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THE GORENSTEIN TRANSPOSE OF COMODULES

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Abstract. Let Γ be a Gorenstein coalgebra over a filed k. We introduce the Gorenstein transpose via a minimal Gorenstein injective copresentation of a quasi-finite Γ-comodule, and obtain a relation between a Gorenstein transpose of a quasi-finite comodule and a transpose of the same comodule. As an application, we obtain that the almost split sequences are constructed in terms of Gorenstein transpose.

1. Introduction and preliminaries

Auslander-Reiten theory plays an important role in the representation theory. As a key ingredient in this theory, the transpose plays a central role, especially in the construction of the Auslander-Reiten sequence. The notion of the transpose of finitely generated module was introduced by Auslander and Bridger in [2], and the well-known almost split sequences were discovered by Auslander and Reiten [3] for finitely generated modules over a finite-dimensional (artin) algebra. As a generalization of the transpose of finitely generated module, Huang [7] introduced the notion of the Gorenstein transpose of finitely generated modules. Dual to the representation theory of algebras, the researches about the representation theory of coalgebras have been on the rise. Chin, Kleiner and Quinn [4] introduced the notion of the transpose of a comodule which is constructed via a minimal injective copresentation of a quasi-finite comodule. Chin and Simson [5] showed the existence of almost split sequences in the category of finitely copresented comodules over semiperfect coalgebras. In recent years, the relative homological coalgebra has been extensively studied by many mathematicians (see for example [1, 6, 8, 9]). Asensio, López Ramos and Torrecillas [1] introduced the notion of Gorenstein injective comodules and proved the equivalent conditions of Gorenstein injective comodules over an n-Gorenstein coalgebra.

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Inspired by the above researches, in this paper, we introduce the notion of the Gorenstein transpose via a minimal Gorenstein injective copresentation of a comodule over a Gorenstien coalgebra, and establish a relation between a Gorenstein transpose of a comodule and a transpose of the same comodule. We prove that the transpose of a quasi-finite comodule M is an extension of a Gorenstein injective comodule along the Gorenstein transpose of M. In particular, Gorenstein transpose shares many nice homological properties of transpose. Then some applications are given: (1) For the quasi-finite comodules, the Gorenstein transpose of a finite injective dimension comodule can be decomposed into a direct sum of the transpose of the same comodule and a Gorenstein injective comodule. (2) We construct an almost split sequence in terms of the Gorenstein transpose. If a quasi-finite Γ-comodule M is indecomposable, non-Gorenstein injective and $\dim Tr_G M < \infty$, then there exists an almost split sequence of the form $0 \to M \to Y \to DTr_G M \to 0$.

We now fix the terminology and recall some definitions used in this paper. Denote by Γ a k-coalgebra with comultiplication $\Delta : \Gamma \to \Gamma \otimes \Gamma$ and counit $\epsilon : \Gamma \to k$, where $\otimes = \otimes_k$. A right Γ-comodule M is given by a structure map $\rho : M \to M\otimes\Gamma$, the category $\mathcal{M}^\Gamma$ of all right Γ-comodules is an abelian category with enough injectives. We identify the category ${}^\Gamma\mathcal{M}$ of left Γ-comodules with the category $\mathcal{M}^{\Gamma^{op}}$, where Γ^{op} is the opposite coalgebra of Γ.

Recall from [1] that a coalgebra Γ is said to be an n-Gorenstein coalgebra if it is semiperfect on both sides and if $pd(\Gamma) \leq n$ as right and left Γ-comodule. We will call Γ a Gorenstein coalgebra if it is n-Gorenstein for some n. A right Γ-comodule N is called Gorenstein injective (see [1]) if and only if there exists an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective right Γ-comodules with $N = Ker(E^0 \to E^1)$ and such that the functor $Com_\Gamma(E, -)$ leaves it exact for any injective right Γ-comodule E. It is clear that an injective comodule is Gorenstein injective and that in a complete injective resolution, all the kernels and hence all the images and cokernels are Gorenstein injective. Note that if Γ is a Gorenstein coalgebra, then every right Γ-comodule has a Gorenstein injective envelope. Let $\mathcal{GI}$ be the full subcategory of $\mathcal{M}^\Gamma$ of Gorenstein injective comodules and $\mathcal{GP}$ be the class of Gorenstein projective comodules. Let $\mathcal{I}^\Gamma$ and $\mathcal{P}^\Gamma$ denote the full subcategory determined by the injectives and the class of projective right Γ-comodules, respectively.

