

# 이변량 성과척도를 가지는 시물레이션 결과 분석을 위한 BMORE 도표의 활용

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## Application of the BMORE Plot to Analyze Simulation Output Data with Bivariate Performance Measures

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### ABSTRACT

Bivariate measure of risk and error(BMORE) plot is originally designed to depict bivariate output data and related statistics obtained from a stochastic simulation such as sample mean, median, outliers, and a boundary of a certain percentile of simulation data. When compared to the static numbers, the plot has a big advantage in visualization that enables scholars and practitioners to understand the potential variability and risk in the simulation data. In this study, beyond just the construction of the plot to depict the variability of a certain system, we add a chance constraint to the plot and apply it for decision making such as checking the feasibility of systems, comparing performances of the systems on statistical background, and also analyzing the sensitivity of the problem parameters. In order to demonstrate an application of the plot, we employ an inventory management problem as an example. However, the techniques and algorithms suggested in this paper can be applied to any other problems comparing systems on bivariate performance measures with simulation/experiment results.

**Key words** : Simulation Output Analysis, Bivariate Measures, Graphical Method, Chance Constraint

### 요약

BMORE(Bivariate measure of risk and error) 도표는 본래 확률적 시물레이션에서의 이변량 성과 데이터와, 그와 관계된 표본 평균, 중앙값, 이상점, 특정 백분위 내의 결과 데이터의 범위 등과 같은 통계량들을 시각화하기 위해 설계되었으며, 정적인 정보들만 제공하는 것에 비해 사용자가 시물레이션 결과 데이터에 담긴 가변성을 더욱 직관적으로 쉽게 이해할 수 있도록 돕는다. 본 연구에서는 BMORE 도표를 단순히 한 시스템의 가변성을 시각화하기 위한 것이 아닌, 특정 확률적 제약 아래 대상 시스템들의 실행 가능성을 타진하고, 통계적 배경 아래 여러 시스템들의 성과 비교를 행하며, 매개변수들의 민감도 분석을 실시하기 위한 의사결정 도구로써 활용할 수 있음을 보이고 이를 위한 여러 방안을 제안한다. 그 활용 예시를 보이고자 본 연구에서는 간단한 재고 관리 문제를 차용하였으나, 제시된 방안들 자체는 이변량 성과 척도를 가지는 시물레이션/실험 데이터를 근거로하여 여러 시스템들을 비교하는 문제라면 어디든 사용될 수 있음을 밝혀둔다.

**주요어** : 시물레이션 결과 분석, 이변량 척도, 도해법, 확률적 제약

\* 본 논문은 2019학년도 홍익대학교 학술연구지원비 (관리번호 520150064)와, 2019년도 정부(과학기술정보통신부)의 재원으로 한국연구재단의 지원을 받아 수행된 연구임 (No. NRF-2019R1H1A2080136).

**Received:** 4 May 2020, **Revised:** 26 May 2020,

**Accepted:** 29 May 2020

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## 1. Introduction

In these days, simulation gets more attention as a methodology to understand complicated systems under various uncertainties, and thus various tools for simulation software have been developed to reflect the uncertainties. Nevertheless, results from the simulation software are

often limited to static numbers such as estimates of mean, variance, and percentiles and long-run average values that can be difficult to interpret for users without professional knowledge. Moreover, many researchers have warned that depending on the aggregated mean performance values is not desirable to understand and analyze the simulation model and outputs (Nelson, 2013; Alexopoulos&Kelton, 2017). Thus, in order to overcome such weakness, scholars and practitioners have studied or adopted the methods to visualize not only the mean performances but also systems' risks and errors more effectively and intuitively (Cohen&Cohen, 2006; Potter et al., 2006; Potter, 2010; Chambers, 2017).

Especially, along with the efforts, Nelson(2008) introduces a very intuitive and easy-to-implement plot, namely the Measure of Risk and Error (MORE) plot, for univariate simulation output data. The MORE plot shows not only the sample mean and percentiles but also their confidence intervals based on the histogram of the output data. With the MORE plot, users can grasp fundamental ideas about risks and errors inherent in the simulated system instantly. A commercial simulation software, SIMIO, adopts the plot and provides options to draw the MORE plot with simulation results from multiple across-replications (Kelton et al., 2014).

