Canal Surfaces in Pseudo-Galilean 3-Spaces

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ABSTRACT. In this paper, we define admissible canal surfaces with isotropic radius vectors in pseudo-Galilean 3-spaces and we obtain their position vectors. We also attain some important results by considering their Gauss and mean curvatures.

1. Introduction

A canal surface is defined as an envelope of a one-parameter set of spheres, centered at a spine curve \( \gamma(s) \) with radius \( r(s) \). When \( r(s) \) is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, and robotic path planning. An envelope of a 1-parameter family of surfaces is constructed in the same way as we construct a 1-parameter family of curves. The family is described by a differentiable function \( F(x, y, z, \lambda) = 0 \), where \( \lambda \) is a parameter. When \( \lambda \) can be eliminated from the equations

\[
F(x, y, z, \lambda) = 0
\]

and

\[
\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0
\]

we get the envelope, which is a surface described implicitly as \( G(x, y, z) = 0 \). For example, for a 1-parameter family of planes, we get a developable surface [3, 5].

A general canal surface is an envelope of a 1-parameter family of surfaces. The envelope of a 1-parameter family \( s \rightarrow S^2(s) \) of spheres in \( \mathbb{R}^3 \) is called a general canal surface [3]. The curve formed by the centers of the spheres is called center curve of the canal surface. The radius of general canal surface is the function \( r \) such that \( r(s) \) is the radius of the sphere \( S^2(s) \).

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Suppose that the center curve of a canal surface is a unit speed curve $\gamma : I \to \mathbb{R}^3$. The general canal surface can be parametrized by the formula

$$C(s, t) = \gamma(s) - R(s)T(s) - Q(s)\cos(t)N(s) + Q(s)\sin(t)B(s)$$

where

$$R(s) = r(s)r'(s)$$
$$Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}$$

and $T(s)$, $N(s)$, $B(s)$ are the unit tangent, the principal normal, the binormal vectors of the center curve $\gamma(s)$. All the tubes and the surfaces of revolution are subclass of the general canal surface.

**Theorem 1.1.** Let $M$ be a canal surface. The center curve of $M$ is a straight line if and only if $M$ is a surface of revolution for which no normal line to the surface is parallel to the axis of revolution. The following conditions are equivalent for a canal surface $M$:

(i) $M$ is a tube parametrized by (1.1);
(ii) the radius of $M$ is constant;
(iii) the radius vector of each sphere in family that defines the canal surface $M$ meets the center curve orthogonally [3].

2. Canal Surfaces in Pseudo-Galilean Space

Pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature $(0, 0, +, -)$ [6]. The absolute of Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where $w$ is the ideal (absolute) plane, $f$ is line in $w$ and $I$ is the fixed hyperbolic involution of points of $f$. The Pseudo-Galilean scalar product $g$ can be written as

$$g(A, B) = \begin{cases} a_1b_1, & \text{if } a_1 \neq 0 \lor b_1 \neq 0 \\ a_2b_2 - a_3b_3, & \text{if } a_1 = 0 \land b_1 = 0 \end{cases}$$

where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, and the Pseudo-Galilean norm of the vector $A = (a_1, a_2, a_3)$ is defined by

$$\|A\| = \begin{cases} a_1, & \text{if } a_1 \neq 0 \\ \sqrt{(a_2)^2 - (a_3)^2}, & \text{if } a_1 = 0. \end{cases}$$

The vector $A = (a_1, a_2, a_3)$ is said to be non-isotropic if $a_1 \neq 0$. The Pseudo-Galilean cross product is defined for $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ by

$$A \wedge_{G^3} B = \begin{vmatrix} 0 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
Trihedron is given by whose projections on the absolute plane would be light-like vectors. The associated, \[ a = 1, 2, 4, 7. \] All unit non-isotropic vectors are in the form \( (1 \pm a_1, a_2, a_3) \), for isotropic vectors \( a_1 = 0 \). There are four types of isotropic vectors: spacelike \((a_2)^2 - (a_3)^2 > 0\), timelike \((a_2)^2 - (a_3)^2 < 0\) and two types of lightlike \((a_2 = \pm a_3)\) vectors. A non-lightlike isotropic vector is a unit vector if \((a_2)^2 - (a_3)^2 = \pm 1\).