A comodule $M \in \mathcal{M}^\Gamma$ is quasi-finite if $\dim_k Com_\Gamma(F, M) < \infty$ for all finite-dimensional $F \in \mathcal{M}^\Gamma$. In what follows, $\mathcal{M}_q^\Gamma$ denotes the full subcategory of $\mathcal{M}^\Gamma$ determined by the quasi-finite comodules. Recall from [10] that if $X \in \mathcal{M}_q^\Gamma$ and $Y \in \mathcal{M}^\Gamma$, then $h_\Gamma(X, Y) = \varinjlim DCom_\Gamma(Y_\lambda, X)$, where $\{Y_\lambda\}$ is the set of finite-dimensional subcomodules of Y. $h_\Gamma(-, -)$ is an additive right exact bifunctor, which is called the cohom functor. Let * denote the contravariant functor $(\ )^* = h_\Gamma(-, \Gamma) : \mathcal{M}_q^\Gamma \to \mathcal{M}^{\Gamma^{op}}$, as well as $h_{\Gamma^{op}}(-, \Gamma^{op}) : \mathcal{M}_q^{\Gamma^{op}} \to \mathcal{M}^\Gamma$. We say that an $X \in \mathcal{M}_q^\Gamma$ is strongly quasi-finite if $X^* \in \mathcal{M}_q^{\Gamma^{op}}$ and denote by $\mathcal{M}_{sq}^\Gamma$ the full subcategory of $\mathcal{M}^\Gamma$ determined by all strongly quasi-finite comodules.

Recall from [4] that a comodule $M \in \mathcal{M}^\Gamma$ is quasi-finite copresented if its minimal injective copresentation $0 \to M \to I_0 \to I_1$ satisfies $I_i \in \mathcal{I}^\Gamma$ is quasi-finite for $j = 0, 1$; in the following, $\mathcal{M}^\Gamma_{qc}$ denotes the full subcategory of $\mathcal{M}^\Gamma$ determined by the quasi-finite copresented comodules.

Throughout this paper, all comodules in $\mathcal{M}^\Gamma$ are quasi-finite.

2. The transpose

Firstly, we recall the notation of transpose in [4] and list some results in order to make the article self-contained.

Definition ([4])**.** If $0 \to M \to E_0 \to E_1$ is a minimal injective copresentation of $M \in \mathcal{M}^\Gamma$, then we define TrM as a left Γ-comodule which makes the sequence

$$0 \to TrM \to E_1^* \to E_0^*$$

exact. We call $Ker(E_1^* \to E_0^*)$ a transpose of M.

Remark 2.1. The transpose TrM of M is determined uniquely up to isomorphism, and $0 \to TrM \to E_1^* \to E_0^*$ is a minimal injective copresentation of $TrM \in \mathcal{M}_q^{\Gamma^{op}}$.

Proposition 2.2 ([4])**.** (a) *The map* $\eta :^{**}= h_{\Gamma^{op}}(h_\Gamma(-, \Gamma), \Gamma) \to E$ *given by* $\eta_X : X^{**} \to X$ *is a natural transformation of functors, where* $E : \mathcal{M}^\Gamma_{sq} \to \mathcal{M}^\Gamma$ *is the natural embedding.*

(b) *The restriction of* η *to* $\mathcal{I}^\Gamma$ *is a natural isomorphism* $^{**} \to 1_{\mathcal{I}^\Gamma}$.