For bivariate simulation output data, Lee&Park (2015) introduce the bivariate-MORE (BMORE) plot as an extension of the MORE plot. The BMORE plot depicts sample mean and median, minimum/maximum values, outliers, and boundary of a certain percentile of the data that are close to the median in the Cartesian coordinate. The plot also provides confidence region of the true mean and confidence region of the percentile to show how uncertain the estimates of mean and percentile are. By plotting the simulation outputs under two performance measures simultaneously, the BMORE plot helps users to notice the skewness and the correlation among the measures as well.

The BMORE plot is an effective tool to visualize risk and error measures for bivariate data. However, Lee&Park(2015) put their emphasis just on the construction of the BMORE plot instead of the discussion about its applications. Thus, beyond the construction,

we focus on the application aspects of the BMORE plot and study how the plot can be used for i) analysing risk and errors in simulated systems, ii) comparing systems under bivariate measures, and iii) testing the sensitivity of parameters in this paper. In Section 2, we review the BMORE plot construction method from Lee&Park(2015) briefly and discuss what kinds of useful information can be derived from the plot. In Section 3, we define an example problem and use the BMORE plot to check the feasibility of the simulated systems for a chance constraint, compare the performances of the systems, and suggest algorithms for sensitivity analysis of problem parameters. Our concluding remark is shown in Section 4.

## 2. The BMORE Plot

The construction of the BMORE plot is started from drawing a scatter plot of simulation output data. When the desired number of bivariate observations are obtained by simulation, every observation is plotted as a point on a scatter plot in two dimensional space. Each axis of the scatter plot represents each performance measure. Then, the BMORE plot marks the minimum/maximum values for each measure, sample mean, median, outliers,  $\beta$ -bag, and confidence regions of the true mean and the  $\beta$ -bag on the scatter plot. Since it is trivial to find minimum/maximum values for each performance measure, only the other components of the BMORE plot are represented here as follows.

■ *Sample mean*: Let  $X$  and  $Y$  be the two variables representing two performance measures that we are interested in. Define bivariate vectors  $\mathbf{Z} = (X, Y)$  and  $\mathbf{Z}_i = (X_i, Y_i)$ , representing their simulation observation of  $\mathbf{Z}$ . When  $n$  is the number of across-replications of the simulation,  $\mathbf{Z}_i$  for  $i=1, 2, \dots, n$  are independent and identically distributed (i.i.d.). Then, a sample mean vector is defined as  $\bar{\mathbf{Z}}_n = \left( \sum_{i=1}^n X_i/n, \sum_{i=1}^n Y_i/n \right)^T$ .

■ *Confidence region of the true mean*: Let  $\boldsymbol{\mu} = (\mu_X, \mu_Y)^T$  define a true mean vector where  $\mu_X$  and  $\mu_Y$  are the unknown true mean values of  $X$  and  $Y$

respectively. Regardless of the normality of  $Z_i$ ,  $\bar{Z}_n$  follows an approximately bivariate normal distribution by the central limit theorem if  $n$  is large enough. Therefore, the Hotelling's statistics can be used to construct a confidence region of the true mean of. Then, where  $\alpha$  is a significance level,  $\Sigma$  is the variance-covariance matrix of  $Z_i$ , and  $F_{(2,n-2)}(\alpha)$  is the  $1-\alpha$  quantile of the F distribution value with and 2 and  $n-2$  degrees of freedom,

$$(\bar{Z}_n - \mu)^T \Sigma^{-1} (\bar{Z}_n - \mu) \leq \frac{2(n-1)}{n(n-2)} F_{(2,n-2)}(\alpha) \quad (1)$$

from Mason&Young(2002). With the calculated, we can construct  $100 \times (1-\alpha)\%$  confidence region of the true mean by considering all possible  $\mu$  satisfying Eq. (1).

