KYUNGPOOK Math. J. 56(2016), 639-645
http://dx.doi.org/10.5666/KMJ.2016.56.2.639
pISSN 1225-6951 eISSN 0454-8124

Erratum to 'Some Limits of the Colored Jones Polynomials of the Figure-eight Knot'

Hitoshi Murakami
Graduate School of Information Sciences, Tohoku University, Aramaki-aza-Aoba 6-3-09, Aoba-ku, Sendai 980-8579, Japan
e-mail : starshea@tky3.3web.ne.jp

Abstract. In [3] the main theorem was erroneously stated. We needed to assume that the irrationality measure of $1/r$ is finite to prove the theorem.

The statement of Theorem 1.2 in [3] should be as follows:

Theorem 1.2.([3]) *Let r be a real number satifying $5/6 < r < 7/6$. We assume that the irrationality measure of $1/r$ is finite. Then*

$$2\pi \limsup_{N\to\infty} \frac{\log\left|J_N\left(E;\exp(2\pi r\sqrt{-1}/N)\right)\right|}{N} = \frac{2\Lambda\bigl(\pi r+\theta(r)/2\bigr)-2\Lambda\bigl(\pi r-\theta(r)/2\bigr)}{r}.$$

Moreover if r is irrational or $r=1$, then

$$2\pi \lim_{N\to\infty} \frac{\log\left|J_N\left(E;\exp(2\pi r\sqrt{-1}/N)\right)\right|}{N} = \frac{2\Lambda\bigl(\pi r+\theta(r)/2\bigr)-2\Lambda\bigl(\pi r-\theta(r)/2\bigr)}{r},$$

and if $r\neq 1$ and rational, then

$$2\pi \liminf_{N\to\infty} \frac{\log\left|J_N\left(E;\exp(2\pi r\sqrt{-1}/N)\right)\right|}{N} = 0.$$

We also need to add the same codition 'the irrationality measure of $1/r$ is finite' to Proporision 7.1.

Remark. It can be proved that $\mu(1/r)=\mu(r)$. The author thanks Y. Tachiya for pointing this out.

Received May 6, 2016.
2010 Mathematics Subject Classification: Primary 57M27, Secondary 57M25.
Key words and phrases: colored Jones polynomial, knot, figure-eight knot, volume conjecture, Lobachevski function, cone-manifold.
This work was supported by JSPS KAKENHI Grant Number 26400079.

In the following I prove Theorem 1.2 above assuming the finiteness of the irrationality measure of $1/r$.

Put $B := \frac{N(1-r)}{r}$ and $D := \frac{N(2\pi-\theta(r))}{2\pi r}$ with $\theta(r) := \arccos(\cos(2\pi r) - 1/2)$. Here arccos takes its value between 0 and π. Note that $0 \le B < D < 1$ and $g(B) = 0$. The following equality may not hold when $1/r$ has infinite irrationality measure:

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{\lfloor D \rfloor} \log|2\sin(\pi rj/N + \pi r)| = \frac{1}{\pi r} \int_{\pi r}^{\pi-\theta(r)/2+\pi r} \log|2\sin x|\,dx.$$

However, the equatity does hold when the irrationality measure of r is finite. Here the irrationality measure is defined as follows. See for example [2, Definition 9.6, p. 141].

Definition 1. Let α be a real number. The irrationality measure (or the irrationality exponent) $\mu(\alpha)$ is defined to be the infimum of μ such that there exists a constant $C > 0$ with $\left|\alpha - \dfrac{p}{q}\right| \ge \dfrac{C}{q^{\mu}}$ for any rational number $\dfrac{p}{q}$ with $q > 0$.

Note that $\mu(\alpha) = 1$ when α is rational, and that $\mu(\alpha) \ge 2$ if α is irrational. Note also that with respect to the Lebesgue measure, almost all real numbers have irrationallity measure 2 [1, Theorem E.3].

We first prove a couple of lemmas. Put $h(x) := \log|2\sin x|$.