(c) $^* : \mathcal{I}^\Gamma \to \mathcal{I}^{\Gamma^{op}}$ *and* $^* : \mathcal{I}^{\Gamma^{op}} \to \mathcal{I}^\Gamma$ *are dualities.*

Denote by $(\mathcal{M}^\Gamma_q)_\mathcal{I}$ the full subcategory of $\mathcal{M}^\Gamma_q$ whose objects have no nonzero injective summands. For each $M \in \mathcal{M}^\Gamma_q$ there is a unique up to isomorphism decomposition $M = M_\mathcal{I} \oplus M'$ where $M_\mathcal{I} \in (\mathcal{M}^\Gamma_q)_\mathcal{I}$ and $M' \in \mathcal{I}^\Gamma$. The following result is an analog of [3, IV Proposition 1.7].

Proposition 2.3 ([4])**.** *Let* $M \in \mathcal{M}^\Gamma$, *we have the following.*

(a) *If* $M = \oplus_{\alpha \in A} M_\alpha$, *then* $M_\alpha \in \mathcal{M}^\Gamma_{qc}$ *and* $TrM \cong \oplus_{\alpha \in A} TrM_\alpha$.

(b) $TrM = 0$ *if and only if* M *is injective.*

(c) $TrTrM \cong M_\mathcal{I}$.

(d) *If* $M, N \in (\mathcal{M}^\Gamma)_\mathcal{I}$, *then* $TrM \cong TrN$ *if and only if* $M \cong N$.

(e) $Tr : \mathcal{M}^\Gamma \to \mathcal{M}^{\Gamma^{op}}$ *induces a bijection between the isomorphism classes of indecomposable comodules in* $(\mathcal{M}^\Gamma)_\mathcal{I}$ *and* $(\mathcal{M}^{\Gamma^{op}})_\mathcal{I}$.

Proposition 2.4. *Let* $0 \to M \to E_0 \to E_1$ *be a minimal injective copresentation of* $M \in \mathcal{M}^\Gamma$. *If* $X \in \mathcal{M}^\Gamma$, *then there is an exact sequence*

$$0 \to X \square_\Gamma TrM \to h_\Gamma(E_1, X) \to h_\Gamma(E_0, X) \to h_\Gamma(M, X) \to 0$$

with all morphisms functorial in X.

Proof. The exact sequence $0 \to TrM \to E_1^* \to E_0^*$ gives rise to the commutative exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & X\Box_\Gamma TrM \arrow[r] & X\Box_\Gamma E_1^* \arrow[r] \arrow[d, "\cong"] & X\Box_\Gamma E_0^* \arrow[d, "\cong"] & & \\
& & h_\Gamma(E_1,X) \arrow[r] & h_\Gamma(E_1,X) \arrow[r] & h_\Gamma(M,X) \arrow[r] & 0
\end{tikzcd}$$

So, it is easy to get our desired exact sequence from the above commutative diagram. $\square$

If Γ is a right semiperfect coalgebra, then the category of all right Γ-comodules has enough projectives. Let T_Γ be quasi-finite, for any right Γ-comodule M, we consider its projective resolution:

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

In this way, we obtain the left derived functor $L^n(h_\Gamma(T,-))$. We denote it by $ext^n_\Gamma(T,-)$. Similarly, by the quasi-finite injective resolution of M in $\mathcal{M}_q^\Gamma$, we obtain the right derived functor $ext^n_\Gamma(-,X)$ for any right Γ-comodule X. Since $D(h_\Gamma(T,X)) \cong Com_\Gamma(X,T)$ for any $X \in \mathcal{M}^\Gamma$, it follows that $D(ext^n_\Gamma(T,X)) \cong Ext^n_\Gamma(X,T)$ (see Proposition 12.2.2 in [11]). It is easy to show that $id_\Gamma T \leq n$ if and only if $D(ext^{n+1}_\Gamma(T,N)) \cong Ext^{n+1}_\Gamma(N,T) = 0$ for any $N \in \mathcal{M}^\Gamma$, if and only if $ext^{n+1}_\Gamma(T,N) = 0$ for any N in $\mathcal{M}^\Gamma$.