■  *$\beta$ -bag*: Based on the bag plot from Rousseeuw et al.(1999), the  $\beta$ -bag is designed to contain a certain percentage (i.e.,  $100 \times \beta\%$  of data points near the median. In order to draw the  $\beta$ -bag, the location depth function value  $ldepth(\theta, \Omega)$  for some point  $\theta \in R^2$  relative to  $\Omega = \{Z_1, Z_2, \dots, Z_n\}$  should be calculated. (See Tukey(1975) and Rousseeuw&Ruts(1996) for details about the location depth function.) For a non-negative integer  $k$ , let  $A_k$  be a set of  $\theta$  having  $ldepth(\theta, \Omega) \geq k$ , and let  $\#A_k$  be the number of simulation observations in  $A_k$ . A  $\beta$ -bag for given bivariate observations is constructed through the following two steps: (i) Find  $k > 0$  such that  $\#A_k \leq \lfloor n\beta \rfloor \leq \#A_{k-1}$ , (ii) Linearly interpolate between the points in  $A_k$  and  $A_{k-1}$ . Note that a  $\beta$ -bag is necessarily a convex polyhedron by its definition.

■ *Confidence region of  $\beta$ -bag*: A confidence region of a  $\beta$ -bag consists of inner and outer boundaries. Let  $\beta_L$  and  $\beta_U$  denote lower and upper bounds of  $\beta$  based on a significance level  $\alpha$  and  $n$  observations, then  $\beta_L$  and is  $\beta_U$  approximated by the equations from Nelson(2008):

$$\beta_L = \beta - Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\beta(1-\beta)}{n-1}} \quad (2)$$

$$\beta_U = \beta + Z_{1-\frac{\alpha}{2}} \sqrt{\frac{\beta(1-\beta)}{n-1}} \quad (3)$$

where  $Z_{1-\frac{\alpha}{2}}$  is the  $1-\frac{\alpha}{2}$  quantile of the standard normal distribution. By drawing the  $\beta_L$ -bag and  $\beta_U$ -bag, we can obtain inner and outer boundaries of the  $100 \times (1-\alpha)\%$  confidence region of the  $\beta$ -bag, respectively.

■ *Median*: The median is defined as the  $\theta$  with the highest location depth function value if such  $\theta$  is unique. If there are multiple  $\theta$ s with the same highest location depth, the geometric center of the  $\theta$ s is selected as the median.

■ *Outlier*: Enlarge the 0.5-bag by a scale factor 3 relative to the median. Any points outside of the boundary are considered as outliers.

In order to discuss the information that can be delivered by the BMORE plot, we provide an example of the BMORE plot with 50 bivariate simulation observations in Fig. 1. Based on the plot in Fig. 1, one may capture the variability of simulation data immediately and recognize the positive correlation between two performance measures considered. In Fig. 1, plus and white circle symbols represent the sample mean and the sample median, respectively. From the difference between locations of the sample mean and median, one can conjecture how much the distribution of the data is skewed to what direction. Outliers are marked with star symbol and minimum/maximum values for each measure can be read from the dotted lines.

For the convenience for finding outliers, we set our  $\beta=0.5$  for this example and draw the 0.5-bag by using a bold line. If the area of  $\beta$ -bag is large, it means that the variability of the simulation observation data is likely to be large, and thus users can perceive the future risk regarding two performance measures. The region in light grey around the sample mean represents the approximated 95% confidence region of the true mean, and the region in dark grey represents the approximated 95% confidence region of the 0.5-bag. Obviously, the wider width of the confidence region of the 0.5-bag means the larger variability of the 0.5-bag. In Fig. 1,

the confidence regions of the mean and the 0.5-bag are overlapped and it may imply that there is some risk to make a decision related to the mean and 0.5-bag only with current 50 observations. In order to avoid the risk and increase the accuracy of the estimates of the mean and 0.5-bag, users may want to obtain additional observations and then generate the BMORE plot newly.

Since the BMORE plot is intuitive and can be used for any simulation analysis that handles two performance measures, its application is not restricted to a certain fields of study or users with professional knowledge. Some possible problems to apply the plot can be found easily. For example, the three buffer allocation problem considering cycle time and throughput as its performance measures (Buzacott& Shanthikumar, 1993) the maintenance optimization problem dealing with the costs for preventive and corrective maintenances (Belyi&Damien, 2009), and typical  $\{s, S\}$  inventory policy problems whose performance relates to ordering and inventory holding costs.