Lemma 1. *Suppose that* $5/6 < r < 1$. *Then we have*

$$\lim_{N\to\infty} \frac{1}{N} h\left(\frac{\pi r(\lfloor B \rfloor - 1)}{N} + \pi r\right) = \lim_{N\to\infty} \frac{1}{N} h\left(\frac{\pi r(\lfloor B \rfloor + 2)}{N} + \pi r\right) = 0.$$

Proof. Since $N/r - 1 < \lfloor N/r \rfloor \le N/r$, we have

$$\pi - \frac{2\pi r}{N} < \frac{\pi r}{N}\left(\left\lfloor \frac{N}{r} \right\rfloor - 1\right) \le \pi - \frac{\pi r}{N}$$

and

$$\pi + \frac{\pi r}{N} < \frac{\pi r}{N}\left(\left\lfloor \frac{N}{r} \right\rfloor + 2\right) \le \pi + \frac{2\pi r}{N}.$$

Since $B = \frac{N(1-r)}{r}$ and $r < 1$, we have

$$\frac{\pi r}{N}\lfloor B \rfloor + \pi r = \frac{\pi r}{N}\left\lfloor N\left(\frac{1}{r} - 1\right)\right\rfloor + \pi r = \frac{\pi r}{N}\left\lfloor \frac{N}{r} \right\rfloor.$$

Since $\sin x$ is decreasing for $\pi/2 < x < 3\pi/2$, we have

$$\sin\left(\pi - \frac{2\pi r}{N}\right) > \sin\left(\frac{\pi r}{N}\left(\left\lfloor \frac{N}{r} \right\rfloor - 1\right)\right) \ge \sin\left(\pi - \frac{\pi r}{N}\right)$$

and

$$\sin\left(\pi + \frac{\pi r}{N}\right) > \sin\left(\frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor + 2\right)\right) \geq \sin\left(\pi + \frac{2\pi r}{N}\right).$$

So we have

$$\sin\left(\frac{2\pi r}{N}\right) > \sin\left(\frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor - 1\right)\right) \geq \sin\left(\frac{\pi r}{N}\right)$$

and

$$\sin\left(\frac{\pi r}{N}\right) < \left|\sin\left(\frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor + 2\right)\right)\right| \leq \sin\left(\frac{2\pi r}{N}\right)$$

and the required formulas follow since

$$\lim_{N\to\infty} h(\pi r/N)/N = \lim_{N\to\infty} h(2\pi r/N)/N = 0.$$

□

Lemma 2. *Suppose that* $5/6 < r < 1$ *and that the irrationality measure of* $1/r$ *is finite. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} h\left(\frac{\pi r\lfloor B\rfloor}{N} + \pi r\right) = \lim_{N\to\infty} \frac{1}{N} h\left(\frac{\pi r(\lfloor B\rfloor + 1)}{N} + \pi r\right) = 0.$$

Proof. Let μ be the irrationality measure of $1/r$. Then from the definition of the irrationality measure, for any $\varepsilon > 0$ there exists $C > 0$ such that

$$\left|\frac{1}{r} - \frac{\lfloor N/r\rfloor}{N}\right| \geq \frac{C}{N^{\mu+\varepsilon}}. \tag{2}$$

So we have

$$\frac{N}{r} - \left\lfloor\frac{N}{r}\right\rfloor \geq \frac{CN}{N^{\mu+\varepsilon}}.$$

Since $\lfloor N/r\rfloor > N/r - 1$, we have

$$\pi - \frac{\pi r}{N} < \frac{\pi r}{N}\left\lfloor\frac{N}{r}\right\rfloor \leq \pi - \frac{C\pi r}{N^{\mu+\varepsilon}}.$$

Since $\sin x$ is decreasing when $\pi/2 < x < \pi$, we have

$$\sin\left(\pi - \frac{C\pi r}{N^{\mu+\varepsilon}}\right) \leq \sin\left(\frac{\pi r}{N}\left\lfloor\frac{N}{r}\right\rfloor\right) < \sin\left(\pi - \frac{\pi r}{N}\right)$$

and so

$$\sin\left(\frac{C\pi r}{N^{\mu+\varepsilon}}\right) \leq \sin\left(\frac{\pi r}{N}\left\lfloor\frac{N}{r}\right\rfloor\right) < \sin\left(\frac{\pi r}{N}\right).$$

Since $2x/\pi < \sin x < x$ for $0 < x < \pi/2$ we have

$$\frac{1}{N}\log\left(\frac{4Cr}{N^{\mu+\varepsilon}}\right) < \frac{1}{N}h\left(\frac{\pi r\lfloor B\rfloor}{N}+\pi r\right) < \frac{1}{N}\log\left(\frac{2\pi r}{N}\right)$$

and so we have

$$\lim_{N\to\infty}\frac{1}{N}h\left(\frac{\pi r\lfloor B\rfloor}{N}+\pi r\right)=0. \tag{3}$$

Similarly, for any $\varepsilon > 0$ there exists $C' > 0$ such that

$$\left|\frac{1}{r}-\frac{\lfloor N/r\rfloor+1}{N}\right| \geq \frac{C'}{N^{\mu+\varepsilon}}.$$

