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WEIGHTED ESTIMATES FOR CERTAIN ROUGH OPERATORS WITH APPLICATIONS TO VECTOR VALUED INEQUALITIES

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Abstract. Under certain rather weak size conditions assumed on the kernels, some weighted norm inequalities for singular integral operators, related maximal operators, maximal truncated singular integral operators and Marcinkiewicz integral operators in nonisotropic setting will be shown. These weighted norm inequalities will enable us to obtain some vector valued inequalities for the above operators.

1. Introduction

Over the last several years an active topic of research is to investigate the weighted norm inequalities for various of integral operators, such as Hardy-Littlewood maximal operator, rough singular integral operators and so on. A classic example was due to Fefferman and Stein [14], who showed that

$$\int_{\mathbb{R}^n}(Mf(x))^p u(x)dx \leq C_p \int_{\mathbb{R}^n}|f(x)|^p Mu(x)dx,$$

holds for all $1 < p < \infty$ and any weight function u. Here M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$. As an immediate application of the above weighted inequality, the following vector valued inequality for M is valid:

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}(Mf_j)^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)} \leq C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)}, \quad 1<p<q<\infty.$$

The primary motivation of this paper is to establish certain weighted norm inequalities for singular integral operators, related maximal operators and truncated singular integral operators as well as Marcinkiewicz integral operators in nonisotropic setting when their integral kernels are given by $\Omega \in L^q(\Sigma)$ and

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$h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$. The above weighted norm inequalities will enable us to obtain some vector valued inequalities for the above operators.

We now recall some notations and background. Let $n \geq 2$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space with a non-isotropic dilation. Precisely, let P be an $n \times n$ real matrix whose eigenvalues have positive real parts and let $\alpha = \mathrm{trac}P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^P = \exp((\log t)P)$. There is a non-negative function r on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$. The function r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; furthermore it satisfies:

(a) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;

(b) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;

(c) if $\Sigma = \{x \in \mathbb{R}^n \mid r(x) = 1\}$, then $\Sigma = \{\theta \in \mathbb{R}^n \mid \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$; And then, the Lebesgue measure can be written as $dx = t^{\alpha-1} d\sigma dt$, that is,

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_\Sigma f(A_t\theta)t^{\alpha-1}d\sigma(\theta)dt$$

for appropriate functions f, where $d\sigma$ is a $\mathcal{C}^\infty$ measure on Σ.

(d) there are positive constants c_1, c_2, c_3, c_4, α_1, α_2, β_1 and β_2 such that

$$c_1|x|^{\alpha_1} \leq r(x) \leq c_2|x|^{\alpha_2} \quad \text{if } r(x) \geq 1,$$

$$c_3|x|^{\beta_1} \leq r(x) \leq c_4|x|^{\beta_2} \quad \text{if } r(x) \leq 1.$$

See [20, 24] for more details. Let Ω be a locally integrable function and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$. Assume that

$$\int_\Sigma \Omega(\theta)d\sigma(\theta) = 0. \tag{1}$$

We consider a singular integral operator $T_{h,\Omega}$ by

$$T_{h,\Omega}f(x) := \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y')h(r(y))}{r(y)^\alpha}dy,$$

where $y' = A_{r(y)^{-1}}y$, $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class on $\mathbb{R}^n$) and $h \in \Delta_1(\mathbb{R}_+)$. Here $\Delta_\gamma(\mathbb{R}_+)$ ($\gamma \geq 1$) denotes the collection of measurable functions h on $\mathbb{R}_+ := (0, \infty)$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0} \Big(\frac{1}{R}\int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is easy to check that $L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+)$ and $\Delta_{\gamma_1}(\mathbb{R}_+) \subset \Delta_{\gamma_2}(\mathbb{R}_+)$ for any $1 \leq \gamma_2 < \gamma_1 < \infty$, which is a proper inclusion. In addition, we also consider the related maximal functions, maximal truncated singular integrals and Marcinkiewicz integrals. To be precise, we define the maximal functions

$M_{h,\Omega}$ and maximal truncated singular integral operator $T^*_{h,\Omega}$ by

$$M_{h,\Omega}f(x) = \sup_{t>0} \frac{1}{t^\alpha} \int_{r(y)<t} |f(x-y)||\Omega(y')h(r(y))|dy,$$

$$T^*_{h,\Omega}f(x) = \sup_{\epsilon>0} \Big| \int_{r(y)>\epsilon} f(x-y) \frac{\Omega(y')h(r(y))}{r(y)^\alpha} dy \Big|.$$

The parametric Marcinkiewicz integral operator $\mathfrak{M}_{h,\Omega,\varrho}$ is defined as

$$\mathfrak{M}_{h,\Omega,\varrho}f(x) = \Big(\int_0^\infty \Big| \frac{1}{t^\varrho} \int_{r(y)\le t} f(x-y) \frac{\Omega(y')h(r(y))}{r(y)^{\alpha-\rho}} dy \Big|^2 \frac{dt}{t} \Big)^{1/2},$$

where $\varrho = \tau + i\vartheta$ (τ, $\vartheta \in \mathbb{R}$ with $\tau > 0$).

When $A_t = tE$ with E being the identity matrix and $r(x) = |x|$, then Σ reduces to the unit sphere in $\mathbb{R}^n$ denoted by S^{n-1} and we denote $T_{h,\Omega} = \tilde{T}_{h,\Omega}$. When $h \equiv 1$, the operator $\tilde{T}_{h,\Omega}$ recovers the classical singular integral operator $\tilde{T}_\Omega$, which was initiated in the seminal work of Calderón and Zygmund [6]. A celebrated work was due to Calderón and Zygmund [7] who established the L^p boundedness for $\tilde{T}_\Omega$ with $1 < p < \infty$ if $\Omega \in L\log L(\mathrm{S}^{n-1})$ by introducing the "method of rotations". A discovery that the Calderón-Zygmund rotation method is no longer to be available if the operator $\tilde{T}_{h,\Omega}$ is also rough in the radial direction was given by R. Fefferman [13] who proved that $\tilde{T}_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ if $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$ and $h \in L^\infty(\mathbb{R}_+)$. Later on, J. Namazi [19] improved Fefferman's result to the case $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$. Subsequently, J. Duoandikoetxea and J. L. Rubio de Francia [11] used the Littlewood-Paley theory to improve the above radial kernel condition $h \in L^\infty(\mathbb{R}_+)$ to the case $h \in \Delta_2(\mathbb{R}_+)$. Since then, the above results have been improved and extended by many authors (see [2, 12]). For the nonisotropic singular integrals, we can consult [5, 20–22, 24], among others. Particularly, Sato [22] proved the following result.

Theorem A ([22])**.** *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1,2]$. Then*

$$\|T_{h,\Omega}f\|_{L^p(\mathbb{R}^n)} \le C_p(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $p \in (1,\infty)$, where the constant $C_p > 0$ is independent of h, Ω, q, γ.

Here $L^q(\Sigma)$ for $q > 1$ denote as the space of all those functions Ω on Σ which satisfy $\|\Omega\|_q = (\int_\Sigma |\Omega(\theta)|^q d\sigma(\theta))^{1/q} < \infty$.

When $A_t = tE$ with E being the identity matrix and $r(x) = |x|$ (the Euclidean norm), then $\mathfrak{M}_{h,\Omega,\varrho}$ reduces to the classic parametric Marcinkiewicz integral operator, which was initiated in the seminal work of E. M. Stein [23] for $\rho \equiv h \equiv 1$ and has been studied by many authors (see [1, 3, 8, 10, 25, 26]). We can consult [18] for the nonisotropic Marcinkiewicz integral operators.

The primary purpose of this paper is to establish some weighted norm inequalities for the operators $T_{h,\Omega}$, $T^*_{h,\Omega}$, $M_{h,\Omega}$ and $\mathfrak{M}_{h,\Omega,\varrho}$ when their integral kernels are given by $\Omega \in L^q(\Sigma)$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$. Our main results can be formulated as follows:

Theorem 1.1. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$.*

(i) *Let $p \in [2, \infty)$. Then for any nonnegative measurable function u on $\mathbb{R}^n$ and $s > 1$, the following inequality holds*:

$$\|T_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u)}. \tag{2}$$

(ii) *Let $1 < p < 2$ and $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying*

$$t_1 = \frac{2}{p}, \quad \frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1 - \frac{1}{t_k}\Big), \quad k = 1, 2, \dots.$$

Then for any nonnegative measurable function u on $\mathbb{R}^n$ and any fixed $k \in \mathbb{N}$ and $s > t_k$, the following inequality holds:

$$\|T_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)}. \tag{3}$$

Here M *denotes the Hardy-Littlewood maximal operator with respect to the function $r(\cdot)$. For $s > 1$, $\mathrm{M}_s u = (\mathrm{M} u^s)^{1/s}$, $M_s^{\tilde{\sigma}}(u) = (M^{\tilde{\sigma}} u^s)^{1/s}$, $M^{\tilde{\sigma}} f(x) = \sup_{k\in\mathbb{Z}} ||\tilde{\sigma}_k| * f(x)|$, where $|\tilde{\sigma}_k|$ is defined by*

$$\int_{\mathbb{R}^n} f(x) d|\tilde{\sigma}_k|(x) = \int_{2^{k-1} < r(y) \leq 2^k} f(-y) \frac{|\Omega(y') h(r(y))|}{r(y)^\alpha} dy.$$

