta_x^m, u_t^m) = 0.$$

Combining (2.7) and (2.8) and by hypothesis (H_2) , we derive

$$(2.9) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ \|u_t^m\|^2 + \hat{M}(\|u_x^m\|^2) + \|\theta^m\|^2 + \frac{1}{r} |h^m(t)|^2 \} \\ & + \mu \|u_t^m\|^2 + \beta \|u_{xt}^m\|^2 + k \|\theta_x^m\|^2 \leq 0. \end{aligned}$$

Integrating (2.9) over $(0, t)$ and using (2.5), (2.6), we get

$$(2.10) \quad \begin{aligned} & \|u_t^m\|^2 + \hat{M}(\|u_x^m\|^2) + \|\theta^m\|^2 + \frac{1}{r} |h^m(t)|^2 \\ & + \mu \int_0^t \|u_t^m\|^2 d\tau + \beta \int_0^t \|u_{xt}^m\|^2 d\tau + k \int_0^t \|\theta_x^m\|^2 d\tau \\ & \leq \|u_1^m\|^2 + \hat{M}(\|u_{0x}^m\|^2) + \|\theta_0^m\|^2 + \frac{1}{r} |h_0^m|^2 \leq c, \end{aligned}$$

where and in the sequel, c denotes a generic positive constant.

Estimate II. First of all, we estimate the L^2 -norms of $u_{tt}^m(0)$ and $\theta_t^m(0)$. Considering $t = 0$, $w_i = u_{tt}^m(0)$ and $v_i = \theta_t^m(0)$ in (2.2) and (2.3), respectively, we obtain

$$\begin{aligned} \|u_{tt}^m(0)\|^2 & \leq M(\|u_x^m(0)\|^2) \|u_{xx}^m(0)\| \|u_{tt}^m(0)\| + \alpha \|\theta_x^m(0)\| \|u_{tt}^m(0)\| \\ & + \beta \|u_{xxt}^m(0)\| \|u_{tt}^m(0)\| + \|g(u_t^m(0))\| \|u_{tt}^m(0)\|, \\ \|\theta_t^m(0)\|^2 & \leq k \|\theta_{xx}^m(0)\| \|\theta_t^m(0)\| + \alpha \|\theta_{xx}^m(0)\| \|\theta_t^m(0)\|. \end{aligned}$$

This and (2.5)-(2.6) imply that

$$(2.11) \quad \|u_{tt}^m(0)\| \leq c, \quad \text{and} \quad \|\theta_t^m(0)\| \leq c.$$

Now, differentiating (2.2) and (2.3), writing the equations with $w_i = y_{tt}^m(t)$ and $v_i = \theta_t^m(t)$ and adding the results, we obtain

$$(2.12) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|u_{tt}^m\|^2 + M(\|u_x^m\|^2) \|u_{xt}^m\|^2 + \frac{r}{2} |u_t^m(L, t)|^4 + \|\theta_t^m\|^2 \right] \\ & + \beta \|u_{xtt}^m\|^2 + k \|\theta_{xt}^m\|^2 \\ & = -2M'(\|u_x^m\|^2) (u_x^m, u_{xtt}^m) + M'(\|u_x^m\|^2) (u_x^m, u_{xt}^m) \|u_{xt}^m\|^2 \\ & - (g'(u_t^m) u_{tt}^m, u_{tt}^m) - h^m(t) |u_{tt}^m(L, t)|^2. \end{aligned}$$

Integrating this over $(0, t)$ and making use of (2.11), (2.5)-(2.6), it follows that

$$\begin{aligned}
 & \frac{1}{2} \|u_{tt}^m\|^2 + \frac{1}{2} M(\|u_x^m\|^2) \|u_{xt}^m\|^2 + \frac{r}{4} |u_t^m(L, t)|^4 + \frac{1}{2} \|\theta_t^m\|^2 \\
 & \quad + \beta \int_0^t \|u_{xtt}^m\|^2 ds + k \int_0^t \|\theta_{xt}^m\|^2 ds \\
 (2.13) \quad & \leq c - 2 \int_0^t M'(\|u_x^m\|^2) (u_x^m, u_{xtt}^m) ds + \int_0^t M'(\|u_x^m\|^2) (u_x^m, u_{xt}^m) \|u_{xt}^m\|^2 ds \\
 & \quad - \int_0^t (g'(u_t^m) u_{tt}^m, u_{tt}^m) ds - \int_0^t h^m(s) |u_{tt}^m(L, s)|^2 ds.
 \end{aligned}$$