Since $\lfloor N/r\rfloor \leq N/r$, we have

$$\pi+\frac{C'\pi r}{N^{\mu+\varepsilon}} \leq \frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor+1\right) \leq \pi+\frac{\pi r}{N}.$$

Since $\sin x$ is decreasing for $\pi < x < 3\pi/2$, we have

$$\sin\left(\pi+\frac{C'\pi r}{N^{\mu+\varepsilon}}\right) \geq \sin\left(\frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor+1\right)\right) \geq \sin\left(\pi+\frac{\pi r}{N}\right)$$

and so

$$\sin\left(\frac{C'\pi r}{N^{\mu+\varepsilon}}\right) \leq \left|\sin\left(\frac{\pi r}{N}\left(\left\lfloor\frac{N}{r}\right\rfloor+1\right)\right)\right| \leq \sin\left(\frac{\pi r}{N}\right).$$

Since $2x/\pi < \sin x < x$ for $0 < x < \pi/2$ we have

$$\log\left(\frac{4C'r}{N^{\mu+\varepsilon}}\right) \leq \log 2\left|\sin\left(\frac{\pi r(\lfloor B\rfloor+1)}{N}+\pi r\right)\right| \leq \log\left(\frac{2\pi r}{N}\right).$$

Therefore we have

$$\lim_{N\to\infty}\frac{1}{N}h\left(\frac{\pi r(\lfloor B\rfloor+1)}{N}+\pi r\right)=0,$$

proving the lemma. □

Now we prove (1) assuming the finiteness of $\mu(1/r)$.

Proof of (1).

Put $h(x) := \log|2\sin x|$ and let r is an irrational number with $5/6 < r < 1$. We will show

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{\lfloor B\rfloor}h(\pi rj/N+\pi r)=\frac{1}{\pi r}\int_{\pi r}^{\pi}h(x)\,dx \tag{4}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=\lfloor B\rfloor+1}^{\lfloor D\rfloor} h(\pi rj/N+\pi r) = \frac{1}{\pi r}\int_{\pi}^{\pi-\theta(r)/2+\pi r} h(x)\,dx \tag{5}$$

First we prove (4).

Since $h(x)$ is decreasing when $\pi r < x < \pi$, we have

$$\frac{\pi r}{N}\sum_{j=1}^{\lfloor B\rfloor} h(\pi rj/N+\pi r) < \int_{\pi r}^{\pi r\lfloor B\rfloor/N+\pi r} h(x)\,dx \tag{6}$$

and

$$\int_{\pi r}^{\pi r(\lfloor B\rfloor-1)/N+\pi r} h(x)\,dx < \frac{\pi r}{N}\sum_{j=0}^{\lfloor B\rfloor-2} h(\pi rj/N+\pi r). \tag{7}$$

Since $\lfloor B\rfloor = \lfloor N/r\rfloor - N$, we have

$$\pi - \frac{\pi r}{N} < \frac{\pi r\lfloor B\rfloor}{N} + \pi r \le \pi.$$

Now we choose δ so that

$$\pi r + \frac{\pi r(\lfloor B\rfloor - 1)}{N} < \pi - \delta < \pi r + \frac{\pi r\lfloor B\rfloor}{N}.$$

Since $h(x) < 0$ when $\pi r < x < \pi$, we have

$$\int_{\pi r}^{\pi r\lfloor B\rfloor/N+\pi r} h(x)\,dx < \int_{\pi r}^{\pi-\delta} h(x)\,dx.$$

So from (6) we have

$$\frac{1}{N}\sum_{j=1}^{\lfloor B\rfloor} h(\pi rj/N+\pi r) < \frac{1}{\pi r}\int_{\pi r}^{\pi-\delta} h(x)\,dx.$$

Similarly since

$$\int_{\pi r}^{\pi-\delta} h(x)\,dx < \int_{\pi r}^{\pi r(\lfloor B\rfloor-1)/N+\pi r} h(x)\,dx,$$

we have

$$\frac{1}{\pi r}\int_{\pi r}^{\pi-\delta} h(x)\,dx < \frac{1}{N}\sum_{j=0}^{\lfloor B\rfloor-2} h(\pi rj/N+\pi r)$$

from (7).