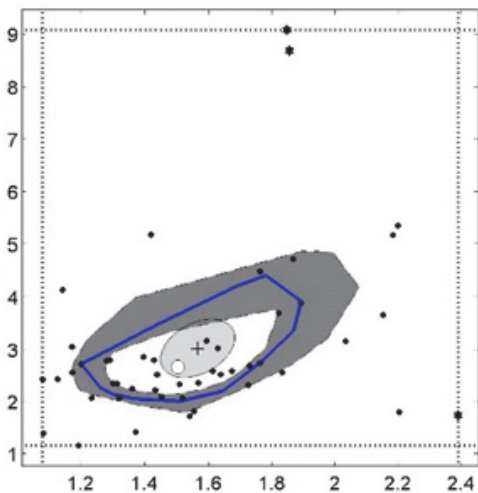


Fig. 1. An example of the BMORE plot (Lee&Park, 2015).

### 3. Application and Analysis

The various symbols and lines marked in the BMORE plot deliver useful information about the central tendency, skewness and variability of the data, while the confidence regions show how accurate the information can be. In

an aspect of visualizing possible risk and error measures inherent in simulation data, the BMORE plot seems to play a good role enough already. However, it is not all we can perform with the BMORE plot. The BMORE plot can be utilized for valuable analyses such as checking if a simulated system is feasible to meet a chance constraint, comparing systems' performances, and doing sensitivity analysis for the parameters. In order to demonstrate such analyses throughout Section 3, we first define an example problem in the next subsection 3.1.

#### 3.1 An Example Problem

We consider a  $\{s, S\}$  inventory problem for a warehouse without delivery lag or backlogging. If the stock level at the end of each day is lower than  $s$ , the stock is replenished up to the level  $S$ . Otherwise, no order is placed. Demand for each day is a Poisson random variable with mean 25. After operating the warehouse with the  $\{s, S\}$  policy via simulation for 30 days, the average number of items in inventory at each day,  $\bar{X}\{s, S\}$ , and the average number of ordered items at each day,  $\bar{Y}\{s, S\}$ , are obtained as a sample observation for a run. Then, an average total cost per day is calculated as

$$\overline{TC}\{s, S\} = h\bar{X}\{s, S\} + c\bar{Y}\{s, S\} \quad (4)$$

where  $h$  is an inventory holding cost per unit and  $c$  is an ordering cost per unit. Note that the total cost is related to the two performance measures  $\bar{X}\{s, S\}$  and  $\bar{Y}\{s, S\}$ . The objective of this problem is to select a proper  $\{s, S\}$  policy that can meet a chance constraint as follows:

$$P[\overline{TC}\{s, S\} \leq 160] \geq 0.9 \quad (5)$$

In order to depict the results and analysis effectively,  $\{22, 60\}$  and  $\{48, 60\}$  are considered as examples of the possible alternatives for the  $\{s, S\}$ . We suppose that the holding cost  $h$  is fixed to \$2/unit but the ordering cost  $c$  has some possibility to change so that  $c \in \{3.8,$

4.0, 4.2}. The significance level  $\alpha$  is set to 0.05.

The simulation itself is coded by using MATLAB to get 200 observations from each policy, and it takes only a few seconds under a personal computer with Intel Core i7-5500U CPU at 2.4GHz and 8GB RAM.

### 3.2 Performance Comparison

The feasibility to Eq. (5) and the performance of the policies can be checked and compared effectively by using the  $\beta$ -bags in the BMORE plot.

Since the minimum probability required in Eq. (5) is 0.9,  $\beta$  is set to 0.9 for the  $\beta$ -bags of the BMORE plots for both policies. Fig. 2 depicts the BMORE plots of the two policies based on 200 simulation observations each. The group of points placed on left side corresponds to the 200 outputs from {22, 60} while the group on right side corresponds to those from {48, 60}. The  $x$  and  $y$  axes of the plot represent  $\bar{X}\{s, S\}$  and  $\bar{Y}\{s, S\}$ , respectively.

In both policies, we can confirm that the average number of items in inventory is negatively correlated with the average number of ordered items. In addition, we can check that the variability of the outputs from the {22, 60} policy is much larger than that from the {48, 60} policy at a glance.