Using the conditions $(H_2) - (H_3)$ and the first estimate (2.10), we see that

$$\begin{aligned}
 & \left| -2 \int_0^t M'(\|u_x^m\|^2) (u_x^m, u_{xtt}^m) ds \right| \leq c \int_0^t \|u_{xt}^m\|^2 \|u_{xtt}^m\|^2 ds \\
 & \leq \frac{\beta}{2} \int_0^t \|u_{xtt}^m\|^2 ds + C_\beta \int_0^t \|u_{xt}^m\|^2 ds,
 \end{aligned}$$

$$\left| \int_0^t M'(\|u_x^m\|^2) (u_x^m, u_{xt}^m) \|u_{xt}^m\|^2 ds \right| \leq c \int_0^t \|u_{xt}^m\| \|u_{xt}^m\|^2 ds$$

and

$$\left| \int_0^t (g'(u_t^m) u_{tt}^m, u_{tt}^m) ds \right| \leq c \int_0^t \|u_{tt}^m\|^2 ds.$$

Adapting these estimates to (2.13) and noting that $\int_0^t h^m(s) |u_{tt}^m(L, s)|^2 ds > 0$, we have

$$\begin{aligned}
 (2.14) \quad & \frac{1}{2} \|u_{tt}^m\|^2 + \frac{1}{2} M(\|u_x^m\|^2) \|u_{xt}^m\|^2 + \frac{r}{4} |u_t^m(L, t)|^4 + \frac{1}{2} \|\theta_t^m\|^2 \\
 & + \frac{\beta}{2} \int_0^t \|u_{xtt}^m\|^2 ds + k \int_0^t \|\theta_{xt}^m\|^2 ds \leq c + c \int_0^t \|u_{tt}^m\|^2 ds \\
 & + c \int_0^t (1 + \|u_{xt}^m\|) \|u_{xt}^m\|^2 ds.
 \end{aligned}$$

Applying Gronwall's Lemma, we have

$$(2.15) \quad \|u_{tt}^m\|^2 + \|u_{xt}^m\|^2 + |u_t^m(L, t)|^4 + \|\theta_t^m\|^2 + \int_0^t \|u_{xtt}^m\|^2 ds + \int_0^t \|\theta_{xt}^m\|^2 ds \leq c.$$

From (2.10) and (2.15), there exist subsequence of (u^m) and (θ^m) , still denoted

by (u^m) and (θ^m) , such that

$$(2.16) \quad \begin{cases} u^m \rightarrow u \text{ weakly star in } L^\infty(0, T; V), \\ u_t^m \rightarrow u_t \text{ weakly star in } L^\infty(0, T; V), \\ u_{tt}^m \rightarrow u_{tt} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V), \\ u_t^m(L, t) \rightarrow u_t(L, t) \text{ weakly in } L^2(0, T), \\ h^m \rightarrow h \text{ weakly star in } L^\infty(0, T), \\ h_t^m \rightarrow h_t \text{ weakly star in } L^\infty(0, T), \\ \theta^m \rightarrow \theta \text{ weakly in } L^2(0, T; H_0^1(0, L)), \\ \theta_t^m \rightarrow \theta_t \text{ weakly in } L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H_0^1(0, L)). \end{cases}$$

Due to the compact embedding $V \hookrightarrow L^2(0, L)$, we have

$$u_t^m \rightarrow u_t \text{ strongly in } L^2(0, T; L^2(0, L)).$$

Thus, we get from (H_2) that

$$g(u_t^m) \rightarrow g(u_t) \text{ a.e. in } x \in (0, L), \quad t > 0.$$

From the above convergence and the boundedness of $(g(u_t^m))$ in $L^2(0, T; L^2(0, L))$, we conclude by Lion's lemma that

$$g(u_t^m) \rightarrow g(u_t) \text{ weakly in } L^2(0, T; L^2(0, L)).$$

Moreover, by the Sobolev embedding theorem and (2.16), we see that

$$h \in C^1[0, T] \quad \text{and} \quad h^m(t)u_t^m(L, t) \rightarrow h(t)u_t(L, t) \text{ weakly in } L^2(0, T).$$

From (2.16) and Arzela-Ascoli theorem, there exists a continuous function ϕ such that

$$\|u_x^m\|^2 \rightarrow \phi \quad \text{uniformly in } [0, T].$$

Thus, letting limit $m \rightarrow \infty$ in (2.2) and substituting $w_i = u$, we get

$$(2.17) \quad \begin{aligned} \int_0^T M(\phi)\|u_x\|^2 ds &= - \int_0^T (u_{tt}, u) ds - \alpha \int_0^T (\theta_x, u) ds \\ &\quad - \frac{\beta}{2}\|u_x(T)\|^2 + \frac{\beta}{2}\|u_x(0)\|^2 - \int_0^T (g(u_t), u) ds - \int_0^T h(s)u_t(L, s)u(L, s) ds. \end{aligned}$$