Therefore we have

$$
\begin{aligned}
&\frac{1}{\pi r}\int_{\pi r}^{\pi-\delta} h(x)\,dx - \frac{1}{N}h(\pi r) + \frac{1}{N}h\left(\frac{\pi r\lfloor B\rfloor}{N}+\pi r\right) + \frac{1}{N}h\left(\frac{\pi r(\lfloor B\rfloor-1)}{N}+\pi r\right)\\
<&\frac{1}{N}\sum_{j=1}^{\lfloor B\rfloor} h\left(\frac{\pi r j}{N}+\pi r\right)\\
<&\frac{1}{\pi r}\int_{\pi r}^{\pi-\delta} h(x)\,dx.
\end{aligned}
$$

From Lemmas 1 and 2 we have

$$
\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{\lfloor B\rfloor} h\left(\frac{\pi r j}{N}+\pi r\right) = \frac{1}{\pi r}\int_{\pi r}^{\pi} h(x)\,dx.
$$

Next we prove (5). We choose δ so that

$$
\pi r + \frac{\pi r(\lfloor B\rfloor+1)}{N} < \pi+\delta < \pi r + \frac{\pi r(\lfloor B\rfloor+2)}{N}.
$$

Since $h(x)$ is increasing when $\pi < x < \pi+\pi r$, $h(x) < 0$ when $\pi < x < 7\pi/6$, and $h(x) > 0$ when $7\pi/6 < x < \pi+\pi r$, we have

$$
\begin{aligned}
\int_{\pi+\delta}^{\pi-\theta(r)/2+\pi r} h(x)\,dx &> \int_{\pi r(\lfloor B\rfloor+1)/N+\pi r}^{\pi-\theta(r)/2+\pi r} h(x)\,dx\\
&> \int_{\pi r(\lfloor B\rfloor+1)/N+\pi r}^{\pi r\lfloor D\rfloor/N+\pi r} h(x)\,dx\\
&> \frac{\pi r}{N}\sum_{j=\lfloor B\rfloor+1}^{\lfloor D\rfloor-1} h\left(\frac{\pi r j}{N}+\pi r\right)
\end{aligned}
$$

if N is sufficiently large.

Similarly we have

$$
\begin{aligned}
\int_{\pi+\delta}^{\pi-\theta(r)/2+\pi r} h(x)\,dx &< \int_{\pi r(\lfloor B\rfloor+2)/N+\pi r}^{\pi-\theta(r)/2+\pi r} h(x)\,dx\\
&< \int_{\pi r(\lfloor B\rfloor+2)/N+\pi r}^{\pi r(\lfloor D\rfloor+1)+\pi r} h(x)\,dx\\
&< \frac{\pi r}{N}\sum_{j=\lfloor B\rfloor+3}^{\lfloor D\rfloor+1} h\left(\frac{\pi r j}{N}+\pi r\right).
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
&\frac{1}{\pi r}\int_{\pi+\delta}^{\pi-\theta(r)/2+\pi r} h(x)\,dx - \frac{1}{N}h\left(\frac{\pi r(\lfloor D\rfloor+1)}{N}+\pi r\right) \\
&+\frac{1}{N}h\left(\frac{\pi r(\lfloor B\rfloor+1)}{N}+\pi r\right)+\frac{1}{N}h\left(\frac{\pi r(\lfloor B\rfloor+2}{N}+\pi r\right) \\
<&\frac{1}{N}\sum_{j=\lfloor B\rfloor+1}^{\lfloor D\rfloor} h\left(\frac{\pi r j}{N}+\pi r\right) \\
<&\frac{1}{\pi r}\int_{\pi+\delta}^{\pi-\theta(r)/2+\pi r} h(x)\,dx + \frac{1}{N}h\left(\frac{\pi r(\lfloor D\rfloor)}{N}+\pi r\right).
\end{aligned}$$

From Lemmas 1 and 2 we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=\lfloor B\rfloor+1}^{\lfloor D\rfloor} h\left(\frac{\pi r j}{N}+\pi r\right)=\frac{1}{\pi r}\int_{\pi+\delta}^{\pi-\theta(r)/2+\pi r} h(x)\,dx$$

since $h(x)$ is continuous near $\pi-\theta(r)+\pi r$. □

In a similar way we can prove the formula when $1<r<7/6$.

Acknowledgements. The author wishes to thank J. Cho and J. Murakami for letting him know the error. He is also grateful to N. Adachi and Y. Tachiya for helpful explanation about the irrationality measure.

References

[1] Y. Bugeaud, *Distribution modulo one and Diophantine approximation*, Cambridge Tracts in Mathematics, vol. 193, Cambridge University Press, Cambridge, 2012.

[2] D. Duverney, *Number theory*, Monographs in Number Theory, vol. 4, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010, An elementary introduction through Diophantine problems.

[3] H. Murakami, *Some limits of the colored Jones polynomials of the figure-eight knot*, Kyungpook Math. J., **44(3)**(2004), 369–383.