An admissible curve \( \gamma : I \subseteq \mathbb{R} \rightarrow G_3^1 \) is defined by
\[
\gamma(s) = (s, y(s), z(s)).
\]
where \( s \) is arc length parameter. The curvature \( \kappa(s) \) and the torsion \( \tau(s) \) are defined by
\[
\kappa(s) = \sqrt{(y''(s))^2 - (z''(s))^2}, \quad \tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^3(s)}.
\]

An admissible curve has no inflection points, no isotropic tangents or normals whose projections on the absolute plane would be light-like vectors. The associated trihedron is given by
\[
T(s) = \gamma'(s) = (1, y'(s), z'(s))
\]
\[
N(s) = \frac{1}{\kappa(s)} (0, y''(s), z''(s))
\]
\[
B(s) = \frac{1}{\kappa(s)} (0, \epsilon z''(s), \epsilon y''(s))
\]
where \( \epsilon = \mp 1 \), chosen by criterion \( \det (T(s), N(s), B(s)) = 1 \) means that
\[
\left| (y''(s))^2 - (z''(s))^2 \right| = \epsilon \left( (y''(s))^2 - (z''(s))^2 \right).
\]

The curve \( \gamma(s) \) given in (2.2) is timelike (resp. spacelike) if \( N(s) \) is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if \( \epsilon = 1 \) and timelike if \( \epsilon = -1 \). For derivatives of the tangent (vector) \( T(s) \), the normal \( N(s) \) and the binormal \( B(s) \), respectively, the following Serret-Frenet formulas hold
\[
T'(s) = \kappa(s)N(s), \quad N'(s) = \tau(s)B(s), \quad B'(s) = \tau(s)N(s).
\]

On the other hand, a \( C^r \)-surface, \( r \geq 2 \), is a subset \( \Phi \subset G_3^1 \) for which there exists an open subset \( D \) of \( \mathbb{R}^2 \) and \( C^r \)-mapping \( X : D \rightarrow G_3^1 \) satisfying \( \Phi = X(D) \). A \( C^r \) surface \( \Phi \subset G_3^1 \) is called regular if \( X \) is an immersion, and \( \Phi \) is called simple if \( X \) is an embedding. It is admissible if it does not have pseudo-Euclidean tangent planes. If we denote
\[
X = X(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))
\]
\[
x_{,i} = \frac{\partial x}{\partial u_i}, \quad y_{,i} = \frac{\partial y}{\partial u_i}, \quad z_{,i} = \frac{\partial z}{\partial u_i}, \quad i = 1, 2
\]
then, a surface is admissible if and only if \( x_i \neq 0 \), for some \( i = 1, 2 \).

Let \( \Phi \subset G^1_3 \) be a regular admissible surface. Then, the unit normal vector field of a surface \( X(u, v) \) is equal to

\[
\eta(u, v) = \frac{(0, x_1 z_2 - x_2 z_1, x_1 y_2 - x_2 y_1)}{W(u, v)},
\]

\[
W(u, v) = \sqrt\left| (x_1 y_2 - x_2 y_1)^2 - (x_1 z_2 - x_2 z_1)^2 \right|.
\]

The function \( W(u, v) \) is equal to the pseudo-Galilean norm of the isotropic vector \( x_1 X_2 - x_2 X_1 \). Vector defined by

\[
\sigma = \frac{(x_1 X_2 - x_2 X_1)}{W}
\]

is called a side tangential vector. We will not consider surfaces with \( W(u, v) = 0 \), i.e. surfaces having lightlike side tangential vector (lightlike surfaces).