Theorem 1.2. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$.*

(i) *Let $p \in [2, \infty)$. Then for any nonnegative measurable function u on $\mathbb{R}^n$ and $s > 1$, the following inequality holds*:

$$\|M_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)}. \tag{4}$$

(ii) *Let $p \in (1, 2)$ and $\{t_k\}_k$ be given as in* (ii) *of Theorem 1.1. Then for any nonnegative measurable function u on $\mathbb{R}^n$ and any fixed $k \in \mathbb{N}$ and $s > t_k$, the following inequality holds*:

$$\|M_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)}. \tag{5}$$

Here M^k denotes the Hardy-Littlewood maximal operator M *iterated k times for all $k \in \mathbb{N}$ and $\mathrm{M}_s^2 u = (\mathrm{M}^2 u^s)^{1/s}$.*

Theorem 1.3. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$.*

(i) *Let $p \in [2, \infty)$. Then for any nonnegative measurable function u on $\mathbb{R}^n$ and $s > 1$, the following inequality holds*:

$$\|T^*_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)}. \tag{6}$$

(ii) *Let $p \in (1, 2)$ and $\{t_k\}_k$ be given as in* (ii) *of Theorem 1.1. Then for any nonnegative measurable function u on $\mathbb{R}^n$ and any fixed $k \in \mathbb{N}$ and $s > t_k$, the following inequality holds*:

$$\|T^*_{h,\Omega} f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)}. \tag{7}$$

Theorem 1.4. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$. Then for any nonnegative measurable function u on $\mathbb{R}^n$, $s > 1$ and $p \in (1, \infty)$, the following inequality holds*:

$$\|\mathfrak{M}_{h,\Omega,\varrho} f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,\varrho,p,s} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{p}} \mathrm{M}_s u)}. \tag{8}$$

As applications of Theorems 1.1-1.4, we can get the following vector valued inequalities for the above operators, which are listed as follows:

Corollary 1.5. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $q, \gamma \in (1, \infty]$. Then for $1 < p, \tilde{p} < \infty$, the following inequality holds*:

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |T_{h,\Omega} f_j|^{\tilde{p}} \Big)^{1/\tilde{p}} \Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p}} \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \Big)^{1/\tilde{p}} \Big\|_{L^p(\mathbb{R}^n)}. \tag{9}$$

Corollary 1.6. *Let $\Omega \in L^q(\Sigma)$ satisfy* (1) *and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some q, $\gamma \in (1, \infty]$. Then for $1 < \tilde{p} \le p < \infty$, the following inequality holds*:

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |T^*_{h,\Omega} f_j|^{\tilde{p}} \Big)^{1/\tilde{p}} \Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p,\tilde{p}} \Big\| \Big(\sum_{j \in \mathbb{Z}} |f_j|^{\tilde{p}} \Big)^{1/\tilde{p}} \Big\|_{L^p(\mathbb{R}^n)}. \tag{10}$$

The same results hold for the operator $M_{h,\Omega}$ and $\mathfrak{M}_{h,\Omega,\varrho}$.

Remark 1.7. It was shown in [22] that both $T^*_{h,\Omega}$ and $M_{h,\Omega}$ are bounded on $L^p(\mathbb{R}^n)$ if h, Ω satisfy the conditions of Theorem 1.1. The same conclusion also holds for $\mathfrak{M}_{h,\Omega,\varrho}$ (see [18]).

Remark 1.8. Our main results are new even in the special case $h(t) \equiv 1$.

The rest of this paper is organized as follows. Section 2 contains some preliminary notations and lemmas, which are the basis of our proofs. In Section 3 we shall prove Theorems 1.1-1.4. The proofs of Corollaries 1.5 and 1.6 will be presented in Section 4. It should be pointed out that the main idea in the proofs of our main results is a combination of ideas and arguments from [15, 17, 18, 22]. The main tools of our proofs are the weighted Littlewood-Paley theory followed from [15] and some iteration arguments, which are originated from [11] and developed in [15, 17].

Throughout this note, for any $p \in (1, \infty)$, let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. For any function f, we define $\tilde{f}$ by $\tilde{f}(x) = f(-x)$. For $f \in L^p(u)$, we set

$$\|f\|_{L^p(u)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p u(x) dx \Big)^{1/p}.$$

2. Some notations and lemmas

Following the notation in [22], let P^* denote the adjoint of the matrix P. Then $A_t^* = \exp((\log t) P^*)$. We can define a non-negative function s from $\{A_t^*\}$ exactly in the same way as we define r from $\{A_t\}$. It was shown in [24] that

$$d_1 |\xi|^{a_1} < s(\xi) < d_2 |\xi|^{a_2}, \quad \text{if } s(\xi) \ge 1; \tag{11}$$

$$d_3|\xi|^{b_1} < s(\xi) < d_4|\xi|^{b_2}, \quad \text{if } 0 < s(\xi) \le 1, \tag{12}$$

where d_j $(j = 1, 2, 3, 4)$, a_k, b_k $(k = 1, 2)$ are positive constants. From (11) and (12) we have that there exist two positive constants C_1, C_2 such that

$$|\xi| \le C_1(s(\xi)^{1/a_1} + s(\xi)^{1/b_1}), \tag{13}$$

$$|\xi|^{-1} \le C_2(s(\xi)^{-1/a_2} + s(\xi)^{-1/b_2}). \tag{14}$$

For $k \in \mathbb{Z}$, we define the measures σ_k and $|\sigma_k|$ respectively by

$$\widehat{\sigma_k}(x) = \int_{2^{k-1}<r(y)\le 2^k} \exp(-2\pi i y\cdot x)\frac{\Omega(y')h(r(y))}{r(y)^\alpha}dy, \tag{15}$$

$$\widehat{|\sigma_k|}(x) = \int_{2^{k-1}<r(y)\le 2^k} \exp(-2\pi i y\cdot x)\frac{|\Omega(y')h(r(y))|}{r(y)^\alpha}dy. \tag{16}$$

To estimate some measures, we need the following estimate of oscillatory integral, which follows from [22, Corollary 4.2].

Lemma 2.1. ([22]) *Let L be the degree of the minimal polynomial of P and $\Psi \in \mathcal{C}^1([a,b])$ with $0 < a < b$. Then for ξ, $\eta \in \mathbb{R}^n \setminus \{0\}$, there exists a constant $C > 0$ independent of ξ, η and Ψ such that*

$$\Big|\int_a^b \exp(i\eta\cdot A_t\xi)\Psi(t)dt\Big| \le C|\eta\cdot P\xi|^{-1/L}\Big(\sup_{t\in[a,b]}|\Psi(t)| + \int_a^b|\Psi'(t)|dt\Big).$$

Applying Lemma 2.1, we have:

Lemma 2.2. *Let Ω satisfy (1). Suppose that $\Omega \in L^q(\Sigma)$ for some $q \in (1,\infty)$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,\infty)$. Then for all $k \in \mathbb{Z}$, there exists a constant $C > 0$ independent of h, Ω, q, γ such that*

$$\max\big\{|\widehat{\sigma_k}(\xi)|, \big|\widehat{|\sigma_k|}(\xi) - \widehat{|\sigma_k|}(0)\big|\big\} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{2^k}\xi|^{1/(4q'\gamma' L)}; \tag{17}$$

$$\max\big\{|\widehat{\sigma_k}(\xi)|, \big|\widehat{|\sigma_k|}(\xi)\big|\big\} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\min\{1, |A^*_{2^k}\xi|^{-1/(4q'\gamma' L)}\}. \tag{18}$$

Proof. By a change of variables and Hölder's inequality, we get

$$\begin{aligned}\max\{|\widehat{\sigma_k}(\xi)|, |\widehat{|\sigma_k|}(\xi)|\} &\le \int_{2^{k-1}<r(y)\le 2^k}\frac{|\Omega(y')h(r(y))|}{r(y)^\alpha}dy\\ &= \int_{2^{k-1}}^{2^k}\int_\Sigma|\Omega(\theta)|d\sigma(\theta)\frac{|h(u)|}{u}du\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}.\end{aligned} \tag{19}$$

By a change of variables, (1) and Hölder's inequality, one has

$$\begin{aligned}
|\widehat{\sigma_k}(\xi)| &= \Big|\int_{2^{k-1}}^{2^k}\int_\Sigma (\exp(-2\pi i\xi\cdot A_u\theta)-1)\Omega(\theta)d\sigma(\theta)\frac{h(u)}{u}du\Big| \\
&\le C\int_{2^{k-1}}^{2^k}\int_\Sigma |\Omega(\theta)||\xi\cdot A_u\theta|d\sigma(\theta)|h(u)|\frac{du}{u} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{2^{k-1}}^{2^k}\Big|\int_\Sigma|\Omega(\theta)||\xi\cdot A_u\theta|d\sigma(\theta)\Big|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{1/2}^{1}\Big|\int_\Sigma|\Omega(\theta)||A^*_{2^k}\xi\cdot A_u\theta|d\sigma(\theta)\Big|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{2^k}\xi|,
\end{aligned}$$

which together with (19) implies that

$$|\widehat{\sigma_k}(\xi)|\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{2^k}\xi|^{1/(4q'\gamma'L)}. \tag{20}$$

Similarly, we can get

$$\big|\widehat{|\sigma_k|}(\xi)-\widehat{|\sigma_k|}(0)\big|\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{2^k}\xi|^{1/(4q'\gamma'L)}. \tag{21}$$

Combining (21) with (20) yields (17).