Proposition 2.5. *Let Γ be a coalgebra and $M \in \mathcal{M}_{sq}^{\Gamma^{op}}$. Then for each X in $\mathcal{M}^\Gamma$ we have an exact sequence*

$$0 \to ext^2_\Gamma(TrM,X) \to h_\Gamma(M^*,X) \xrightarrow{\alpha_X} X\Box_\Gamma M \to ext^1_\Gamma(TrM,X) \to 0$$

where all morphisms are functorial in X.

Proof. Let $0 \to M \xrightarrow{f} E_0 \xrightarrow{g} E_1$ be a minimal injective copresentation of M. Then we have an exact sequence

$$0 \to TrM \to E_1^* \xrightarrow{g^*} E_0^* \xrightarrow{f^*} M^* \to 0.$$

Let $K = Img^*$ and applying the functor $(\ )^* = h_\Gamma(-,\Gamma)$ on the exact sequences both $0 \to K \to E_0^* \xrightarrow{f^*} M^* \to 0$ and $0 \to TrM \to E_1^* \to K \to 0$. Since the E_i^* is an injective Γ-comodule for $i = 0,1$ by Proposition 2.2, it is not hard to see the following for all X in $\mathcal{M}^\Gamma$.

(a) $0 \to ext^2_\Gamma(TrM,X) \to h_\Gamma(M^*,X) \xrightarrow{h_\Gamma(f^*,X)} h_\Gamma(E_0^*,X)$ is an exact sequence with all morphisms functorial in X.

(b) $0 \to ext^1_\Gamma(TrM,X) \to Coker h_\Gamma(f^*,X) \to h_\Gamma(E_1^*,X)$ is an exact sequence with all morphisms functorial in X.

By Propositions 1.13 and 1.14 in [10], we have that $h_\Gamma(E_i^*,X) \cong X\Box_\Gamma E_i$, for $i = 0,1$. Using these observations, it is not difficult to deduce our desired exact sequence from the commutative diagram with exact second row:

$$\begin{tikzcd}
 & h_\Gamma(M^*, X) \arrow[r, "h_\Gamma(f^*,X)"] \arrow[d, "\alpha_X"] & h_\Gamma(E_0^*, X) \arrow[r, "h_\Gamma(g^*,X)"] \arrow[d, "\cong"] & h_\Gamma(E_1^*, X) \arrow[d, "\cong"] \\
0 \arrow[r] & X\Box_\Gamma M \arrow[r, "\varphi"] & X\Box_\Gamma E_0 \arrow[r, "\psi"] & X\Box_\Gamma E_1
\end{tikzcd}$$

□

Remark 2.6. Proposition 2.5 is a generalization of a result by Auslander and Reiten [3] (Chapter IV. Proposition 3.2).

Corollary 2.7. *Let* Γ *be a coalgebra and* $M \in \mathcal{M}_{sq}^{\Gamma^{op}}$. *Then we have an exact sequence*

$$0 \to ext_\Gamma^2(TrM, \Gamma) \to M^{**} \xrightarrow{\alpha_\Gamma} M \to ext_\Gamma^1(TrM, \Gamma) \to 0.$$

Definition. Let $M \in \mathcal{M}_{sq}^{\Gamma^{op}}$. M is said to be reflexive if $\alpha_\Gamma : M^{**} \longrightarrow M$ is an isomorphism.

We have the following immediate consequence of Proposition 2.5.

Corollary 2.8. *Let* Γ *be a coalgebra. Then*
(1) $M \in \mathcal{M}_{sq}^{\Gamma^{op}}$ *is reflexive if and only if* $ext_\Gamma^i(TrM, \Gamma) = 0$ *for* $i = 1, 2$.
(2) Γ *is selfprojective if and only if every* $M \in \mathcal{M}_{sq}^{\Gamma^{op}}$ *is reflexive.*

3. The Gorenstein transpose

In this section, we introduce the notion of Gorenstein transpose of comodules. Throughout this section, Γ will be a Gorenstein coalgebra.