On the other hand, we have

$$(2.18) \quad \begin{aligned} \int_0^T M(\|u_x^m\|^2)\|u_x^m\|^2 ds &= - \int_0^T (u_{tt}^m, u^m) ds - \alpha \int_0^T (\theta_x^m, u^m) ds - \frac{\beta}{2}\|u_x^m(T)\|^2 \\ &\quad + \frac{\beta}{2}\|u_x^m(0)\|^2 - \int_0^T (g(u_t^m), u^m) ds - \int_0^T h^m(s)u_t^m(L, s)u^m(L, s) ds. \end{aligned}$$

From (2.17) and (2.18), we derive that

$$(2.19) \quad \limsup_{m \rightarrow \infty} \int_0^T M(\|u_x^m\|^2) \|u_x^m\|^2 ds \leq \int_0^T M(\phi) \|u_x\|^2 ds.$$

It follows that $\sqrt{M(\|u_x^m\|^2)} \|u_x^m\|$ converges strongly in $L^2(0, T; L^2(0, L))$. Hence we conclude that u_x^m converges strongly to u_x in $L^2(0, L)$. Therefore, the above convergence are sufficient to pass to the limit in (2.2)-(2.6). Then it is a matter of routine to deduce the existence of global solutions in $[0, T]$. \square

3. Exponential Stability

Having established global existence of solution to (1.9)-(1.14), we focus our attention on exponential decay that can be obtained for the energy function. We define the energy $E(t)$ of problem (1.9)-(1.14) by

$$(3.1) \quad E(t) = \frac{1}{2} \int_0^L |u_t(t)|^2 dx + \frac{1}{2} \int_0^L |\theta(t)|^2 dx + \frac{1}{2} \hat{M} \left(\int_0^L |u_x(t)|^2 dx \right).$$

Then the derivative of the energy is given by

$$(3.2) \quad E'(t) = -\beta \|u_{xt}(t)\|^2 - k \|\theta_x(t)\|^2 - (g(u_t(t), u_t(t)) - h(t)[u_t(L, t)]^2) \leq 0.$$

Theorem 3.1. *Let (u, θ) be the solution given by Theorem 2.1 and assume that (H_4) holds. Then we have*

$$\lim_{t \rightarrow \infty} E(t) = 0, \quad \lim_{t \rightarrow \infty} h(t) \leq \sqrt{2rE(0) + [h(0)]^2}.$$

We define the perturbed energy by

$$(3.3) \quad E_\epsilon(t) = E(t) + \epsilon \psi(t),$$

where

$$(3.4) \quad \psi(t) = (\theta(t), \theta(t)) + (u_t(t), u(t)).$$

Then we have the following propositions.

Proposition 3.1. *There exists $C_1 > 0$ such that $|E_\epsilon(t) - E(t)| \leq \epsilon C_1 E(t)$, $\forall t \geq 0$ and $\epsilon > 0$.*

Proof. Using Young inequality and Sobolev embedding theorem, there exists $C_1 > 0$ such that $|E_\epsilon(t) - E(t)| \leq \epsilon |\psi(t)| \leq \epsilon C_1 E(t)$. \square

Proposition 3.2. *There exist positive constants C_2, C_3 such that*

$$(3.5) \quad \frac{d}{dt} E_\epsilon(t) \leq -\epsilon C_2 E(t) + \epsilon C_3 h(t) [u(L, t)]^2, \quad \forall t \geq 0, \quad \epsilon \in (0, \epsilon_1].$$

Proof. Using (1.1), (1.2) and (2.14), Young inequality and Sobolev embedding theorem, we deduce that