On the other hand, by a change of variables and Hölder's inequality,

$$\begin{aligned}
&|\widehat{\sigma_k}(\xi)| \\
&= \Big|\int_{2^{k-1}}^{2^k}\int_\Sigma \exp(-2\pi i\xi\cdot A_u\theta)\Omega(\theta)d\sigma(\theta)\frac{h(u)}{u}du\Big| \\
&\le \int_{2^{k-1}}^{2^k}\Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_u\theta)\Omega(\theta)d\sigma(\theta)\Big||h(u)|\frac{du}{u} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{2^{k-1}}^{2^k}\Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_u\theta)\Omega(\theta)d\sigma(\theta)\Big|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}^{\max\{0,1-2/\gamma'\}} \\
&\quad\times\Big(\int_{2^{k-1}}^{2^k}\Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_u\theta)\Omega(\theta)d\sigma(\theta)\Big|^{2}\frac{du}{u}\Big)^{1/\max\{2,\gamma'\}}.
\end{aligned}\tag{22}$$

Invoking Lemma 2.1 and using Hölder's inequality, we have

$$\begin{aligned}
&\int_{2^{k-1}}^{2^k}\Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_u\theta)\Omega(\theta)d\sigma(\theta)\Big|^{2}\frac{du}{u} \\
&= \int_{2^{k-1}}^{2^k}\iint_{\Sigma\times\Sigma}\exp(-2\pi i A^*_u\xi\cdot(\theta-w))\Omega(\theta)\overline{\Omega(w)}d\sigma(\theta)d\sigma(w)\frac{du}{u} \\
&= \int_{1/2}^{1}\iint_{\Sigma\times\Sigma}\exp(-2\pi i A^*_{2^ku}\xi\cdot(\theta-w))\Omega(\theta)\overline{\Omega(w)}d\sigma(\theta)d\sigma(w)\frac{du}{u} \\
&\le \iint_{\Sigma\times\Sigma}\Big|\int_{1/2}^{1}\exp(-2\pi i A^*_{2^k}\xi\cdot A_u(\theta-w))\frac{du}{u}\Big| \\
&\quad\times|\Omega(\theta)\overline{\Omega(w)}|d\sigma(\theta)d\sigma(w)
\end{aligned}\tag{23}$$

$$\le C\iint_{\Sigma\times\Sigma}|A_{2^k}^*\xi\cdot P(\theta-w))|^{-\epsilon}|\Omega(\theta)\overline{\Omega(w)}|d\sigma(\theta)d\sigma(w)$$
$$\le C\|\Omega\|_{L^q(\Sigma)}^2\Big(\iint_{\Sigma\times\Sigma}|P^*A_{2^k}^*\xi\cdot(\theta-w)|^{-\epsilon q'}d\sigma(\theta)d\sigma(w)\Big)^{1/q'}$$
$$\le C\|\Omega\|_{L^q(\Sigma)}^2|A_{2^k}^*\xi|^{-\epsilon},$$

for any $0<\epsilon<\min\{1/(2q'),1/L\}$, where the last inequality follows from [11, p. 553] (see also [22, p. 418]). In light of (22) and (23) we would have

$$|\widehat{\sigma_k}(\xi)|\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A_{2^k}^*\xi|^{-1/(2q'\max\{2,\gamma'\}L)}. \tag{24}$$

Similarly, we have

$$\big|\widehat{|\sigma_k|}(\xi)\big|\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A_{2^k}^*\xi|^{-1/(2q'\max\{2,\gamma'\}L)}. \tag{25}$$

Combining (25) with (24) and (19) implies (18). □

The following result is an application of iteration argument followed from [11], which will play a key role in the proofs of our main results.

Lemma 2.3. *Let h,Ω be given as in Lemma 2.2. Then, for all $p\in(1,\infty)$, it holds that*

$$\|M^\sigma f\|_{L^p(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p(\mathbb{R}^n)}. \tag{26}$$

Proof. Let ψ be a nonnegative $\mathcal{C}_0^\infty(\mathbb{R}^n)$ function such that $\operatorname{supp}(\psi)\subset\{x\in\mathbb{R}^n: s(x)\le 1\}$ and $\psi(t)\equiv 1$ when $s(x)<1/2$. For $k\in\mathbb{Z}$, we define the function $\psi_k(x)=2^{-k\alpha}\psi^\vee(A_{2^{-k}}x)$. It is clear that $\hat{\psi}_k(x)=\psi(A_{2^k}^*x)$. Define the measures $\{\nu_k\}_k$ by

$$\nu_k(\xi)=|\sigma_k|(\xi)-\psi_k(\xi)\widehat{|\sigma_k|}(0). \tag{27}$$

Applying Lemma 2.2 and (13)-(14), we get

$$\begin{aligned}|\widehat{\nu_k}(\xi)|\le{}& C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\\&\times\big((2^ks(\xi))^{1/(4q'\gamma'a_1L)}+(2^ks(\xi))^{1/(4q'\gamma'b_1L)}\big);\end{aligned} \tag{28}$$

$$\begin{aligned}|\widehat{\nu_k}(\xi)|\le{}& C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\\&\times\min\big\{1,(2^ks(\xi))^{-1/(4q'\gamma'a_2L)}+(2^ks(\xi))^{-1/(4q'\gamma'b_2L)}\big\},\end{aligned} \tag{29}$$

where $C>0$ is independent of h,Ω,q,γ. Moreover, it is easy to verify that

$$M^\sigma f\le G_\nu(f)+\widehat{|\sigma_k|}(0)\mathrm{M}|f|, \tag{30}$$

$$M^\nu f\le M^\sigma f+\widehat{|\sigma_k|}(0)\mathrm{M}|f|, \tag{31}$$

where $M^\nu f=\sup_{k\in\mathbb{Z}}||\nu_k|*f|$ and $G_\nu(f)=(\sum_{k\in\mathbb{Z}}|\nu_k*f|^2)^{1/2}$. Applying the L^p boundedness of M, we get

$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)} \tag{32}$$

for all $p\in(1,\infty)$. A standard iteration argument from [11] together with (28)-(32) may yield (26). The details are omitted. □

3. Proofs of Theorems 1.1-1.4

This section is devote to proving Theorems 1.1-1.4. In what follows, we fix a nonnegative measurable function u on $\mathbb{R}^n$.

Proof of Theorem 1.1. We split the proof into two steps:

Step 1: Proof of part (i). Let $\{\sigma_k\}_k$ be defined as in (15). We define the maximal operator $M^{\tilde{\sigma}}$ by

$$M^{\tilde{\sigma}} f(x) = \sup_{k\in\mathbb{Z}} \big||\tilde{\sigma}_k| * f(x)\big|,$$

where

$$\int_{\mathbb{R}^n} f(x) d|\tilde{\sigma}_k|(x) = \int_{\mathbb{R}^n} f(-x) d|\sigma_k|(x).$$

One can easily verify that

$$M^{\tilde{\sigma}} f(x) = M^{\sigma} \tilde{f}(x); \tag{33}$$

$$T_{h,\Omega} f(x) = \sum_{k\in\mathbb{Z}} \sigma_k * f(x). \tag{34}$$

Let $\Psi \in \mathcal{C}_c^\infty((1/4,1))$ such that $0 \le \Psi \le 1$ and $\sum_{k\in\mathbb{Z}} (\Psi(2^k s(\xi)))^3 = 1$ for every $\xi \in \mathbb{R}^n$. Define the Fourier multiplier operators $\{S_k\}_k$ by $S_k f(x) = \Theta_k * f(x)$, where $\widehat{\Theta_k}(\xi) = \Psi(2^k s(\xi))$. It was proved in [15] that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |S_k f|^2\Big)^{1/2}\Big\|_{L^p(w)} \le C_{p,w} \|f\|_{L^p(w)} \tag{35}$$

and

$$\Big\|\sum_{k\in\mathbb{Z}} S_k f_k\Big\|_{L^p(w)} \le C_{p,w} \Big\|\Big(\sum_{k\in\mathbb{Z}} |f_k|^2\Big)^{1/2}\Big\|_{L^p(w)} \tag{36}$$

for all $p \in (1,\infty)$ and $w \in A_p$ (the Muckenhoupt weight class).