Since the normal vector field satisfies \( g(\eta, \eta) = \epsilon = \pm 1 \), we distinguish two basic types of admissible surfaces: spacelike surfaces having timelike surface normals \( (\epsilon = -1) \) and timelike surfaces having spacelike normals \( (\epsilon = 1) \).

The first fundamental form of a surface is induced from the metric of the ambient space \( G^1_3 \)

\[
ds^2 = (g_1 du_1 + g_2 du_2)^2 + \delta(h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2),
\]

where \( g_i = x_i, h_{ij} = g\left(\tilde{X}_i, \tilde{X}_j \right) \) and

\[
\delta = \begin{cases} 
0 & \text{if direction } du_1 : du_2 \text{ is non-isotropic} \\
1 & \text{if direction } du_1 : du_2 \text{ is isotropic.}
\end{cases}
\]

By \( (\tilde{\ } ) \) above of a vector is denoted the projection of a vector on the pseudo-Euclidean \( yz \)-plane. The Gaussian curvature of a surface is defined by means of the coefficients of the second fundamental form

\[
K = -\epsilon L_{11} L_{22} - L_{12}^2 W^2.
\]

The second fundamental form \( II \) is given by

\[
II = L_{11} du_1^2 + 2L_{12} du_1 du_2 + L_{22} du_2^2
\]

where \( L_{ij} \) are the normal components of \( X_{i,j} \), \( i, j = 1, 2 \). It holds

\[
L_{ij} = \epsilon g\left(\frac{x_i \tilde{X}_{i,j} - x_{i,j} \tilde{X}_i}{x_1}, \eta \right) = \epsilon g\left(\frac{x_j \tilde{X}_{i,j} - x_{i,j} \tilde{X}_j}{x_2}, \eta \right).
\]
The mean curvature of a surface is defined by [4, 7]

$$H = -\epsilon \frac{(g_2)^2 L_{11} - 2g_1 g_2 L_{12} + (g_1)^2 L_{22}}{2W^2}. \quad (2.10)$$

In pseudo-Galilean geometry, there are two types of sphere depending radius vector whether it is an isotropic vector or it is a non-isotropic vector. Spheres with non-isotropic radius vector are pseudo-Euclidean circles in $yz$-plane and spheres with isotropic radius vector are parallel planes such as $x = \pm r$. Pseudo-Euclidean circles intersect the absolute line $f$. There are three kinds of pseudo-Euclidean circles; circles with timelike radius vector ($H^1_\pm(r)$), spacelike radius vector ($S^1_\pm(r)$) and lightlike radius vector, where

$$S^1_\pm(r) = \{ X \in yz-plane | g(X,X) = r^2 \}$$
and

$$H^1_\pm(r) = \{ X \in yz-plane | g(X,X) = -r^2 \}.$$

**Definition 2.1.** The envelope of a 1-parameter family $r \to S^1_\pm(r)$ (or $r \to H^1_\pm(r)$) of pseudo-Euclidean circles in $G^1_3$ is called a canal surface in pseudo-Galilean 3-space. The curve formed by the centers of the pseudo-Euclidean circles is called center curve of the canal surface. The radius of the canal surface is the function $r(s)$ such that $r(s)$ is the radius of the pseudo-Euclidean circles $S^1_\pm(s)$ (or $H^1_\pm(s)$).