Definition. If $0 \to M \to I_0 \xrightarrow{f} I_1$ is a minimal Gorenstein injective copresentation of M in $\mathcal{M}^\Gamma$, that is, $0 \to M \to I_0$ and $Imf \to I_1$ are Gorenstein injective envelope, then we define $Tr_G M$ as a Γ^{op}-comodule which makes the sequence

$$0 \to Tr_G M \to I_1^* \xrightarrow{f^*} I_0^*$$

exact. We call $Ker(I_1^* \to I_0^*)$ a Gorenstein transpose of M.

Remark 3.1. It is trivial that a transpose of M in $\mathcal{M}^\Gamma$ is a Gorenstein transpose of M, but the converse is not true in general. For example, if a comodule $N \in \mathcal{M}^\Gamma$ is Gorenstein injective but not injective, then the Gorenstein transpose of N is zero, and any transpose of N is Gorenstein injective (see Corollary 3.5 below) but non-zero (otherwise, if a transpose of N is zero, then N is injective, which is a contradiction). So it is interesting to study the connections between Gorenstein transposes and transposes.

Lemma 3.2. *Let* $M \in \mathcal{M}^\Gamma$. *If* M *is Gorenstein injective, then so is* M^* *in* $\mathcal{M}^{\Gamma^{op}}$.

Proof. Let $M \in \mathcal{M}^\Gamma$ be Gorenstein injective. Then by the definition, there is an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n$$

where each E_i is an injective Γ-comodule. Now applying the functor $(\ )^*$ on the above sequence, we get an exact sequence

$$E_n^* \to \cdots \to E_1^* \to E_0^* \to M^* \to 0$$

where each E_i^* is an injective Γ^{op}-comodule by Proposition 2.2. Thus $M^* \in \mathcal{M}^{\Gamma^{op}}$ is Gorenstein injective by Theorem 3.5 in [1]. $\square$

The following theorem establishes a relation between a Gorenstein transpose of a comodule and a transpose of the same comodule.

Theorem 3.3. *Suppose that $M \in \mathcal{M}^\Gamma$. Then, for any Gorenstein transpose of M, there exists an exact sequence*

$$0 \to K \to TrM \to Tr_G M \to 0$$

in $\mathcal{M}^{\Gamma^{op}}$ with K Gorenstein injective.

Proof. Let $0 \to M \to I_0 \xrightarrow{f} I_1$ be a minimal Gorenstein injective copresentation of M. Let $K_1 = Imf$, $K_2 = Cokerf$ and $f = i\alpha$ be the natural epic-monic factorization of f. Then we have an exact sequence $0 \to Tr_G M \to I_1^* \xrightarrow{f^*} I_0^* \to M^* \to 0$. Since I_0 is a Gorenstein injective comodule, there exists an exact sequence $0 \to I_0 \to E_0 \to I_0' \to 0$ in $\mathcal{M}^\Gamma$ with E_0 injective and I_0' Gorenstein injective. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & I_0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & E_0 \arrow[r] \arrow[d] & K_1' \arrow[r] \arrow[d] & 0 \\
 & & I_0' \arrow[r, equal] \arrow[d] & I_0' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Diagram (1)

Now consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & K_2 \arrow[r] & 0 \\
 & I_0' \arrow[r, equal] \arrow[d] & I_0' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram (2)

Since I_1, I_0' are Gorenstein injective and the class $\mathcal{GI}$ is closed under extension (see Theorem 2.8 in [9]), it follows that I is Gorenstein injective. Thus there exists an exact sequence $0 \to I \to E_0' \to I' \to 0$ in $\mathcal{M}^{\Gamma}$ with E_0' injective and I' Gorenstein injective. Then we have the following push-out diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1' \arrow[r] & E_0' \arrow[r] \arrow[d] & K_2' \arrow[r] \arrow[d] & 0 \\
 & & I' \arrow[r, equal] \arrow[d] & I' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Diagram (3)