We can write

$$\begin{aligned} T_{h,\Omega} f(x) &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} S_{j+k}^3 (\sigma_k * f)(x) \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} S_{j+k}^3 (\sigma_k * f)(x) =: \sum_{j\in\mathbb{Z}} T_j f(x). \end{aligned} \tag{37}$$

Applying Lemma 2.2, we get from (13) and (14) that

$$|\widehat{\sigma_k}(\xi)| \le C_{h,\Omega} \big((2^k s(\xi))^{1/(4q'\gamma' a_1 L)} + (2^k s(\xi))^{1/(4q'\gamma' b_1 L)}\big); \tag{38}$$

$$|\widehat{\sigma_k}(\xi)| \le C_{h,\Omega} \min\big\{1, (2^k s(\xi))^{-1/(4q'\gamma' a_2 L)} + (2^k s(\xi))^{-1/(4q'\gamma' b_2 L)}\big\}. \tag{39}$$

By (38), (39) and Plancherel's theorem, we have

$$\|\sigma_k * S_{j+k} w\|_{L^2(\mathbb{R}^n)} \le C_{h,\Omega} 2^{-\delta_1 |j|} \|w\|_{L^2(\mathbb{R}^n)} \tag{40}$$

for some $\delta_1 > 0$ and any arbitrary function w on $\mathbb{R}^n$. Here δ_1 depends on q, γ, L.

Fix $s>1$. One may get that

$$\begin{aligned}&\|\sigma_k * S_{j+k}w\|_{L^2(u^s)}\\ &\le (\|\sigma_k\|\|\Theta_{j+k}\|_{L^1(\mathbb{R}^n)})^{1/2}\Big(\int_{\mathbb{R}^n}|\sigma_k|*|\Theta_{j+k}|*|w|^2(x)u^s(x)dx\Big)^{1/2}\\ &\le C_{h,\Omega}\|w\|_{L^2(\mathrm{M}M^{\tilde{\sigma}}u^s)}.\end{aligned}\tag{41}$$

An interpolation of L^2-spaces with change of measure ([4, Theorem 5.4.1]) between (40) and (41) implies that

$$\|\sigma_k * S_{j+k}w\|_{L^2(u)} \le C_{h,\Omega}2^{-\delta_1(1-1/s)|j|}\|w\|_{L^2(\mathrm{M}_s M_s^{\tilde{\sigma}}u)}.\tag{42}$$

Note that $\mathrm{M}_s M_s^{\tilde{\sigma}}u\in A_1$. By (42) with $w=S_{j+k}f$ and (36),

$$\begin{aligned}\|T_jf\|_{L^2(u)} &= \Big\|\sum_{k\in\mathbb{Z}}S^3_{j+k}\sigma_k * f\Big\|_{L^2(u)}\\ &\le C\Big(\sum_{k\in\mathbb{Z}}\|\sigma_k * S^2_{j+k}f\|^2_{L^2(u)}\Big)^{1/2}\\ &\le C_{h,\Omega}2^{-\delta_1(1-1/s)|j|}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j+k}f|^2\Big)^{1/2}\Big\|_{L^2(\mathrm{M}_s M_s^{\tilde{\sigma}}u)}\\ &\le C_{h,\Omega}2^{-\delta_1(1-1/s)|j|}\|f\|_{L^2(\mathrm{M}_s M_s^{\tilde{\sigma}}u)}.\end{aligned}\tag{43}$$

We now prove that

$$\|T_jf\|_{L^p(u)} \le C_{h,\Omega,s}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u)}\tag{44}$$

for all $p\in(2,\infty)$. Note that $\|\sigma_k\|\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}$. By Lemma 2.3 and the arguments similar to those used in getting [17, (3.25)], we obtain

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k * g_k|^2\Big)^{1/2}\Big\|_{L^p(u)} \le C_{h,\Omega,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M_s^{\tilde{\sigma}}u)}\tag{45}$$

for all $p\in(2,\infty)$ and any $s>1$. It was noted that $u\le \mathrm{M}_s u$ and $\mathrm{M}_s u\in A_1$ (see [9]). By (35), (36), (45) and the fact $M_s^{\tilde{\sigma}}u\le \mathrm{M}_s M_s^{\tilde{\sigma}}u\in A_1$,

$$\begin{aligned}\|T_jf\|_{L^p(u)} &= \Big\|\sum_{k\in\mathbb{Z}}S^3_{j+k}\sigma_k * f\Big\|_{L^p(u)}\\ &\le \Big\|\sum_{k\in\mathbb{Z}}S^3_{j+k}\sigma_k * f\Big\|_{L^p(\mathrm{M}_s u)}\\ &\le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_k * S^2_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_s u)}\\ &\le C_{h,\Omega,s,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^2_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(M_s^{\tilde{\sigma}}\mathrm{M}_s u)}\\ &\le C_{h,\Omega,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u)}\end{aligned}$$

for all $p\in(2,\infty)$. This proves (44). By an interpolation between (43) and (44) (see [4, Corollary 5.5.4]) and the fact that $u\le \mathrm{M}_s u$, we have

$$\|T_jf\|_{L^p(u)} \le C_{h,\Omega,s,p}2^{-\delta_1(1-1/s)|j|}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u)}\tag{46}$$

for some $c > 0$. Combining (46) with (37) yields (2) and completes the proof of part (i).

Step 2: Proof of part (ii). We now prove (3). Let $\{t_k\}_k$ be given as in (ii) of Theorem 1.1. To prove (3), it suffices to show that

$$\|T_{h,\Omega}f\|_{L^p(u^{1/s})} \le C_{h,\Omega,q,\gamma,p,s}\|f\|_{L^p((\mathrm{M}M^{\tilde{\sigma}}u+\mathrm{M}^2u)^{1/s})} \tag{47}$$

for any fixed $k \in \mathbb{N}$, all $s > t_k$ and $p \in (1,2)$. Let $\{\nu_k\}_k$ be given as in (27). By Minkowski's inequality,

$$\begin{aligned} G_\nu f &= \Big(\sum_{k\in\mathbb{Z}}\Big|\nu_k * \sum_{j\in\mathbb{Z}} S^3_{j+k}f\Big|^2\Big)^{1/2} \\ &\le \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}|\nu_k * S^3_{j+k}f|^2\Big)^{1/2} =: \sum_{j\in\mathbb{Z}} G_j f. \end{aligned} \tag{48}$$

It is clear that

$$\|\nu_k * f\|_{L^\infty(\mathbb{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^\infty(\mathbb{R}^n)}; \tag{49}$$

$$\|\nu_k * f\|_{L^1(u)} \le C\|f\|_{L^1(M^{\tilde{\nu}}u)}. \tag{50}$$

An interpolation between (49) and (50) gives us that

$$\|\nu_k * f\|_{L^p(u)} \le C_{h,\Omega}\|f\|_{L^p(M^{\tilde{\nu}}u)} \le C_{h,\Omega}\|f\|_{L^p(\mathrm{M}M^{\tilde{\nu}}u)} \tag{51}$$

for all $p \in (1,2)$. From (51) we get

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * f_k|^p\Big)^{1/p}\Big\|_{L^p(u)} \le C_{h,\Omega}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^p\Big)^{1/p}\Big\|_{L^p(\mathrm{M}M^{\tilde{\nu}}u)} \tag{52}$$

for all $p \in (1,2)$. On the other hand, we get from (26), (30) and (31) that

$$\|M^\nu f\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega}\|f\|_{L^p(\mathbb{R}^n)} \tag{53}$$

for all $p \in (1,\infty)$. From (53) we get

$$\Big\|\sup_{k\in\mathbb{Z}}|\nu_k * f_k|\Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega}\Big\|\sup_{k\in\mathbb{Z}}|f_k|\Big\|_{L^p(\mathbb{R}^n)} \tag{54}$$

for all $p \in (1,2)$. An interpolation between (52) and (54) yields that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * f_k|^2\Big)^{1/2}\Big\|_{L^p(u^{1/t_1})} \le C_{h,\Omega}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p((\mathrm{M}M^{\tilde{\nu}}u)^{1/t_1})} \tag{55}$$

for all $p \in (1,2)$, where $t_1 = 2/p$. By Substituting u^{t_1} for u in (55),

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * f_k|^2\Big)^{1/2}\Big\|_{L^p(u)} \le C_{h,\Omega}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_{t_1}M^{\tilde{\nu}}_{t_1}u)}. \tag{56}$$

Since $\mathrm{M}_{t_1} M_{t_1}^{\tilde{\nu}} u \in A_1$, we get by the weighted Littlewood-Paley theory and (56) that

$$\begin{aligned}\|G_j f\|_{L^p(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}} |\nu_k * S^3_{j+k} f|^2\Big)^{1/2}\Big\|_{L^p(u)} \\ &\le C_{h,\Omega}\Big\|\Big(\sum_{k\in\mathbb{Z}} |S^3_{j+k} f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_{t_1} M_{t_1}^{\tilde{\nu}} u)} \\ &\le C_{h,\Omega}\|f\|_{L^p(\mathrm{M}_{t_1} M_{t_1}^{\tilde{\nu}} u)}\end{aligned} \tag{57}$$

for all $p \in (1, 2)$. By substituting u^{1/t_1} for u in (57),

$$\|G_j f\|_{L^p(u^{1/t_1})} \le C\|f\|_{L^p((\mathrm{M} M^{\tilde{\nu}} u)^{1/t_1})} \tag{58}$$

for all $p \in (1, 2)$. On the other hand, by (28), (29) and the arguments as in getting (42), we have

$$\|\nu_k * S_{j+k} w\|_{L^2(u)} \le C_{h,\Omega} 2^{-\delta_2(1-1/s)|j|}\|w\|_{L^2(\mathrm{M}_s M_s^{\tilde{\nu}} u)} \tag{59}$$

for some $\delta_2 > 0$, any function w and any $s > 1$. By (59) with $w = S^2_{j+k} f$ and (35), we obtain

$$\begin{aligned}\|G_j f\|_{L^2(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}} |\nu_k * S^3_{j+k} f|^2\Big)^{1/2}\Big\|_{L^2(u)} \\ &\le \Big(\sum_{k\in\mathbb{Z}} \|\nu_k * S^3_{j+k} f\|^2_{L^2(u)}\Big)^{1/2} \\ &\le C_{h,\Omega} 2^{-\delta_2(1-1/s)|j|}\Big\|\Big(\sum_{k\in\mathbb{Z}} |S^2_{j+k} f|^2\Big)^{1/2}\Big\|_{L^2(\mathrm{M}_s M_s^{\tilde{\nu}} u)} \\ &\le C_{h,\Omega} 2^{-\delta_2(1-1/s)|j|}\|f\|_{L^2(\mathrm{M}_s M_s^{\tilde{\nu}} u)}.\end{aligned} \tag{60}$$