$$\begin{aligned}
 \frac{d}{dt}\psi(t) &= 2(\theta(t), \theta_t(t)) + (u(t), u_{tt}(t)) + \|u_t(t)\|^2 \\
 &= -2k\|\theta_x(t)\|^2 - 2\alpha(\theta(t), u_{xt}(t)) - M(\|u_x(t)\|^2)\|u_x(t)\|^2 - \alpha(u(t), \theta_x(t)) \\
 &\quad -\beta(u_x(t), u_{xt}(t)) - (g(u_t(t)), u(t)) - u(L, t)h(t)u_t(L, t) + \|u_t(t)\|^2 \\
 &\leq -2k\|\theta_x(t)\|^2 + \alpha\|\theta(t)\|^2 + \alpha\|u_{xt}(t)\|^2 - M(\|u_x(t)\|^2)\|u_x(t)\|^2 \\
 &\quad + \frac{\alpha\lambda}{2}\|u_x(t)\|^2 + \frac{\alpha}{2}\|\theta_x(t)\|^2 + \frac{\beta}{2}\|u_x(t)\|^2 + \frac{\beta}{2}\|u_{xt}(t)\|^2 \\
 &\quad - (g(u_t(t)), u(t)) - u(L, t)h(t)u_t(L, t) + \|u_t(t)\|^2 \\
 &\leq -(2k - \frac{\alpha}{2})\|\theta_x(t)\|^2 + \alpha\|\theta(t)\|^2 + (\alpha + \frac{\beta}{2})\|u_{xt}(t)\|^2 - M(\|u_x(t)\|^2)\|u_x(t)\|^2 \\
 (3.6) \quad &+ (\frac{\alpha\lambda}{2} + \beta)\|u_x(t)\|^2 - u(L, t)h(t)u_t(L, t) + (1 + C_\beta)\|u_t(t)\|^2,
 \end{aligned}$$

where we used

$$|(g(u_t(t)), u(t))| \leq C_\beta\|u_t(t)\|^2 + \frac{\beta}{2}\|u_x(t)\|^2,$$

where C_β is a positive constant depending on β . On the other hand, the assumption (H_3) gives

$$\|u_x(t)\|^2 \leq \frac{1}{\gamma}M(\|u_x(t)\|^2)\|u_x(t)\|^2.$$

Substituting this to (3.6) and keeping in mind (H_4) , we obtain

$$\begin{aligned}
 (3.7) \quad \frac{d}{dt}\psi(t) &\leq -2c_1E(t) - (2k - \frac{\alpha}{2})\|\theta_x(t)\|^2 + c_2\|\theta(t)\|^2 \\
 &\quad + c_3\|u_{xt}(t)\|^2 + c_4\|u_t(t)\|^2 - u(L, t)h(t)u_t(L, t),
 \end{aligned}$$

where $c_1 = 1 - \frac{1}{\gamma}(\frac{\alpha\lambda}{2} + \beta) > 0$, $c_2 = \alpha + c_1$, $c_3 = \alpha + \frac{\beta}{2}$, $c_4 = 1 + c_1 + C_\beta$.

Thus, we summarize from (3.1)-(3.3) and (3.7) that

$$\begin{aligned}
 (3.8) \quad \frac{d}{dt}E_\epsilon(t) &= \frac{d}{dt}E(t) + \epsilon \frac{d}{dt}\psi(t) \\
 &\leq -2c_1\epsilon E(t) - (\mu_1 - c_4\epsilon)\|u_t(t)\|^2 - (\beta - c_3\epsilon)\|u_{xt}(t)\|^2 \\
 &\quad - \{k + (2k - \frac{\alpha}{2})\epsilon - c_2\lambda\epsilon\}\|\theta_x(t)\|^2 - h(t)(1 - \frac{\epsilon}{2})(u_t(L, t))^2 + \frac{\epsilon}{2}h(t)(u(L, t))^2.
 \end{aligned}$$

Now, we define

$$(3.9) \quad \epsilon_1 = \min \left\{ \frac{\mu_1}{c_4}, \frac{\beta}{c_3}, \frac{k}{c_2\lambda}, 2 \right\},$$

and considering $\epsilon \in [0, \epsilon_1]$, then from (3.8) and (3.9), we get

$$(3.10) \quad \frac{d}{dt}E_\epsilon(t) \leq -c_1\epsilon E(t) + \frac{\epsilon}{2}h(t)(u(L, t))^2, \quad \forall t \geq 0.$$

This ends the proof of Proposition 3.2. \square

Proof of Theorem 3.1. From Proposition 3.1, we see that

$$(3.11) \quad (1 - \epsilon C_1)E(t) \leq E_\epsilon(t) \leq (1 + \epsilon C_1)E(t), \quad \forall t \geq 0.$$

By (3.10) and (3.11), we have

$$(3.12) \quad \frac{d}{dt}E_\epsilon(t) \leq \frac{-C_1\epsilon}{1+C_1\epsilon}E_\epsilon(t) + \frac{\epsilon}{2}h(t)(u(L,t))^2, \quad \forall t \geq 0.$$

Let $C_\epsilon = \frac{C_1\epsilon}{1+C_1\epsilon}$ and apply Gronwall's inequality to (3.12), we obtain

$$(3.13) \quad \begin{aligned} E_\epsilon(t) &\leq e^{-C_\epsilon t}E_\epsilon(0) + \frac{\epsilon}{2} \int_0^t e^{-C_\epsilon(t-\tau)} h(\tau)(u(L,\tau))^2 d\tau \\ &\leq e^{-C_\epsilon t}E_\epsilon(0) + \frac{\epsilon}{2} \sup_{t \geq 0} |h(t)| \int_0^t e^{-C_\epsilon(t-\tau)} (u(L,\tau))^2 d\tau, \quad \forall t \geq 0. \end{aligned}$$