Let us consider $C(s,t) - \gamma(s)$ is a isotropic vector of $H^1_\pm(r)$ then, the envelope of a 1-parameter family $r \to H^1_\pm(r)$ in $G^1_3$ is spacelike canal surface and since $C(s,t) - \gamma(s) \in Sp\{T(s), N(s), B(s)\}$ and $C(s,t)$ is non-isotropic then, we have

$$C(s,t) = \gamma(s) + \psi(s,t) T(s) + \varphi(s,t) N(s) + \omega(s,t) B(s) \quad (2.11)$$
and $\psi(s,t) = 0$. In the case that the centered curve is a spacelike curve, we can write

$$g(C(s,t) - \gamma(s), C(s,t) - \gamma(s)) = -\varphi^2(s,t) + \omega^2(s,t) = -r(s)^2. \quad (2.12)$$

By differentiating (2.12) with respect to $s$ and $t$, we get

$$\varphi(s,t) \varphi_s(s,t) - \omega(s,t) \omega_s(s,t) = r'(s) r(s) \quad (2.13)$$
and

$$\varphi(s,t) \varphi_t(s,t) - \omega(s,t) \omega_t(s,t) = 0 \quad (2.14)$$
from the equations (2.12), (2.13) and (2.14), we obtain

$$\omega(s,t) = r(s) \sinh(t) , \varphi(s,t) = r(s) \cosh(t).$$

Thus, we have the following corollary.
Corollary 2.2. Let \( \gamma(s) \) be an admissible spacelike curve with arclength parameter in pseudo-Galilean 3-space. Then, position vector of spacelike canal surface with spacelike centered curve is

\[
C(s, t) = \gamma(s) + r(s) \cosh(t) N(s) + r(s) \sinh(t) B(s).
\]

By using (2.5) and (2.15), natural bases \( \{C_s, C_t\} \) are

\[
C_s = T(s) + \{r' \cosh(t) + r \tau \sinh(t)\} N(s) + \{r' \sinh(t) + r \tau \cosh(t)\} B(s)
\]
\[
C_t = r \sinh(t) N(s) + r \cosh(t) B(s)
\]
and from (2.7) the coefficients \( h_{ij} \) and \( g_i \) are

\[
h_{11} = r^2(s) \tau^2(s) - (r'(s))^2, \quad h_{12} = h_{21} = r^2(s) \tau(s), \quad h_{22} = r^2(s)\]
\[
g_1 = 1, \quad g_2 = 0.
\]

Thus, the first fundamental form of spacelike canal surface is

\[
I_C = \left(1 + r^2(s) \tau^2(s) - (r'(s))^2\right) ds^2 + 2r^2(s) \tau(s) ds dt + r^2(s) dt^2.
\]

By using (2.5), the second derivations (2.15) are

\[
C_{ss} = \left\{\kappa + (2r' \tau + r \tau') \sinh(t) + (r \tau^2 + r'') \cosh(t)\right\} N(s)
\]
\[
+ \left\{(2r' \tau + r \tau') \cosh(t) + (r \tau^2 + r'') \sinh(t)\right\} B(s)
\]
\[
C_{tt} = r \cosh(t) N(s) + r \sinh(t) B(s)
\]
\[
C_{ts} = \{r' \sinh(t) + r \tau \cosh(t)\} N(s) + \{r \tau \sinh(t) + r' \cosh(t)\} B(s)
\]
and the unit normal vector is

\[
\eta(s, t) = \cosh(t) N(s) + \sinh(t) B(s).
\]

From (2.9) coefficients \( L_{ij} \) are

\[
L_{11} = r(s) \tau^2(s) + r''(s) + \kappa(s) \cosh(t), \quad L_{12} = L_{21} = r(s) \tau(s), \quad L_{22} = r(s)
\]
and the second fundamental form is

\[
H_C = (r(s) \tau^2(s) + r''(s) + \kappa(s) \cosh(t)) ds^2 + 2r(s) \tau(s) ds dt + r(s) dt^2.
\]

Thus, from (2.8) and (2.10), Gauss and mean curvatures are

\[
K(s, t) = \frac{r''(s) + \kappa(s) \cosh(t)}{r(s)}, \quad H(s, t) = \frac{1}{2r(s)}.
\]

In the case that \( K(s, t) = 0 \), the centered curve has to be planar and there are two K-flat canal surfaces for \( r(s) = c_1 s + c_2 \) and \( r(s) = c \).