Combining the above commutative diagrams (2) and (3), we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1' \arrow[r] & E_0' \arrow[r] & K_2' \arrow[r] & 0
\end{tikzcd}$$

Diagram (4)

Then we have the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d] & E_0' \arrow[r] \arrow[d] & K_2' \arrow[r] \arrow[d] & 0 \\
 & I_0' \arrow[d] & H_2 \arrow[d] & I' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram (5)

where $H_2 = Coker(I_1 \to E_0')$. By the snake lemma, we get the exact sequence $0 \to I_0' \to H_2 \to I' \to 0$. Since I' and I_0' are Gorenstein injective, H_2 is Gorenstein injective. Combining the above diagram (5) with the diagram (1) in this proof, we obtain the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & E_0 \arrow[r] \arrow[d] & E'_0 \arrow[r] \arrow[d] & K'_2 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & I'_0 \arrow[r, "g"] \arrow[d] & H_2 \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram (6)

Applying the functor $(\ )^* = h_\Gamma(-, \Gamma)$ on above diagram (6), we get the following commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & kerg^* \arrow[r] \arrow[d] & H_2^* \arrow[r, "g^*"] \arrow[d] & I_0'^* \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & TrM \arrow[r] \arrow[d] & E_0'^* \arrow[r] \arrow[d] & E_0^* \arrow[r] \arrow[d] & M^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Tr_G M \arrow[r] & I_1^* \arrow[r] \arrow[d] & I_0^* \arrow[r] \arrow[d] & M^* \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Diagram (7)

By the snake lemma, we obtain an exact sequence

$$0 \to kerg^* \to TrM \to Tr_G M \to 0$$

in $\mathcal{M}^{\Gamma^{op}}$ with $kerg^* \cong I'^*$ Gorenstein injective. □

Corollary 3.4. *Let $M \in \mathcal{M}^\Gamma$. If $Tr_G M$ is quasi-finite, then*

$$ext^i_{\Gamma^{op}}(Tr_G M, \Gamma) \cong ext^i_{\Gamma^{op}}(TrM, \Gamma).$$

Proof. By Theorem 3.3 we have an exact sequence $0 \to K \to TrM \to Tr_G M \to 0$ in $\mathcal{M}^{\Gamma^{op}}$ with K Gorenstein injective. Applying the functor $(\ )^*$ on the sequence, we obtain a long exact sequence

$$\begin{aligned}\cdots \to ext^2_{\Gamma^{op}}(K,\Gamma) \to ext^1_{\Gamma^{op}}(Tr_G M,\Gamma) \to ext^1_{\Gamma^{op}}(TrM,\Gamma)\\ \to ext^1_{\Gamma^{op}}(K,\Gamma) \to (Tr_G M)^* \to (TrM)^* \to K^* \to 0.\end{aligned}$$

Since K is Gorenstein injective, it follows that $ext^i_{\Gamma^{op}}(K,\Gamma) \cong DExt^i_{\Gamma^{op}}(\Gamma,K) = 0$ for $i = 0,1\dots$. So we have $ext^i_{\Gamma^{op}}(Tr_G M,\Gamma) \cong ext^i_{\Gamma^{op}}(TrM,\Gamma)$. □

By Proposition 2.5 and Corollary 3.4, for any $M \in \mathcal{M}^\Gamma_{sq}$, if $Tr_G M \in \mathcal{M}^{\Gamma^{op}}$ is quasi-finite, then we obtain the following exact sequence:

$$0 \to ext^2_{\Gamma^{op}}(Tr_G M,\Gamma) \to M^{**} \xrightarrow{\alpha_\Gamma} M \to ext^1_{\Gamma^{op}}(Tr_G M,\Gamma) \to 0.$$

It is easy to see that if M is Gorenstein injective, then $M^{**} \cong M$.