In order to consider the chance constraint Eq. (5) with variable  $c$  values, we add the solid, dashed, and dotted lines representing the sets of all points that achieve the exact total cost  $\overline{TC}\{s, S\} = 160$  with  $c = 3.8, 4.0,$  and  $4.2$ , respectively. Note that the lower closed half space of each line is the set of points with  $\overline{TC}\{s, S\} \leq 160$  and the other closed half space is the set of points with  $\overline{TC}\{s, S\} \geq 160$ . Also, note that all lines do not touch the 0.9-bag's confidence region of the {22, 60} policy. Based on that, we can conclude {22, 60} is a feasible policy to satisfy Eq. (5) with any  $c \in \{3.8, 4.0, 4.2\}$ . (If any of the lines touched the 0.9-bag's confidence region of the policy, it would mean that there is some risk to conclude with the feasibility and more observations are needed to tighten the confidence region of the 0.9-bag.)

In addition to the feasibility check, performance

comparison of multiple systems can be performed by using the BMORE plot. If there exists any line or curve that can separate the considered systems into two groups, we can claim that systems in one group is better or worse than those in the other group with some statistical guarantee.

Note that all lines in Fig. 2 separate the two 0.9-bags and do not touch any of the 0.9-bags or their confidence region. This implies that the averaged total cost of the {22, 60} policy is smaller than 160 with 0.9 probability and the averaged total cost of the {48, 60} policy is larger than 160 with 0.9 probability for all  $c \in \{3.8, 4.0, 4.2\}$ . Therefore, we can obtain the lower bound for the probability of that the averaged total cost of the {22, 60} policy is lower than that of the {48, 60} policy as follows:

$$P[\overline{TC}\{22, 60\} \leq \overline{TC}\{48, 60\}] \geq 0.9^2 \quad (6)$$

for any  $c \in \{3.8, 4.0, 4.2\}$ . In sum, we can conclude that the {22, 60} policy is preferred to the {48, 60} policy with 0.81 probability at least.

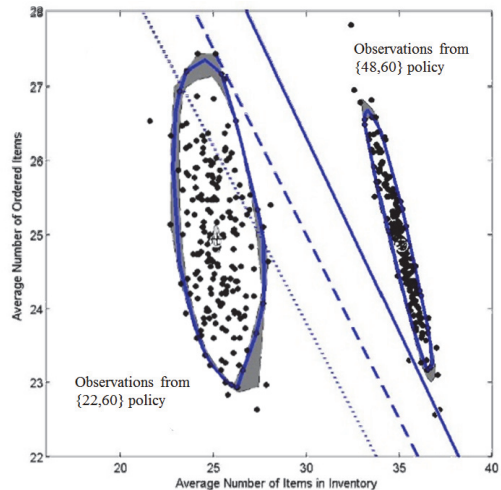


Fig. 2. Comparison of two systems

### 3.3 Sensitivity Analysis on Parameters

In the previous section, we consider only three values of  $c$  and conclude that one policy is better than the other policy with any of the given  $c$  values. Then now,

one may want to know in exactly what range of  $c$  values the conclusion (i.e., Eq. (6)) remains the same. Such sensitivity analyses for parameters are considered in this section. The analysis can be done simply by searching a set of  $c$  values which makes the line  $\overline{TC}\{s, S\} = 160$  locate in between the 0.9-bags of two systems: {22, 60} and {48, 60}.

For the sake of simplicity, let us denote the average number of items in inventory at each day (i.e.,  $\overline{X}\{s, S\}$ ) as  $x$  variable, the average number of ordered items at each day (i.e.,  $\overline{Y}\{s, S\}$ ) as  $y$  variable, and the average total cost per day (i.e.,  $\overline{TC}\{s, S\}$ ) as  $\overline{TC}$  in this section. With the abbreviated notation, Eq. (4) can be re-written as follows:

$$y = -\frac{h}{c} \left( x - \frac{\overline{TC}}{h} \right) \tag{7}$$

Note that the graph of Eq. (7) is a line. When  $\overline{TC}$  and  $h$  are fixed, the  $x$ -intercept of the line is fixed at the coordinate  $\left( \frac{\overline{TC}}{h}, 0 \right)$  and only the slope of the line (i.e.,  $-\frac{h}{c}$ ) changes as  $c$  changes. Therefore, the range of  $c$  values to hold Eq. (5) can be derived easily from the slopes of the lines in between the two  $\beta$ -bags with the fixed  $x$ -intercept. The upper and lower limits of the slope can be read from the lines just touching the two  $\beta$ -bags. Based on this relationship among the graph of Eq. (7), the two  $\beta$ -bags, and  $c$ , we suggest an algorithm represented in Fig. 3 to derive the range of  $c$  values that place the graph of Eq. (7) in between the  $\beta$ -bags of the two different systems with the fixed  $h > 0$ . Fig. 4 depicts the concept of the algorithm to help users to understand the algorithm as well.