Hence, from (2.2), (2.3), (2.4), (2.15) and (2.16), we have the following theorem.

Theorem 2.3. Let \( M \) be a spacelike canal surface with spacelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.
(i) There is no minimal spacelike canal surface with spacelike centered curve,

(ii) Gauss and mean curvatures of $M$ satisfy the relation

$$K(s,t) - 2H(s,t)(r''(s) + \kappa(s) \cosh(t)) = 0,$$

(iii) $M$ is a $K$-flat if and only if $M$ is a parabolic cone and its position vector is

$$C(s,t) = (s, (c_1s + c_2)(c_3 \cosh(t) \mp \sqrt{(c_3)^2 + 1 \sinh(t)})$$

$$, (c_1s + c_2)(\mp \sqrt{(c_3)^2 + 1 \cosh(t) + c_3 \sinh(t)})$$

where $c_1 \neq 0$, $c_2, c_3 \in \mathbb{R}$, (see figure 1.a),

(iv) $M$ is a $K$-flat spacelike tubular surface if and only if $M$ is a parabolic cylinder and its position vector is

$$C(s,t) = (s, c_1c_2 \cosh(t) \mp c_1 \sqrt{(c_2)^2 + 1 \sinh(t)} + c_1c_2 \sinh(t))$$

where $c_1 \in \mathbb{R}^+$, $c_2 \in \mathbb{R}$, (see figure 1.b),

(v) All the spacelike tubes with spacelike centered curve are positive-constant mean curvature surfaces.

In the case that $C(s,t)$ is spacelike canal surface and centered curve is a timelike curve, we can write:

$$g(C(s,t) - \gamma(s), C(s,t) - \gamma(s)) = \varphi^2(s,t) - \omega^2(s,t) = -r^2(s).$$

By differentiating (2.17) with respect to $s$ and $t$, we get

$$\omega(s,t)\omega_s(s,t) - \varphi(s,t)\varphi_s(s,t) = r'(s)r(s)$$

and

$$\omega(s,t)\omega_t(s,t) - \varphi(s,t)\varphi_t(s,t) = 0$$

then, we obtain

$$\omega(s,t) = r(s) \cosh(t), \varphi(s,t) = r(s) \sinh(t)$$

by using (2.17), (2.18) and (2.19).

Thus, we have the following corollary.

**Corollary 2.4.** Let $\gamma(s)$ be an admissible timelike curve with arclength parameter in pseudo-Galilean 3-space. Then, position vector of spacelike canal surface with timelike centered curve is