Corollary 3.5. *Let $M \in \mathcal{M}^\Gamma$. Then we have*

(a) $Tr_G M = 0$ *if and only if M is Gorenstein injective.*

(b) *If M is Gorenstein injective, then TrM is Gorenstein injective.*

(c) *If $M \in \mathcal{M}^\Gamma_{sq}$ is Gorenstein injective and $Tr_G M \in \mathcal{M}^{\Gamma^{op}}_q$, then $M \cong M^{**}$.*

(d) *If $Tr_G M \in \mathcal{M}^{\Gamma^{op}}_q$, then $M \in \mathcal{M}^\Gamma_{sq}$ is a reflexive comodule if and only if $ext^i_{\Gamma^{op}}(Tr_G M,\Gamma) = 0$ for $i = 1,2$.*

Proof. By Theorem 3.3 and Corollary 3.4, it is easy to check. □

Proposition 3.6. *Let $\mathcal{A}$ be the full subcategory of $\mathcal{M}^\Gamma$ consisting of all right Γ-comodules A with $Gid_\Gamma A \leq 1$. Then the contravariant functors $Tr_G : \mathcal{A}^\Gamma/\mathcal{GI} \to \mathcal{M}^{\Gamma^{op}}$ and $ext^1_\Gamma(-,\Gamma) : \mathcal{A}^\Gamma/\mathcal{GI} \to \mathcal{M}^{\Gamma^{op}}$ are isomorphic, where $\mathcal{A}^\Gamma/\mathcal{GI}$ is the category $\mathcal{A}^\Gamma$ modulo Gorenstein injectives.*

Proof. Let $0 \to A \to I_0 \to I_1 \to 0$ be a minimal Gorenstein injective resolution for A in $\mathcal{A}^\Gamma$. Then

$$0 \to ext^1_\Gamma(A,\Gamma) \to I_1^* \to I_0^* \to A^* \to 0$$

is exact. In fact, $ext^1_\Gamma(I_0,\Gamma) = 0$ since I_0 is Gorenstein injective comodule and $pd_\Gamma\Gamma < \infty$. This gives an isomorphism $Tr_G A \cong ext^1_\Gamma(A,\Gamma)$ in $\mathcal{M}^{\Gamma^{op}}$ which is not difficult to be checked functorial in A. □

Next, we give a new relation between the Gorenstein transpose of a comodule and the transpose of the same comodule.

Theorem 3.7. *Let $M \in \mathcal{M}^\Gamma$. If $N \in \mathcal{M}^{\Gamma^{op}}$ is a Gorenstein transpose of M, then N is a transpose of L, where $0 \to I \to L \to M \to 0$ is an exact sequence in $\mathcal{M}^\Gamma$ with I Gorenstein injective.*

Proof. Let N be a Gorenstein transpose of M. Then there exists a minimal Gorenstein injective copresentation $0 \to M \to I_0 \xrightarrow{f} I_1$ of M. Applying the functor $(\ )^*$ on the above sequence, we obtain an exact sequence $0 \to Tr_G M \to$

$I_1^* \xrightarrow{f^*} I_0^* \to M^* \to 0$. Since I_1 is Gorenstein injective, there exists an exact sequence $0 \to I_1' \to E_1 \to I_1 \to 0$ in $\mathcal{M}^\Gamma$ with E_1 injective and I_1' Gorenstein injective. Let $K_1 = Imf$ and $K_2 = Cokerf$, then we get the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & I_1' \arrow[r, equal] \arrow[d] & I_1' \arrow[d] & & \\
0 \arrow[r] & K_1' \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram (8)

Hence we have the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I_1' \arrow[r, equal] \arrow[d] & I_1' \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d, "\pi"] & K_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & I_0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Diagram (9)

Since I_0, I_1' are Gorenstein injective and the class $\mathcal{GI}$ is closed under extension, it follows that I is also Gorenstein injective. Thus there is an exact sequence $0 \to I' \to E_0 \to I \to 0$ in $\mathcal{M}^\Gamma$ with E_0 injective and I' Gorenstein injective.