Referring to the paper [1], we deduce that

$$\int_0^t e^{-C_\epsilon(t-\tau)} (u(L,\tau))^2 d\tau \leq e^{-C_\epsilon t/2} \int_0^\infty (u(L,\tau))^2 d\tau, \quad \forall t \geq 0.$$

By Theorem 2.1 and $H_0^1(0, L) \cap H^2(0, L) \hookrightarrow L^2(0, L)$, $u(L, t) \in L^2(0, \infty)$ we see that

$$(3.14) \quad \int_0^t e^{-C_\epsilon(t-\tau)} (u(L,\tau))^2 d\tau \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

From (3.13) and (3.14), we get $\lim_{t \rightarrow \infty} E_\epsilon(t) = 0$. Let $\epsilon_0 = \min\{\epsilon_1, \frac{1}{2C_1}\}$, where C_1 is given in Proposition 3.1. Consider $\epsilon \in (0, \epsilon_0]$. Since $\epsilon \leq \frac{1}{2C_1}$ and (3.11), we obtain

$$(3.15) \quad \frac{1}{2}E(t) \leq E_\epsilon(t) \leq \frac{3}{2}E(t), \quad \forall t \geq 0.$$

Therefore, we have

$$(3.16) \quad \lim_{t \rightarrow \infty} E(t) = 0.$$

Now, we consider the Lyapunov functional $U(t) = E(t) + \frac{h^2(t)}{2r}$. Then, by (3.2), we see that

$$(3.17) \quad U_t(t) \leq 0.$$

Therefore we obtain

$$\sup_{t \geq 0} \left\{ E(t) + \frac{h^2(t)}{2r} \right\} \leq M_3,$$

where $M_3 > 0$ is a constant depending on the initial data. From (3.17), we deduce that

$$E(\infty) + \frac{h^2(\infty)}{2r} \leq E(0) + \frac{h^2(0)}{2r}.$$

Thus, we have

$$h(\infty) \leq \sqrt{2rE(0) + (h(0))^2}.$$

Since $h(t)$ is nondecreasing, we obtain

$$(3.18) \quad h(t) \leq \sqrt{2rE(0) + (h(0))^2}.$$

The proof of Theorem 3.1 is completed. \square

We can now proceed to state our exponential stability result.

Theorem 3.2 *Let (u, θ) be the solution of Theorem 2.1, then there exist constants $\kappa > 0$ and $\nu > 0$ depending on the initial data such that*

$$E(t) \leq \kappa e^{-\nu t}, \quad \forall t \geq 0.$$

Proof. From (3.12), we have

$$(3.19) \quad \frac{d}{dt} E_\epsilon(t) \leq -C_\epsilon E_\epsilon(t) + \frac{\epsilon}{2} h(t)(u(L, t))^2, \quad \forall t \geq 0.$$

By integrating over $(0, t)$ in (3.19) and using (3.18), we obtain

$$(3.20) \quad \begin{aligned} E_\epsilon(t) &\leq E_\epsilon(0) - C_\epsilon \int_0^t E_\epsilon(\tau) d\tau + \frac{\epsilon}{2} \int_0^t h(\tau)(u(L, \tau))^2 d\tau \\ &\leq E_\epsilon(0) - C_\epsilon \int_0^t E_\epsilon(\tau) d\tau + \frac{\epsilon}{2} \|h\|_{L^\infty(0, \infty)} \int_0^t (u(L, \tau))^2 d\tau. \end{aligned}$$

Using Gronwall's inequality and since $\int_0^\infty (u(L, \tau))^2 d\tau \leq M$, we get

$$(3.21) \quad E_\epsilon(t) \leq (k_1 + E_\epsilon(0))e^{-C_\epsilon t}, \quad \forall t \geq 0.$$

For sufficiently small ϵ , using the Proposition 3.1, we get

$$(3.22) \quad E(t) \leq \frac{1}{1 - \epsilon C_1} E_\epsilon(t) \leq \frac{k_1 + E_\epsilon(0)}{1 - \epsilon C_1} e^{-\nu t}, \quad \forall t \geq 0.$$

This completes the proof of Theorem 3.2 by putting $\kappa = \frac{k_1 + E_\epsilon(0)}{1 - \epsilon C_1}$ and $\nu = \epsilon C_1$. \square

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