$$C(s,t) = \gamma(s) + r(s) \sinh(t)N(s) + r(s) \cosh(t)B(s).$$
From (2.5) and (2.20), natural bases \( \{ C_s, C_t \} \) are
\[
C_s = T(s) + \{ r' \sinh(t) + r \tau \cosh(t) \} N(s) + \{ r' \cosh(t) + r \tau \sinh(t) \} B(s)
\]
\[
C_t = r \cosh(t)N(s) + r \sinh(t)B(s)
\]
and from (2.7) the coefficients \( h_{ij} \) and \( g_i \) are
\[
h_{11} = r^2(s) \tau^2(s) - (r'(s))^2, \quad h_{12} = h_{21} = r^2(s) \tau(s), \quad h_{22} = r^2(s)
\]
\[
g_1 = 1, \quad g_2 = 0.
\]
Thus, the first fundamental form is
\[
I_C = \left( 1 + r^2(s) \tau^2(s) - (r'(s))^2 \right) ds^2 + 2r^2(s) \tau(s) ds dt + r^2(s) dt^2.
\]
By using (2.5), the second derivations (2.20) are
\[
C_{ss} = \left\{ \kappa + (r'' + r \tau^2) \sinh(t) + (2r' \tau + r \tau') \cosh(t) \right\} N(s)
\]
\[
+ \left\{ (r'' + r \tau^2) \cosh(t) + (2r' \tau + r \tau') \sinh(t) \right\} B(s)
\]
\[
C_{tt} = r \sinh(t)N(s) + r \cosh(t)B(s)
\]
\[
C_{ts} = \{ r' \cosh(t) + r \tau \sinh(t) \} N(s) + \{ r' \sinh(t) + r \tau \cosh(t) \} B(s)
\]
the unit normal vector is
\[
\eta(s,t) = \sinh(t)N(s) + \cosh(t)B(s).
\]
From (2.9), the coefficients \( L_{ij} \) are
\[
L_{11} = \kappa(s) \sinh(t) - r(s) \tau^2(s) - r''(s), \quad L_{12} = L_{21} = -r(s) \tau(s), \quad L_{22} = -r(s)
\]
and so the second fundamental form is
\[
II_C = \left( \kappa(s) \sinh(t) - r(s) \tau^2(s) - r''(s) \right) ds^2 - 2r(s) \tau(s) ds dt - r(s) dt^2.
\]
From (2.8) and (2.10), Gauss and mean curvatures are
\[
K(s,t) = \frac{\kappa(s) \sinh(t) - r''(s)}{r(s)}, \quad H(s,t) = \frac{1}{2r(s)}
\]
respectively. In the case that \( K(s,t) = 0 \), the centered curve has to be planar and there are two K-flat canal surfaces for \( r(s) = c_1 s + c_2 \) and \( r(s) = c \).

Hence, from (2.2), (2.3), (2.4), (2.20) and (2.21), we have the following cases.

**Theorem 2.5.** Let \( M \) be a spacelike canal surface with timelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.

(i) There is no minimal spacelike canal surface with timelike centered curve,
(ii) Gauss and mean curvatures of $M$ satisfy the relation
\[ K(s, t) + 2H(s, t)(\kappa \sinh(t) - r'') = 0, \]

(iii) $M$ is a $K$-flat if and only if $M$ is a parabolic cone and its position vector is
\[ C(s, t) = (s, (c_1s + c_2)(c_3 \sinh(t) \mp \sqrt{(c_3)^2 - 1} \sinh(t) + c_3 \cosh(t))) \]
where $c_1 \neq 0$, $c_2 \in \mathbb{R}$, $c_3 \in \mathbb{R} - [0, 1)$, (see figure 2.a),

(iv) $M$ is a $K$-flat spacelike tubular surface if and only if $M$ is a parabolic cylinder and its position vector is
\[ C(s, t) = (s, c_1c_2 \sinh(t) \mp c_1 \sqrt{(c_2)^2 - 1} \cosh(t) + c_1c_2 \cosh(t)) \]
where $c_1 \in \mathbb{R}^+$, $c_2 \in \mathbb{R} - [0, 1)$, (see figure 2.b),

(v) All the spacelike tubes with timelike centered curve are positive-constant mean curvature surfaces.

Accordingly, in the case that $C(s, t) - \gamma(s)$ is an isotropic radius vector of $S^1_+(r)$ then, the envelope of a 1-parameter family $s \rightarrow S^1_+(r)$ in $G^3_1$ is timelike canal surface and since $C(s, t) - \gamma(s) \in Sp\{T(s), N(s), B(s)\}$ and $C(s, t)$ is non-isotropic then, we have (2.11) and $\psi(s, t) = 0$. If the centered curve is a timelike curve then, the position vector $C(s, t)$ is obtained in the same form of (2.15). From (2.7) and (2.9), coefficients of the first and the second fundamental forms are obtained as
\[ h_{11} = (r'(s))^2 - r^2(s) \tau^2(s), \quad h_{12} = h_{21} = -r^2(s) \tau(s), \quad h_{22} = -r^2(s) \]
\[ L_{11} = \kappa(s) \cosh(t) + r(s) \tau(s)^2 + r''(s), \quad L_{12} = L_{21} = r(s) \tau(s), \quad L_{22} = r(s) \]
and also from (2.8) and (2.10), the Gauss and the mean curvatures are
\[
(2.22) \quad K(s, t) = -\frac{\kappa(s) \cosh(t) + r''(s)}{r(s)}, \quad H(s, t) = -\frac{1}{2r(s)}.
\]

Thus, from (2.2), (2.3), (2.4), (2.15) and (2.22), we can give the following corollary.