In Fig. 3, Steps 1 and 2 are for finding the upper/lower limits of the slope of line from Eq. (7). To be a critical line providing the limit value, the line should satisfies two conditions: i) placing one  $\beta$ -bag on one side of the line while placing the other  $\beta$ -bag on the other side of the line, and ii) touching at least one of the  $\beta$ -bags. Remind that a  $\beta$ -bag is necessarily a convex

polyhedron by its definition. Therefore, to find the line that just touches a  $\beta$ -bag, one needs to check only the lines passing the extreme vertex points of the  $\beta$ -bag instead of checking all points consisting the  $\beta$ -bag. Based on the vertex points of the  $\beta$ -bags, Steps 1 and 2 are designed to find the slope limit of the line that just touches the  $\beta$ -bag of Systems 1 and 2, respectively, while keeping the first condition i). After obtaining the limits of the slope in Steps 1 and 2, the corresponding range of  $c$  is calculated in Step 3. As an output, the algorithm provides the range of  $c$  values to keep Eq. (6). If there is no line that separates the two  $\beta$ -bags, it prints an error message as a result.

By setting  $\overline{TC}\{s, S\} = 160$ ,  $h = 2$ , and  $\beta = 0.9$  for the given two systems {22, 60} and {48, 60}, the  $x$ -intercept of Eq. (7) is fixed at the coordinate (80, 0) and the corresponding range of  $c$  obtained by the algorithm in Fig. 3 is about (3.7554, 4.2979). Therefore, with the current observations and parameter settings, the {22, 60} policy is preferred to the {48, 60} policy at least with 0.81 probability while  $c$  stays between 3.7554 to 4.2979.

Similarly, the range of  $h$  values with a given  $c > 0$  can be derived as well. For the analysis, Eq. (4) can be re-written as follows:

$$y = -\frac{h}{c}x + \frac{\overline{TC}}{c} \tag{8}$$

Note that the shape of the graph of Eq. (8) is also a line. When  $\overline{TC}$  and  $c$  are given, the  $y$ -intercept of Eq. (8) is fixed at the coordinate  $\left( 0, \frac{\overline{TC}}{c} \right)$  and only the slope of Eq. (8) changes as  $h$  varies. Thus, the range of  $h$  values that do not harm the conclusion (i.e., Eq. (6)) can be derived from the slope of the lines located in between the two  $\beta$ -bags with the fixed  $y$ -intercept. Fig. 5 depicts the concept of the algorithm to calculate the range of  $h$  while Fig. 6 provides the detailed algorithm similar to that in Fig. 3. Based on the algorithm, Table 1 provides examples of the ranges of  $h$  to keep Eq. (6) for some given values of  $c$ . As  $c$  changes from 3.7554 to 4.2979,  $h$  can vary between

**Notation**

- $A_1$  A set of vertex points consisting a  $\beta$ -bag of System 1.
- $A_2$  A set of vertex points consisting a  $\beta$ -bag of System 2.

**Step 0 (Initialization)**

Designate  $\overline{TC}$ ,  $h$ , and  $\beta$  values.  
 Set  $\Omega_1 = A_1$  and  $\Omega_2 = A_2$   
 Initialize the slope values  $s_1^* = 0$  and  $s_2^* = 0$ .

**Step 1 (Finding a limit of the slopes from the line touching the  $\beta$ -bag of System 1)**

**While**  $\Omega_1 \neq \emptyset$  **do**

Pick any  $(x_1^*, y_1^*) \in \Omega_1$  and calculate the slope  $s_1^* = y_1^* / (x_1^* - \frac{\overline{TC}}{h})$ .

If either

i)  $y_1 \geq s_1^*(x_1 - \frac{\overline{TC}}{h})$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \leq s_1^*(x_2 - \frac{\overline{TC}}{h})$  for all  $(x_2, y_2) \in A_2$

or

ii)  $y_1 \leq s_1^*(x_1 - \frac{\overline{TC}}{h})$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \geq s_1^*(x_2 - \frac{\overline{TC}}{h})$  for all  $(x_2, y_2) \in A_2$

is satisfied, save the slope value  $s_1^*$  and go to Step 2.