Now consider the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & I' \arrow[r, equal] \arrow[d] & I' \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\varphi"] & K_1' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & K_1' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Diagram (10)

Combining the above commutative diagrams (9) and (10), we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & I' \arrow[d] & H \arrow[d] & I_1' \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\pi\varphi"] & K_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram (11)

where $H = ker(\pi\varphi)$. By the snake lemma, we get the exact sequence $0 \to I' \to H \to I_1' \to 0$. Since I' and I_1' are Gorenstein injective, H is Gorenstein injective. Combining the above diagram (11) with the diagram (8) in this proof, we obtain the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & I' \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I_1' \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\pi\varphi"] & E_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & K_2 \arrow[r] & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

Diagram (12)

Now applying the functor $(\ )^* = h_\Gamma(-, \Gamma)$ to the diagram (12) yields the following exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Tr_G M \arrow[r] & I_1^* \arrow[r] \arrow[d] & I_0^* \arrow[r] \arrow[d] & M^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TrL \arrow[r] & E_1^* \arrow[r] \arrow[d] & E_0^* \arrow[r] \arrow[d] & L^* \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & I_1'^* \arrow[r] \arrow[d] & H^* \arrow[r] \arrow[d] & I'^* \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram (13)

By the snake lemma, it follows that $N = Tr_G M \cong TrL$, where $0 \to I' \to L \to M \to 0$ is an exact sequence in $\mathcal{M}^\Gamma$ with I' Gorenstein injective. $\square$

As an application of Theorem 3.7, we get that the Gorenstein transpose of a comodule can be decomposed into a direct sum of a transpose of the same comodule and a Gorenstein injective comodule.

Corollary 3.8. *If $M \in \mathcal{M}^\Gamma$ has finite injective dimension, then $Tr_G M = TrM \oplus I$, where I is a Gorenstein injective comodule.*

Proof. By Theorem 3.7, we have $Tr_G M = TrL$ for some right Γ-comodule L, and there is an exact sequence $0 \to I \to L \to M \to 0$ in $\mathcal{M}^\Gamma$ with I Gorenstein injective. The finiteness of injective dimension of M implies that $Ext^1_\Gamma(M, I) = 0$ from Theorem 3.5 in [1], which means that the sequence above is split. Hence $L = I \oplus M$, and then $Tr_G M = TrL = TrI \oplus TrM$ by Proposition 2.3, where TrI is a Gorenstein injective comodule by Corollary 3.5. $\square$

Proposition 3.9. *If $M \in \mathcal{M}^\Gamma$ is indecomposable, non-Gorenstein injective and* $\dim Tr_G M < \infty$*, then there exists an almost split sequence of the form*

$$0 \to M \to Y \to DTr_G M \to 0$$

in $\mathcal{M}^\Gamma$.

Proof. By Theorem 3.3, there exists an exact sequence $0 \to K \to TrM \to Tr_G M \to 0$ in $\mathcal{M}^{\Gamma^{op}}$ with K Gorenstein injective. Since M is indecomposable, not Gorenstein injective and $\dim Tr_G M < \infty$, it follows that there is an almost split sequence $0 \to M \to X \to DTrM \to 0$ in $\mathcal{M}^\Gamma$ by Theorem 4.2 in [4]. Thus we obtain the following pull-back diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, "h"] & DK \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & DTr_G M \arrow[r] \arrow[d] & DTrM \arrow[r] \arrow[d] & DK \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Then the first column is the desired sequence. In fact, since M is indecomposable, we only need to show that the morphism $f : Y \to DTr_G M$ is right almost split. It is easy to know that the exact sequence $0 \to M \to Y \to DTr_G M \to 0$ does not split, since the exact sequence $0 \to M \to X \to DTrM \to 0$ is almost split sequence. Since $h : X \to DTrM$ is right almost split, it follows that for every nonisomorphism $g : Z \to DTrM$ with Z indecomposable factors through h and then $Ext^1_\Gamma(Z, M) = 0$. Thus $g' : Z \to DTr_G M$ factors through f. So we have that $f : Y \to DTr_G M$ is right almost split. $\square$

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