Take $s = t_1$. By substituting u^{1/t_1} for u in (60),

$$\|G_j f\|_{L^2(u^{1/t_1})} \le C_{h,\Omega} 2^{-\delta_2(1-1/t_1)|j|}\|f\|_{L^2((\mathrm{M} M^{\tilde{\nu}} u)^{1/t_1})}. \tag{61}$$

By an interpolation between (61) and (58), we obtain

$$\|G_j f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega} 2^{-\delta_2(1-1/t_1)|j|}\|f\|_{L^p((\mathrm{M} M^{\tilde{\nu}} u)} \tag{62}$$

for all $p \in (1, 2]$. Combining (62) with (48) yields that

$$\|G_\nu f\|_{L^p(u^{1/t_1})} \le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M} M^{\tilde{\nu}} u)^{1/t_1})} \tag{63}$$

for all $p \in (1, 2]$. By the well-known Fefferman-Stein inequality for M,

$$\|\mathrm{M} f\|_{L^p(u)} \le C_p\|f\|_{L^p(\mathrm{M}u)} \tag{64}$$

for all $p \in (1, \infty)$. It follows from (63), (64), (30) and (19) that

$$\begin{aligned}\|M^\sigma f\|_{L^p(u^{1/t_1})} &\le \|G^\nu f\|_{L^p(u^{1/t_1})} + \widehat{|\sigma_k|}(0)\|\mathrm{M}|f|\|_{L^p(u^{1/t_1})} \\ &\le C_{h,\Omega,q,\gamma,p}(\|f\|_{L^p((\mathrm{M} M^{\tilde{\nu}} u)^{1/t_1})} + \|f\|_{L^p(\mathrm{M}u^{1/t_1})}) \\ &\le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M} M^{\tilde{\nu}} u + \mathrm{M}u)^{1/t_1})}\end{aligned} \tag{65}$$

for all $p \in (1,2]$. Inequalities (65) together with (31), (19) and (64) imply that

$$\begin{aligned}\Big\|\sup_{k\in\mathbb{Z}}|\nu_k * f|\Big\|_{L^p(u^{1/t_1})} &\le \|M^{\nu}|f|\|_{L^p(u^{1/t_1})} \\ &\le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M}M^{\tilde{\nu}}u+\mathrm{M}u)^{1/t_1})}\end{aligned} \tag{66}$$

for all $p \in (1,2]$. An interpolation between (52) and (66) yields that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * f_k|^2\Big)^{1/2}\Big\|_{L^p(u^{1/t_2})} \\ &\le C_{h,\Omega,q,\gamma,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p((\mathrm{M}M^{\tilde{\nu}}u+\mathrm{M}u)^{1/t_2})}\end{aligned} \tag{67}$$

for all $p \in (1,2]$, where $\frac{1}{t_2} = \frac{1}{t_1} + \frac{p}{2}(1-\frac{1}{t_1})$. Inequality (67) together with the arguments similar to those used in deriving (65) yields that

$$\|M^{\sigma}f\|_{L^p(u^{1/t_2})} \le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M}M^{\tilde{\nu}}u+\mathrm{M}u)^{1/t_2})} \tag{68}$$

for all $p \in (1,2]$. By using the above argument repeatedly, there exists a strictly decreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ by the recursion formula

$$t_1 = \frac{2}{p}, \quad \frac{1}{t_{k+1}} = \frac{1}{t_k} + \frac{p}{2}\Big(1-\frac{1}{t_k}\Big), \quad k = 1,2,\dots$$

such that

$$\|M^{\sigma}f\|_{L^p(u^{1/t_k})} \le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M}M^{\tilde{\nu}}u+\mathrm{M}u)^{1/t_k})} \tag{69}$$

for all $p \in (1,2]$ and all $k \in \mathbb{N}$. It follows from (69), (19) and (31) that

$$\|M^{\sigma}f\|_{L^p(u^{1/t_k})} \le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p((\mathrm{M}M^{\tilde{\sigma}}u+\mathrm{M}^2u)^{1/t_k})} \tag{70}$$

for all $p \in (1,2]$ and all $k \in \mathbb{N}$. Then (47) follows from (70) and the lemma in [27, p. 1574]. □

Proof of Theorem 1.2. We spit the proof into two parts:

Step 1: Proof of part (i). One can easily check that

$$M_{h,\Omega}f(x) \le CM^{\sigma}f(x). \tag{71}$$

Hence, inequality (4) reduces to the following

$$\|M^{\sigma}f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p(\mathrm{M}_sM_s^{\tilde{\sigma}}u+\mathrm{M}_s^2u)} \tag{72}$$

for all $p \in [2,\infty)$ and $s > 1$. By the arguments similar to those used in deriving (45),

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * g_k|^2\Big)^{1/2}\Big\|_{L^p(u)} \le C_{h,\Omega}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M_s^{\tilde{\nu}}u)} \tag{73}$$

for all $p \in (2, \infty)$ and any $s > 1$. An application of the weighted Littlewood-Paley theory together with (73) and the fact that $\mathrm{M}_s M_s^{\tilde{\nu}} u \in A_1$ implies that

$$\begin{aligned} \|G_j f\|_{L^p(u)} &= \Big\| \Big(\sum_{k\in\mathbb{Z}} |\nu_k * S_{j+k}^3 f|^2 \Big)^{1/2} \Big\|_{L^p(u)} \\ &\leq C_{h,\Omega} \Big\| \Big(\sum_{k\in\mathbb{Z}} |S_{j+k}^3 f|^2 \Big)^{1/2} \Big\|_{L^p(\mathrm{M}_s M_s^{\tilde{\nu}} u)} \\ &\leq C_{h,\Omega} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\nu}} u)} \end{aligned} \tag{74}$$

for all $p \in (2, \infty)$ and any $s > 1$. An interpolation between (60) and (74) (see [4, Corollary 5.5.4]) implies that

$$\|G_j f\|_{L^p(u)} \leq C_{h,\Omega} 2^{-\delta_2(1-1/s)|j|} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\nu}} u)} \tag{75}$$

for all $p \in [2, \infty)$ and any $s > 1$. Inequality (75) together with (48) yields that

$$\|G_\nu f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\nu}} u)}$$

for all $p \in [2, \infty)$ and $s > 1$. Above inequality together with (19) and (31) yields that

$$\|G_\nu f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)} \tag{76}$$

for all $p \in [2, \infty)$ and $s > 1$. Combining (30) with (19), (64) and (76) yields that

$$\|M^\sigma f\|_{L^p(u)} \leq \|G_\nu f\|_{L^p(u)} + C\|\mathrm{M}|f|\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)}$$

for all $p \in [2, \infty)$ and $s > 1$. This proves (72).

Step 2: Proof of part (ii). Fix $k \in \mathbb{N}$. Substitute u^{t_k} for u in (70) and by Hölder's inequality, one finds

$$\begin{aligned} \|M^\sigma f\|_{L^p(u)} &\leq C_{h,\Omega,q,\gamma,p} \|f\|_{L^p((\mathrm{M} M^{\tilde{\sigma}} u^{t_k} + \mathrm{M}^2 u^{t_k})^{1/t_k})} \\ &\leq C_{h,\Omega,q,\gamma,s,p} \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)} \end{aligned} \tag{77}$$

for all $p \in (1, 2]$ and $s > t_k$. Combining (77) with (71) yields (5). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We divide the proof into two steps:

Step 1: Proof of part (i). We write

$$T_{h,\Omega}^* f(x) \leq M^\sigma |f|(x) + \sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{\infty} \sigma_j * f(x) \Big|. \tag{78}$$

Hence, to prove (6), by (72), (78) and the fact that $u \leq \mathrm{M}_s u$ for all $s > 1$, it suffices to show that

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{j=k}^{\infty} \sigma_k * f \Big| \Big\|_{L^p(u)} \leq C \|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)} \tag{79}$$

for all $p \in [2, \infty)$ and any $s > 1$. It is easy to see that

$$
\begin{aligned}
&\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_j * f(x)\Big| \\
&= \sup_{k\in\mathbb{Z}}\Big|\psi_k * T_{h,\Omega}f(x) - \psi_k * \sum_{j=-\infty}^{k}\sigma_j * f(x) \\
&\quad + (\delta - \psi_k) * \sum_{j=k+1}^{\infty}\sigma_j * f(x)\Big| \\
&\le \sup_{k\in\mathbb{Z}}|\psi_k * T_{h,\Omega}f(x)| + \sup_{k\in\mathbb{Z}}\Big|\psi_k * \sum_{j=-\infty}^{k}\sigma_j * f(x)\Big| \\
&\quad + \sup_{k\in\mathbb{Z}}\Big|(\delta - \psi_k) * \sum_{j=k+1}^{\infty}\sigma_j * f(x)\Big| \\
&=: I_1 f(x) + I_2 f(x) + I_3 f(x),
\end{aligned}
\tag{80}
$$

where ψ_k is given as in (27) and δ is the Dirac-Delta.