Otherwise, remove  $(x_1^*, y_1^*)$  from  $\Omega_1$ .

**end while**

**Step 2 (Finding a limit of the slopes from the line touching the  $\beta$ -bag of System 2)**

**While**  $\Omega_2 \neq \emptyset$  **do**

Pick any  $(x_2^*, y_2^*) \in \Omega_2$  and calculate the slope  $s_2^* = y_2^* / (x_2^* - \frac{\overline{TC}}{h})$ .

If either

i)  $y_1 \geq s_2^*(x_1 - \frac{\overline{TC}}{h})$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \geq s_2^*(x_2 - \frac{\overline{TC}}{h})$  for all  $(x_2, y_2) \in A_2$

or

ii)  $y_1 \leq s_2^*(x_1 - \frac{\overline{TC}}{h})$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \leq s_2^*(x_2 - \frac{\overline{TC}}{h})$  for all  $(x_2, y_2) \in A_2$

is satisfied, save the slope value  $s_2^*$  and go to Step 3.

Otherwise, remove  $(x_2^*, y_2^*)$  from  $\Omega_2$ .

**end while**

**Step 3: (Finding the range of  $c$  values)**

If  $\Omega_1 = \emptyset$  or  $\Omega_2 = \emptyset$ , print ‘Error: no line separating the two  $\beta$ -bags’ and terminate.

If  $s_1^* \geq s_2^*$ , return the range  $(-\frac{h}{s_2^*}, -\frac{h}{s_1^*})$ . Otherwise, return the range  $(-\frac{h}{s_1^*}, -\frac{h}{s_2^*})$ .

**Fig. 3.** Algorithm to find the range of  $c$  with the given  $\overline{TC}$  and  $h > 0$

2.6483 and 1.6554. While  $h$  stays in the obtained range with the given  $c$ , the {22, 60} policy is preferred to the {48, 60} policy with 0.81 probability at least.

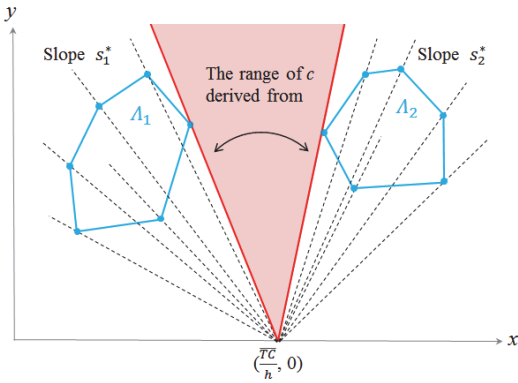


Fig. 4. Concept of the algorithm for  $c$

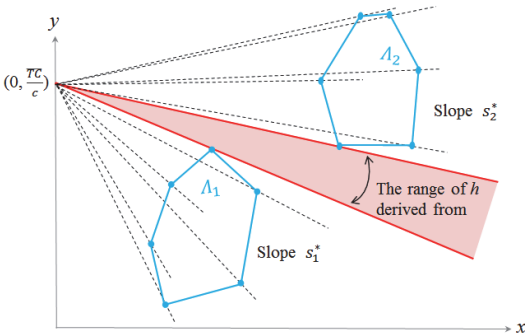


Fig. 5. Concept of the algorithm for  $h$

Table 1. The range of  $h$  corresponding to each  $c$  value

$c$	$h$ (lower limit, upper limit)
3.7554	(2.0000, 2.6483)
3.8000	(1.9717, 2.5971)
4.0000	(1.8446, 2.3608)
4.2000	(1.7176, 2.1212)
4.2979	(1.6554, 2.0000)

#### 4. Concluding Remark

The BMORE plot is a graphical tool to depict the variability in bivariate simulation observation data. Compared to many conventional simulation results

providing a list of static numbers such as sample means, long-run average values, and standard deviations, the BMORE plot considers risk measures in addition to the static results and visualize them intuitively and effectively. The BMORE plot helps users not to spend a great deal of time to analyze the static numbers but to understand the potential risks as well as the setting of simulation experiments at a glance.