**Corollary 2.6.** Let $M$ be a timelike canal surface with timelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.

(i) There is no minimal timelike canal surface with timelike centered curve,

(ii) Gauss and mean curvatures of $M$ satisfy the relation
\[
K(s, t) - 2H(s, t)(\kappa(s) \cosh(t) + r''(s)) = 0,
\]
(iii) \( M \) is a \( K \)-flat if and only if \( M \) is a parabolic cone and its position vector is
\[
C(s, t) = (s, (c_1 s + c_2)(c_3 \cosh(t) \mp \sqrt{(c_3)^2 - 1} \sinh(t))
\]
\[
, (c_1 s + c_2)(\mp \sqrt{(c_3)^2 - 1} \cosh(t) + c_3 \sinh(t)))
\]
where \( c_1 \neq 0 \), \( c_2, c_3 \in \mathbb{R} - [0, 1) \), (see figure 3.a),

(iv) \( M \) is a \( K \)-flat timelike tubular surface if and only if \( M \) is a parabolic cylinder and its position vector is
\[
C(s, t) = (s, c_1 c_2 \cosh(t) \mp c_1 \sqrt{(c_2)^2 - 1} \sinh(t), \mp c_1 \sqrt{(c_2)^2 - 1} \cosh(t) + c_1 c_2 \sinh(t))
\]
where \( c_1 \in \mathbb{R}^+, c_2 \in \mathbb{R} - [0, 1) \), (see figure 3.b),

(v) All the timelike tubes with timelike centered curve are negative-constant mean curvature surfaces.

If the centered curve is a spacelike curve then, the position vector \( C(s, t) \) is obtained in the same form of (2.20) and from (2.7) and (2.9), coefficients of the first and the second fundamental forms are
\[
h_{11} = (r'(s))^2 - r^2(s) \tau^2(s), \ h_{12} = h_{21} = -r^2(s) \tau(s), \ h_{22} = -r^2(s)
\]
\[
L_{11} = \kappa(s) \sinh(t) - r(s) \tau^2(s) - r''(s), \ L_{12} = L_{21} = -r(s) \tau(s), \ L_{22} = -r(s)
\]
and also from (2.8) and (2.10), the Gauss and the mean curvatures are
\[
K(s, t) = \frac{r''(s) - \kappa(s) \sinh(t)}{r(s)}, \ H(s, t) = \frac{-1}{2r(s)}.
\]

We have the following cases, by using the equations (2.2), (2.3), (2.4), (2.20) and (2.23).

**Corollary 2.7.** Let \( M \) be a timelike canal surface with spacelike centered curve in pseudo-Galilean 3-space. Then, the followings are true.

(i) There is no minimal timelike canal surface with spacelike centered curve,

(ii) Gauss and mean curvatures of \( M \) satisfy the relation
\[
K(s, t) + 2H(s, t) (r''(s) - \kappa(s) \sinh(t)) = 0,
\]