For $I_1 f$, we get from (2) and (64) that

$$
\begin{aligned}
\|I_1 f\|_{L^p(u)} &\le \|\mathrm{M}(T_{h,\Omega}f)\|_{L^p(u)} \\
&\le C_p\|T_{h,\Omega}f\|_{L^p(\mathrm{M}u)} \le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}u)}
\end{aligned}
\tag{81}
$$

for all $p \in [2, \infty)$ and $s > 1$.

For $I_2 f$, we can write

$$
I_2 f(x) = \sup_{k\in\mathbb{Z}}\Big|\sum_{j=0}^{\infty}\psi_k * \sigma_{k-j} * f(x)\Big| \le \sum_{j=0}^{\infty}\sup_{k\in\mathbb{Z}}|\psi_k * \sigma_{k-j} * f(x)| =: \sum_{j=0}^{\infty}H_j f(x).
$$

It follows that

$$
\|I_2 f\|_{L^p(u)} \le \sum_{j=0}^{\infty}\|H_j f\|_{L^p(u)}
\tag{82}
$$

for all $p \in (1, \infty)$. By (64) and (72), we obtain

$$
\begin{aligned}
\|H_j f\|_{L^p(u)} &\le \|\mathrm{M}M^{\sigma}|f|\|_{L^p(u)} \\
&\le C_p\|M^{\sigma}|f|\|_{L^p(\mathrm{M}u)} \\
&\le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u + \mathrm{M}_s^3 u)}
\end{aligned}
\tag{83}
$$

for all $p \in [2, \infty)$ and $s > 1$. By (38) and Plancherel's theorem,

$$
\begin{aligned}
\|H_j f\|_{L^2(\mathbb{R}^n)} &\le \Big\|\Big(\sum_{k\in\mathbb{Z}}|\psi_k * \sigma_{k-j} * f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)} \\
&\le \Big(\sum_{k\in\mathbb{Z}}\int_{\{2^k s(\xi)\le 1\}}|\widehat{\sigma_{k-j}}(\xi)|^2|\hat{f}(\xi)|^2 d\xi\Big)^{1/2} \\
&\le C\Big(\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\widehat{\sigma_{k-j}}(\xi)|^2\chi_{\{2^k s(\xi)\le 1\}}|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}
\end{aligned}
\tag{84}
$$

$$\leq C_{h,\Omega}2^{-cj}\Big(\sup_{\xi\in\mathbb{R}^n}\sum_{k\in\mathbb{Z}}((2^k s(\xi))^{1/(2q'\gamma' a_1 L)} + (2^k s(\xi))^{1/(2q'\gamma' b_1 L)})\chi_{\{2^k s(\xi)\leq 1\}}\Big)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}$$
$$\leq C_{h,\Omega,q,\gamma}2^{-cj}\|f\|_{L^2(\mathbb{R}^n)},$$

for some $c>0$, where in the last inequality we have used the properties of lacunary sequence. On the other hand, by (83) with $p=2$ and substitute u^s for u in (83), we get

$$\|H_j f\|_{L^2(u^s)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^2(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u^s+\mathrm{M}_s^3 u^s)} \tag{85}$$

for $s>1$. By interpolating between (84) and (85), we get

$$\|H_j f\|_{L^2(u)} \leq C_{h,\Omega,q,\gamma,s,p}2^{-(1-1/s)cj}\|f\|_{L^2(\mathrm{M}_{s^2} M_{s^2}^{\tilde{\sigma}}\mathrm{M}_{s^2} u+\mathrm{M}_{s^2}^3 u)} \tag{86}$$

for $s>1$. Substitute s^2 for s in (86), we have

$$\|H_j f\|_{L^2(u)} \leq C_{h,\Omega,q,\gamma,s,p}2^{-(1-1/\sqrt{s})cj}\|f\|_{L^2(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{87}$$

for $s>1$. An interpolation between (83) and (87) (see [4, Corollary 5.5.4]) yields

$$\|H_j f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p}2^{-\varsigma(p,s)j}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u+\mathrm{M}_s^3 u)}, \tag{88}$$

for all $p\in[2,\infty)$ and $s>1$, where $\varsigma(p,s)>0$. Inequality (88) together with (82) yields that

$$\|I_2 f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{89}$$

for all $p\in[2,\infty)$ and $s>1$.

Finally we estimate $I_3 f$. It is easy to see that

$$I_3 f(x) = \sup_{k\in\mathbb{Z}}\Big|\sum_{j=1}^{\infty}(\delta-\psi_k)*\sigma_{k+j}*f(x)\Big|$$
$$\leq \sum_{j=1}^{\infty}\sup_{k\in\mathbb{Z}}|(\delta-\psi_k)*\sigma_{k+j}*f(x)| =: \sum_{j=1}^{\infty}J_j f(x).$$

It follows that

$$\|I_3 f\|_{L^p(u)} \leq \sum_{j=1}^{\infty}\|J_j f\|_{L^p(u)} \tag{90}$$

for all $p\in(1,\infty)$. By the argument similar to those used in deriving (83),

$$\|J_j f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{91}$$

for all $p\in[2,\infty)$ and $s>1$.

On the other hand, by (39) and the Plancherel theorem,

$$
\begin{aligned}
&\|J_j f\|_{L^2(\mathbb{R}^n)}\\
&\le \Big\|\Big(\sum_{k\in\mathbb{Z}}|(\delta-\psi_k)*\sigma_{j+k}*f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}\\
&\le \Big(\sum_{k\in\mathbb{Z}}\int_{\{2^k s(\xi)\ge 1\}}|\widehat{\sigma_{j+k}}(\xi)|^2|\hat f(\xi)|^2 d\xi\Big)^{1/2}\\
&\le \Big(\sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}\int_{\{2^{-i}\le s(\xi)\le 2^{-i+1}\}}|\widehat{\sigma_{j+k}}(\xi)|^2|\hat f(\xi)|^2 d\xi\Big)^{1/2}\\
&\le C\Big(\sum_{k\in\mathbb{Z}}\sum_{i=-\infty}^{k}2^{-c(j+k-i)}\int_{\{2^{-i}\le s(\xi)\le 2^{-i+1}\}}|\hat f(\xi)|^2 d\xi\Big)^{1/2}\\
&\le C_{h,\Omega}2^{-cj}\Big(\sum_{k\in\mathbb{Z}}\sum_{i=0}^{\infty}2^{-ci}\int_{\{2^{k-i}\le s(\xi)\le 2^{k-i+1}\}}|\hat f(\xi)|^2 d\xi\Big)^{1/2}\\
&\le C_{h,\Omega}2^{-cj}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{92}
$$

By (91), (92) and the arguments similar to those used to derive (88),

$$\|J_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,s,p}2^{-\tau(p,s)j}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{93}$$

for all $p\in[2,\infty)$ and $s>1$, where $\tau(p,s)>0$. Combining (93) with (90) yields that

$$\|I_3 f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}\mathrm{M}_s u+\mathrm{M}_s^3 u)}, \tag{94}$$

for all $p\in[2,\infty)$ and $s>1$. (80) together with (81), (89) and (94) yields (79).

Step 2: Proof of part (ii). By (77) and (78), to prove (7), it suffices to prove that

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_k*f\Big|\Big\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{95}$$

for all $p\in(1,2)$, any fixed $k\in\mathbb{N}$ and $s>t_k$. For $I_1 f$, we get from (64) and (3) that

$$
\begin{aligned}
\|I_1 f\|_{L^p(u)} &\le C\|\mathrm{M}(T_{h,\Omega}f)\|_{L^p(u)}\\
&\le C_p\|T_{h,\Omega}f\|_{L^p(\mathrm{M}u)} \le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}u+\mathrm{M}_s^2 u)}
\end{aligned}
\tag{96}
$$

for any fixed positive integer k, $s>t_k$ and $p\in(1,2]$.