In addition to such direct advantages of the BMORE plot, we combine the plot with a chance constraint and show that the BMORE plot can be used for decision making such as the feasibility check of simulated systems with a chance constraint, adjustment of the number of simulation runs, and performance comparison of systems on bivariate measures as well. Based on such analyses, we can compare different systems and discuss the preference or priority among the systems on the statistical background.

We also suggest algorithms to perform sensitivity analyses for problem parameters. With the algorithms, we help users to understand in what ranges of the parameter values the conclusion (i.e., the preference or priority on the considered systems) remains the same. If the parameters get out of the ranges, it implies that one may need to simulate and reevaluate the systems once again with the changed parameters.

In order to demonstrate all the analyses, we employ a specific inventory problem in industrial engineering. However, the techniques and algorithms discussed in this paper can be applied to any other problems comparing systems with simulation/experiment results on two performance measures.

If applied to different pairs of performance measures repeatedly, the BMORE plot can be used as a tool for comparing systems under more-than-two performance measures. However, to meet the increasing demand for complicated data analysis in these days, we are considering expanding the plot itself to deal with more than two measures (out of the two-dimensional space). Also, altering the algorithms to deal with objective functions in nonlinear format (instead of the lines in this paper) can be an interesting future work to study.



**Notation**

$A_1$  A set of vertex points consisting a  $\beta$ -bag of System 1.

$A_2$  A set of vertex points consisting a  $\beta$ -bag of System 2.

**Step 0 (Initialization)**

Designate  $\overline{TC}$ ,  $c$ , and  $\beta$  values.

Set  $\Omega_1 = A_1$  and  $\Omega_2 = A_2$

Initialize the slope values  $s_1^* = 0$  and  $s_2^* = 0$ .

**Step 1 (Finding a limit of the slopes from the line touching the  $\beta$ -bag of System 1)**

**While**  $\Omega_1 \neq \emptyset$  **do**

Pick any  $(x_1^*, y_1^*) \in \Omega_1$  and calculate the slope  $s_1^* = (y_1^* - \frac{\overline{TC}}{c})/x_1^*$ .

If either

i)  $y_1 \geq s_1^*x_1 + \frac{\overline{TC}}{c}$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \leq s_1^*x_2 + \frac{\overline{TC}}{c}$  for all  $(x_2, y_2) \in A_2$

or

ii)  $y_1 \leq s_1^*x_1 + \frac{\overline{TC}}{c}$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \geq s_1^*x_2 + \frac{\overline{TC}}{c}$  for all  $(x_2, y_2) \in A_2$

is satisfied, save the slope value  $s_1^*$  and go to Step 2.

Otherwise, remove  $(x_1^*, y_1^*)$  from  $\Omega_1$ .

**end while**

**Step 2 (Finding a limit of the slopes from the line touching the  $\beta$ -bag of System 2)**

**While**  $\Omega_2 \neq \emptyset$  **do**

Pick any  $(x_2^*, y_2^*) \in \Omega_2$  and calculate the slope  $s_2^* = (y_2^* - \frac{\overline{TC}}{c})/x_2^*$ .

If either

i)  $y_1 \geq s_2^*x_1 + \frac{\overline{TC}}{c}$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \leq s_2^*x_2 + \frac{\overline{TC}}{c}$  for all  $(x_2, y_2) \in A_2$

or

ii)  $y_1 \leq s_2^*x_1 + \frac{\overline{TC}}{c}$  for all  $(x_1, y_1) \in A_1$  and  $y_2 \geq s_2^*x_2 + \frac{\overline{TC}}{c}$  for all  $(x_2, y_2) \in A_2$

is satisfied, save the slope value  $s_2^*$  and go to Step 3.

Otherwise, remove  $(x_2^*, y_2^*)$  from  $\Omega_2$ .

**end while**

**Step 3: (Finding the range of  $h$  values)**

If  $\Omega_1 = \emptyset$  or  $\Omega_2 = \emptyset$ , print ‘Error: no line separating the two  $\beta$ -bags’ and terminate.

If  $s_1^* \geq s_2^*$ , return the range  $(-cs_1^*, -cs_2^*)$ . Otherwise, return the range  $(-cs_2^*, -cs_1^*)$ .

**Fig. 6.** Algorithm to find the range of  $h$  with the given  $\overline{TC}$  and  $c > 0$

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