(iii) \( M \) is a \( K \)-flat if and only if \( M \) is a parabolic cone and its position vector is
\[
C(s, t) = (s, (c_1 s + c_2)(c_3 \sinh(t) \mp \sqrt{(c_3)^2 + 1} \cosh(t))
\]
\[
, (c_1 s + c_2)(\mp \sqrt{(c_3)^2 + 1} \sinh(t) + c_3 \cosh(t)))
\]
where \( c_1 \neq 0, c_2 \in \mathbb{R}, c_3 \in \mathbb{R} \), (see figure 4.a),
(iv) \( M \) is a \( K \)-flat timelike tubular surface if and only if \( M \) is a parabolic cyclinder and its position vector is
\[
C(s,t) = (s, c_1 c_2 \sinh(t) \mp c_1 \sqrt{(c_2)^2 + 1} \cosh(t), \mp c_1 \sqrt{(c_2)^2 + 1} \sinh(t) + c_1 c_2 \cosh(t))
\]
where \( c_1 \in \mathbb{R}^+, \ c_2 \in I \), (see figure 4.b),

(v) All the timelike tubes with spacelike centered curve are negative-constant mean curvature surfaces.

Now, we can summarise our study as in following theorem.

**Theorem 2.8.** Let \( \gamma : (a, b) \rightarrow G_{1}^{3} \) be an admissible curve in \( G_{1}^{3} \) and \( M \) be a canal surface with the centered curve \( \gamma(s) \) then, there are two types canal surfaces in \( G_{1}^{3} \) such that,

**type-1:** \( M \) is spacelike (timelike) canal surface and \( \gamma(s) \) is spacelike (timelike) curve then, \( M \) is parametrized by
\[
C_\mu (s,t) = \gamma(s) + r(s) \cosh(t)N(s) + r(s) \sinh(t)B(s),
\]

**type-2:** \( M \) is spacelike (timelike) canal surface and \( \gamma(s) \) is timelike (spacelike) curve then, \( M \) is parametrized by
\[
C_\sigma (s,t) = \gamma(s) + r(s) \sinh(t)N(s) + r(s) \cosh(t)B(s).
\]

In consideration of above theorem, we can give coefficients of the first fundamental forms, Gauss and mean curvatures as follow by taking \( g_1 = 1 \), \( g_2 = 0 \).

For the type-1 canal surfaces,
\[
h_{11} = \mu r(s)^2 \tau(s)^2, \quad h_{21} = h_{12} = \mu r(s)^2 \tau(s), \quad h_{22} = \mu r(s)^2,
\]
Gauss and mean curvatures are
\[
K(s,t) = \frac{\mu (r''(s) + \kappa(s) \cosh(t))}{r(s)}, \quad H(s,t) = \frac{\mu}{2r(s)}.
\]

For the type-2 canal surfaces,
\[
h_{11} = \sigma r(s)^2 \tau(s)^2, \quad h_{12} = h_{21} = \sigma r(s)^2 \tau(s), \quad h_{22} = \sigma r(s)^2,
\]
Gauss and mean curvatures are
\[
K(s,t) = \frac{\sigma (r''(s) + \kappa(s) \cosh(t))}{r(s)}, \quad H(s,t) = \frac{\sigma}{2r(s)}
\]
where
\[
\mu = \begin{cases} 
1, \quad \text{if } M \text{ is a spacelike canal surface with spacelike centered curve} \\
-1, \quad \text{if } M \text{ is a timelike canal surface with timelike centered curve}
\end{cases}
\]
and
\[
\sigma = \begin{cases} 
1, \quad \text{if } M \text{ is a spacelike canal surface with timelike centered curve} \\
-1, \quad \text{if } M \text{ is a timelike canal surface with spacelike centered curve}
\end{cases}
\]
Figure 1: For (a); $c_1 = 2, c_2 = 1, c_3 = 0$, $\text{sign} : (-)$, for (b); $c_1 = 2, c_2 = 1$.

Figure 2: For (a); $c_1 = c_2 = 1, c_3 = 0$, for (b); $c_1 = 1, c_2 = 2$.

Figure 3: For (a); $c_1 = c_2 = 1, c_3 = 2$, for (b); $c_1 = 1, c_2 = 2$. 
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Figure 4: For (a); $c_1 = 2, c_2 = c_3 = 1$, for (b); $c_1 = 2, c_2 = 0$.

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