For $I_2 f$, by (64) and (77) we have

$$
\begin{aligned}
\|H_j f\|_{L^p(u)} &\le C\|\mathrm{M}M^{\sigma}f\|_{L^p(u)}\\
&\le C_p\|M^{\sigma}f\|_{L^p(\mathrm{M}u)} \le C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}u+\mathrm{M}_s^2 u)}
\end{aligned}
\tag{97}
$$

for any fixed $k\in\mathbb{N}$, all $s>t_k$ and $p\in(1,2]$. An interpolation between (87) and (97) (see [4, Corollary 5.5.4]) yields that

$$\|H_j f\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,s,p}2^{-\delta(p,s)j}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde\sigma}\mathrm{M}_s u+\mathrm{M}_s^3 u)} \tag{98}$$

for any fixed positive integer k, all $s > t_k$ and $p \in (1,2]$. Here $\delta(p,s) > 0$. Inequality (98) together with (82) yields that

$$\|I_2 f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)} \tag{99}$$

for any fixed positive integer k, all $s > t_k$ and $p \in (1,2]$.

For $I_3 f$. by (68) and (82) we have

$$\begin{aligned} \|J_j f\|_{L^p(u)} &\leq C\|\mathrm{M}M^\sigma f\|_{L^p(u)} \\ &\leq C_p\|M^\sigma f\|_{L^p(\mathrm{M}u)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} u + \mathrm{M}_s^2 u)} \end{aligned} \tag{100}$$

for any fixed positive integer k, all $s > t_k$ and $p \in (1,2]$. By interpolating between (100) and (93) (see [4, Corollary 5.5.4]),

$$\|J_j f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p} 2^{-\delta(p,s)j}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)} \tag{101}$$

for any fixed positive integer k, all $s > t_k$ and $p \in (1,2]$. Here $\delta(p,s) > 0$. We get from (101) and (90) that

$$\|I_3 f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,s,p}\|f\|_{L^p(\mathrm{M}_s M_s^{\tilde{\sigma}} \mathrm{M}_s u + \mathrm{M}_s^3 u)} \tag{102}$$

for any fixed positive integer k, all $s > t_k$ and $p \in (1,2]$. Then (95) follows from (80), (96), (89) and (102). $\square$

Proof of Theorem 1.4. We define two families of measures $\{\tau_{k,t}\}_{k\in\mathbb{Z}}$ and $\{|\tau_{k,t}|\}_{k\in\mathbb{Z}}$ respectively by

$$\int_{\mathbb{R}^n} f(x)d\tau_{k,t}(x) = \frac{1}{(2^k t)^\varrho}\int_{2^{k-1}t<r(x)\leq 2^k t} f(x)\frac{\Omega(x')h(r(x))}{r(x)^{\alpha-\varrho}}dx$$

and

$$\int_{\mathbb{R}^n} f(x)d|\tau_{k,t}|(x) = \frac{1}{(2^k t)^\varrho}\int_{2^{k-1}t<r(x)\leq 2^k t} f(x)\frac{|\Omega(x')h(r(x))|}{r(x)^{\alpha-\varrho}}dx.$$

We also define the maximal operators M^τ and $M^{\tilde{\tau}}$ respectively by

$$M^\tau f(x) = \sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}||\tau_{k,t}| * f(x)| \quad \text{and} \quad M^{\tilde{\tau}} f(x) = \sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}||\tilde{\tau}_{k,t}| * f(x)|,$$

where

$$\int_{\mathbb{R}^n} f(x)d|\tilde{\tau}_{k,t}|(x) = \int_{\mathbb{R}^n} f(-x)d|\tau_{k,t}|(x).$$

One can easily check that

$$M^\tau f(x) \leq 2M^\sigma|f|(x), \quad M^{\tilde{\tau}} f(x) \leq 2M^{\tilde{\sigma}}|f|(x). \tag{103}$$

Invoking Lemma 2.3, we get from (103) that

$$\|M^\tau f\|_{L^p(\mathbb{R}^n)} \leq C_{h,\Omega,q,\gamma}\|f\|_{L^p(\mathbb{R}^n)} \tag{104}$$

for all $p \in (1,\infty)$. By the arguments similar to those used to derive[18, Lemma 2.3],

$$\begin{aligned} &\max\left\{|\widehat{\tau_{k,t}}(\xi)|, \left|\widehat{|\tau_{k,t}|}(\xi) - \widehat{|\tau_{k,t}|}(0)\right|\right\} \\ &\leq C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{2^k t}\xi|^{1/(4q'\gamma' L)}; \end{aligned} \tag{105}$$

$$\begin{aligned}(106)\qquad &\max\big\{|\widehat{\tau_{k,t}}(\xi)|, \big|\widehat{|\tau_{k,t}|}(\xi)\big|\big\}\\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\min\{1,|A^*_{2^kt}\xi|^{-1/(4q'\gamma'L)}\}.\end{aligned}$$

By (13), (14), (105) and (106), we get

$$\begin{aligned}(107)\qquad |\widehat{\tau_{k,t}}(\xi)| &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\\ &\quad\times\big((2^ks(\xi))^{1/(4q'\gamma'a_1L)}+(2^ks(\xi))^{1/(4q'\gamma'b_1L)}\big);\end{aligned}$$

$$\begin{aligned}(108)\qquad |\widehat{\tau_{k,t}}(\xi)| &\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|_{L^q(\Sigma)}\\ &\quad\times\min\big\{1,(2^ks(\xi))^{-1/(4q'\gamma'a_2L)}+(2^ks(\xi))^{-1/(4q'\gamma'b_2L)}\big\},\end{aligned}$$

where $C>0$ is independent of h,Ω,q,γ. An argument similar to those used in deriving [16, (3.2)] yields that

$$(109)\quad \mathfrak{M}_{h,\Omega,\varrho}f(x)\le\frac{1}{1-2^{-\tau}}\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\tau_{k,t}*f(x)|^2\frac{dt}{t}\Big)^{1/2}=:\frac{1}{1-2^{-\tau}}\mathfrak{M}f(x).$$

In what follows, we fix a nonnegative measurable function u on $\mathbb{R}^n$. We divide the proof into two steps:

Step 1: Prove (8) for the case $p\in[2,\infty)$. Let S_k be given as in the proof of Theorem 1.1. By Minkowski's inequality, it holds that

$$\begin{aligned}(110)\qquad \mathfrak{M}f(x)&=\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S^3_{j+k}(\tau_{k,t}*f)(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\le\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_1^2|S^3_{j+k}(\tau_{k,t}*f)(x)|^2\frac{dt}{t}\Big)^{1/2}=:\sum_{j\in\mathbb{Z}}A_jf(x).\end{aligned}$$

By (104) and the arguments similar to those used in deriving (45),

$$(111)\quad \Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\tau_{k,t}*g_k|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}\le C_{h,\Omega,q,\gamma,p,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M_s^{\tilde{\tau}}u)}$$

for all $2<p<\infty$ and any $s>1$. By (35), (36), (111) and the fact $M_s^{\tilde{\tau}}u\le \mathrm{M}_sM_s^{\tilde{\tau}}u\in A_1$, one finds

$$\begin{aligned}(112)\qquad \|A_jf\|_{L^p(u)}&=\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\tau_{k,t}*S^3_{j+k}f|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p,s}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(M_s^{\tilde{\tau}}u)}\\ &\le C_{h,\Omega,q,\gamma,p,s}\|f\|_{L^p(\mathrm{M}_sM_s^{\tilde{\tau}}u)}\end{aligned}$$

for all $2<p<\infty$ and any $s>1$. On the other hand, fix $t\in[1,2]$, by (107), (108) and Plancherel's theorem, we have

$$(113)\qquad \|\tau_{k,t}*S_{j+k}w\|_{L^2(\mathbb{R}^n)}\le C_{h,\Omega,q,\gamma}2^{-\delta_3|j|}\|w\|_{L^2(\mathbb{R}^n)}$$

for some $\delta_3 > 0$ and any arbitrary function w on $\mathbb{R}^n$. Here δ_3 depends on q, γ, L. One can easily check that

$$\begin{aligned}(114)\qquad &\|\tau_{k,t} * S_{j+k}w\|_{L^2(u^s)}\\ &\leq (\|\tau_{k,t}\|\|\Theta_{j+k}\|_{L^1(\mathbb{R}^n)})^{1/2}\||\tau_{k,t}| * |\Theta_{j+k}| * |w|^2\|_{L^1(u^s)}\\ &\leq C_{h,\Omega,q,\gamma}\|w\|_{L^2(\mathrm{M}^{\tilde{\tau}}u^s)}\end{aligned}$$

for any $s > 1$. By (113), (114) and the interpolation of L^2-spaces with change of measure ([4, Theorem 5.4.1]), we obtain

$$(115)\qquad \|\tau_{k,t} * S_{j+k}w\|_{L^2(u)} \leq C_{h,\Omega,q,\gamma,s}2^{-\delta_3(1-1/s)|j|}\|w\|_{L^2(\mathrm{M}_sM_s^{\tilde{\tau}}u)}$$

for any $s > 1$. By (115) with $w = S_{j+k}f$ and (35), we obtain

$$\begin{aligned}\|A_jf\|_{L^2(u)}^2 &= \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_1^2 |S_{j+k}^3\tau_{k,t} * f(x)|^2\frac{dt}{t}u(x)dx\\ &= \int_1^2\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|S_{j+k}^3\tau_{k,t} * f(x)|^2u(x)dx\frac{dt}{t}\\ &\leq C\sum_{k\in\mathbb{Z}}\int_1^2\int_{\mathbb{R}^n}|\tau_{k,t} * S_{j+k}^2f(x)|^2u(x)dx\frac{dt}{t}\\ &\leq C_{h,\Omega,q,\gamma,s}2^{-2\delta_3(1-1/s)|j|}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|S_{j+k}f(x)|^2\mathrm{M}_sM_s^{\tilde{\tau}}u(x)dx\\ &\leq C_{h,\Omega,q,\gamma,s}2^{-2\delta_3(1-1/s)|j|}\|f\|_{L^2(\mathrm{M}_sM_s^{\tilde{\tau}}u)}^2.\end{aligned}$$

It follows that

$$(116)\qquad \|A_jf\|_{L^2(u)} \leq C_{h,\Omega,q,\gamma,s}2^{-\delta_3(1-1/s)|j|}\|f\|_{L^2(\mathrm{M}_sM_s^{\tilde{\tau}}u)}$$

for any $s > 1$. An interpolation between (116) and (112) (see [4, Corollary 5.5.4]) yields that

$$(117)\qquad \|T_jf\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s}2^{-\beta(p,q,\gamma,s)|j|}\|f\|_{L^p(\mathrm{M}_sM_s^{\tilde{\tau}}u)}$$

for $2 \leq p < \infty$ and $s > 1$. Here $\beta(p, q, \gamma, s) > 0$ depends only on p, q, γ and s. Combining (117) with (103), (109) and (110) yields (8).

Step 2: Prove (8) for the case $p \in (1, 2)$. We want to show that for any $1 < p < 2$ and $s > 1$, there exists a constant $\alpha(p, q, \gamma) > 0$ independent of j such that

$$(118)\qquad \|A_jf\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p,s}2^{-\alpha(p,q,\gamma)|j|}\|f\|_{L^p(\mathrm{M}_sM_s^{\tilde{\tau}}u)}.$$

Fix $t \in [1, 2]$. It is clear that

$$(119)\qquad \|\tau_{k,t} * f\|_{L^\infty(\mathbb{R}^n)} \leq C_{h,\Omega,q,\gamma}\|f\|_{L^\infty(\mathbb{R}^n)};$$

$$(120)\qquad \|\tau_{k,t} * f\|_{L^1(u)} \leq C\|f\|_{L^1(M^{\tilde{\tau}}u)}.$$

An interpolation between (119) and (120) implies that

$$(121)\qquad \|\tau_{k,t} * f\|_{L^p(u)} \leq C_{h,\Omega,q,\gamma,p}\|f\|_{L^p(M^{\tilde{\tau}}u)}$$

for all $1 < p < 2$. Combining (121) with (103) yields that

$$\begin{aligned}(122)\qquad &\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\tau_{k,t}*f_k\|^p_{L^p([1,2],t^{-1}dt)}\Big)^{1/p}\Big\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^p\Big)^{1/p}\Big\|_{L^p(M^{\tilde{\sigma}}u)}\end{aligned}$$

for all $1 < p < 2$. We get from (104) that

$$(123)\qquad \Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}|\tau_{k,t}*f_k|\Big\|_{L^p(\mathbb{R}^n)} \le C_{h,\Omega,q,\gamma,p}\Big\|\sup_{k\in\mathbb{Z}}|f_k|\Big\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < 2$. By interpolating between (122) and (123),

$$\begin{aligned}(124)\qquad &\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\tau_{k,t}*f_k\|^2_{L^2([1,2],t^{-1}dt)}\Big)^{1/2}\Big\|_{L^p(u^{1/t_1})}\\ &\le C_{h,\Omega,q,\gamma,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde{\sigma}}u)^{1/t_1}}\end{aligned}$$

for all $1 < p < 2$, where $t_1 = 2/p$. Substitute u^{t_1} for u in (124), we obtain

$$\begin{aligned}(125)\qquad &\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\tau_{k,t}*f_k\|^2_{L^2([1,2],t^{-1}dt)}\Big)^{1/2}\Big\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde{\sigma}}_{t_1}u)}.\end{aligned}$$

Since $\mathrm{M}_{t_1}M^{\tilde{\tau}}_{t_1}u \in A_1$, by the weighted Littlewood-Paley theory and (125),

$$\begin{aligned}(126)\qquad \|A_jf\|_{L^p(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}}\|\tau_{k,t}*S^3_{j+k}f\|^2_{L^2([1,2],t^{-1}dt)}\Big)^{1/2}\Big\|_{L^p(u)}\\ &\le C_{h,\Omega,q,\gamma,p}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde{\tau}}_{t_1}u)}\\ &\le C_{h,\Omega,q,\gamma,p}\|f\|_{L^p(\mathrm{M}_{t_1}M^{\tilde{\tau}}_{t_1}u)}\end{aligned}$$

for all $1 < p < 2$. Using (116) with $s = t_1$ and (103), we get

$$(127)\qquad \|A_jf\|_{L^2(u)} \le C_{h,\Omega,q,\gamma}2^{-\delta_3|j|/t_1'}\|f\|_{L^2(\mathrm{M}_{t_1}M^{\tilde{\sigma}}_{t_1}u)}.$$

An interpolation between (126) and (127) implies that for any $p \in (1,2)$, $s \in (1,2)$, there exist $q \in (1,2)$ and $\theta \in [0,1]$ such that $s = \frac{2}{q}$, $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$ and

$$(128)\qquad \|A_jf\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p}2^{-\alpha(p,q,\gamma)|j|}\|f\|_{L^p(\mathrm{M}_sM^{\tilde{\sigma}}_s u)},$$

where $\alpha(p,q,\gamma) > 0$. Note that $\mathrm{M}_sM^{\tilde{\tau}}_s u \le C\mathrm{M}_tM^{\tilde{\tau}}_t u$ for $t > s$ by Hölder's inequality. This together with (128) yields that

$$(129)\qquad \|A_jf\|_{L^p(u)} \le C_{h,\Omega,q,\gamma,p}2^{-\alpha(p,q,\gamma)|j|}\|f\|_{L^p(\mathrm{M}_tM^{\tilde{\sigma}}_t u)}$$

for all $1 < p < 2$ and $t > 1$. Then (8) follows from (109), (110) and (129). This completes the proof of Theorem 1.4. □

4. Proofs of Corollaries 1.5 and 1.6

In this section we shall prove Corollaries 1.5 and 1.6. To prove our main results, we need the following proposition.

Proposition 4.1. *Let $1 < p < \infty$ and $s_0 \geq 1$. Assume that T is a linear or sublinear operator such that*

$$\|Tf\|_{L^p(u)} \leq C_{p,s,s_0}\|f\|_{L^p(\mathcal{H}_s(u))} \tag{130}$$

for all $s > s_0$ and any nonnegative measurable function u on $\mathbb{R}^n$, where for a fixed $s > s_0$, the operator $\mathcal{H}_s$ is a bounded operator from $L^r(\mathbb{R}^n)$ to itself for all $r \in (s, \infty)$. Then for any $q \in (p, \frac{ps_0}{s_0-1})$, the following inequality holds:

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)} \leq C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)}. \tag{131}$$

Proof. Fix $q \in (p, \frac{ps_0}{s_0-1})$ and write $r = \frac{q}{q-p}$. Let $\{f_j\} \in L^q(\mathbb{R}^n, \ell^p)$ and fix $s \in (s_0, r)$. By assumption (130) and Hölder's inequality, one has

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)}\\
&= \sup_{\substack{g\in L^r(\mathbb{R}^n),\\ g\geq 0, \|g\|_{L^r(\mathbb{R}^n)}\leq 1}} \int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|Tf_j(x)|^p g(x)dx\\
&\leq C_{p,s,s_0} \sup_{\substack{g\in L^r(\mathbb{R}^n),\\ g\geq 0, \|g\|_{L^r(\mathbb{R}^n)}\leq 1}} \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}|f_j(x)|^p\mathcal{H}_s(g)(x)dx\\
&\leq C_{p,s,s_0} \sup_{\substack{g\in L^r(\mathbb{R}^n),\\ g\geq 0, \|g\|_{L^r(\mathbb{R}^n)}\leq 1}} \int_{\mathbb{R}^n}\Big\|\sum_{j\in\mathbb{Z}}|f_j|^p\Big\|_{L^{q/p}(\mathbb{R}^n)}\|\mathcal{H}_s(g)\|_{L^r(\mathbb{R}^n)}\\
&\leq C_{p,s,s_0}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^p\Big)^{1/p}\Big\|_{L^q(\mathbb{R}^n)}^p,
\end{aligned}$$

which gives (131) and completes the proof. □

Proofs of Corollaries 1.5 and 1.6. By Lemma 2.3 and (32), we have

$$\|\mathrm{M}_s M_s^{\tilde{\sigma}}\mathrm{M}_s u + \mathrm{M}_s^3 u\|_{L^r(\mathbb{R}^n)} \leq C_{h,\Omega,q,\gamma,r}\|u\|_{L^r(\mathbb{R}^n)} \tag{132}$$

for any $1 < s < \infty$ and $r > s$. By (132), Remark 1.7, Proposition 4.1, Theorems 1.1-1.4 and the arguments similar to those used to derive [17, Corollaries 1.3-1.5], we can get the conclusions of Corollaries 1.5 and 1.6. The details are omitted. □

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