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EVEN 2-UNIVERSAL QUADRATIC FORMS OF RANK 5

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Abstract. A (positive definite integral) quadratic form is called *even* 2-*universal* if it represents all even quadratic forms of rank 2. In this article, we prove that there are at most 55 even 2-universal even quadratic forms of rank 5. The proofs of even 2-universalities of some candidates will be given so that exactly 20 candidates remain unproven.

1. Introduction

A positive definite integral quadratic form

$$f(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \quad (a_{ij} = a_{ji} \in \mathbb{Z})$$

of rank n is called *universal* if it represents all positive integers, that is, the diophantine equation $f(x_1, x_2, \ldots, x_n) = N$ has an integer solution for any positive integer N. After Lagrange's celebrated four square theorem, which implies that the quaternary quadratic form $x^2+y^2+z^2+t^2$ is universal, a number of universal quaternary quadratic forms are known (see, for example, [19] and [21]). One may easily show that there does not exist a positive definite integral universal quadratic form of rank 3. In 2002, Conway and Schneeberger proved that there are exactly 204 positive definite integral universal quadratic forms of rank 4. Furthermore, they proved the so called "15-Theorem", which states that every positive definite integral quadratic form that represents $1, 2, 3, 5, 6, 7, 10, 14$, and 15 is, in fact, universal, irrespective of its rank (see [1]). Recently, Bhargava and Hanke [2] proved the "290-Theorem", which states that every positive definite

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integer-valued quadratic form represents all positive integers if it represents

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 13,\ 14,\ 15,\ 17,\ 19,\ 21,\ 22,\ 23,\ 26,\ 29,$$
$$30,\ 31,\ 34,\ 35,\ 37,\ 42,\ 58,\ 93,\ 110,\ 145,\ 203,\ \text{ and }\ 290.$$

Here, a quadratic form $f(x_1,x_2,\dots,x_n)$ is called "integer-valued", if $f(x_1,\dots,x_n)$ is always an integer for any integral vector $(x_1,x_2,\dots,x_n)\in\mathbb{Z}^n$. Hence any integral quadratic form is integer-valued, whereas the converse is not true in general.

From now on, we always assume that a quadratic form is "positive definite" and "integral".

As a natural generalization, a quadratic form is called *n-universal* if it represents all quadratic forms of rank n. In 1998, Kim and his collaborators proved in [11] that there are exactly eleven 2-universal quinary quadratic forms (for higher rank cases, see [10] and [16]). To generalize this result to the integer-valued case, we consider even quadratic forms obtained from integer-valued quadratic forms by scaling 2. A quadratic form $f(\mathbf{x})$ is called *even* if $f(\mathbf{x})$ is even for any vector $\mathbf{x}$. A quadratic form is called *even 2-universal* if it represents all even binary quadratic forms. In this article, we show that there are at most 55 even 2-universal even quinary quadratic forms. Furthermore, we prove even 2-universalities of some candidates so that exactly 20 candidates remain unproven. Even 2-universal even quinary quadratic forms and their candidates are listed in Tables 4 and 5. We conjecture that the remaining 20 candidates are also even 2-universal.

To explain more precisely, we adopt lattice-theoretic language. A $\mathbb{Z}$-lattice L is a finitely generated free $\mathbb{Z}$-module equipped with a nondegenerate symmetric bilinear form B such that $B(L,L)\subset\mathbb{Z}$. The corresponding quadratic map Q is defined by $Q(\mathbf{v})=B(\mathbf{v},\mathbf{v})$ for any $\mathbf{v}\in L$.

Let $L=\mathbb{Z}\mathbf{x}_1+\mathbb{Z}\mathbf{x}_2+\cdots+\mathbb{Z}\mathbf{x}_n$ be a $\mathbb{Z}$-lattice. The quadratic form f_L corresponding to L is defined by $f_L(x_1,x_2,\dots,x_n)=\sum B(\mathbf{x}_i,\mathbf{x}_j)x_ix_j$. Furthermore, the corresponding symmetric matrix M_L is defined by $M_L=(B(\mathbf{x}_i,\mathbf{x}_j))$, which is called the *matrix presentation* of L. If L admits an orthogonal basis $\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$, then we call L *diagonal* and simply write $L=\langle Q(\mathbf{x}_1),\dots,Q(\mathbf{x}_n)\rangle$. The $\mathbb{Z}$-lattice L is called positive definite or simply positive if $Q(\mathbf{v})>0$ for any $\mathbf{v}\in L-\{\mathbf{0}\}$. The ideal of $\mathbb{Z}$ generated by $B(L,L)$ is called *the scale of* L, which is denoted by $\mathfrak{s}(L)$, and the ideal generated by $Q(\mathbf{v})$ for $\mathbf{v}\in L$ is called the *the norm of* L, which is denoted by $\mathfrak{n}(L)$. A $\mathbb{Z}$-lattice L is called *integral* if $\mathfrak{s}(L)\subseteq\mathbb{Z}$, and is called *integer-valued* if $\mathfrak{n}(L)\subseteq\mathbb{Z}$. We say L is *even* if $\mathfrak{n}(L)\subseteq 2\mathbb{Z}$. As mentioned above, we always assume that any $\mathbb{Z}$-lattice is *positive definite* and *integral*, unless stated otherwise. For any positive integer a, L^a is the $\mathbb{Z}$-lattice obtained from L by scaling $L\otimes\mathbb{Q}$ by a. For any prime p, we define $L_p=L\otimes\mathbb{Z}_p$, which is a $\mathbb{Z}_p$-lattice. We say L is a *primitive* $\mathbb{Z}$-lattice if there does not exist an integral $\mathbb{Z}$-lattice in $L\otimes\mathbb{Q}$ properly containing L. The primitiveness of a $\mathbb{Z}_p$-lattice L_p is defined similarly.

For a $\mathbb{Z}$-lattice ℓ, we say that L represents ℓ, and we write $\ell \longrightarrow L$, if there is an injective $\mathbb{Z}$-linear map σ from ℓ to L such that

$$B(\sigma(\mathbf{v}), \sigma(\mathbf{w})) = B(\mathbf{v}, \mathbf{w}) \text{ for any } \mathbf{v}, \mathbf{w} \in \ell.$$

Such a linear map σ is called a *representation.* If the linear map σ is bijective, then we say ℓ is *isometric* to L, and we write $\ell \simeq L$. The representation and the isometry between $\mathbb{Z}_p$-lattices are defined similarly for any prime p. We say ℓ is *locally isometric* to L if $\ell_p \simeq L_p$ for any prime p. The *genus* $\text{gen}(L)$ of the $\mathbb{Z}$-lattice L is the set of $\mathbb{Z}$-lattices which are locally isometric to L. The set of isometric classes in the genus of L is denoted by $\text{gen}(L)/\sim$. The *class number* $h(L)$ of L is the number of isometric classes in the genus of L. It is well known that $h(L)$ is finite for any $\mathbb{Z}$-lattice L. We say L is *locally* (*even*) 2*-universal* if L_p represents all (even, respectively) $\mathbb{Z}_p$-lattices of rank 2 for any prime p. It is well known that any $\mathbb{Z}$-lattice L that is locally (even) 2-universal and $h(L) = 1$ is (even) 2-universal, which is called *strongly* (even, respectively) 2-universal. We say L is *almost* 2*-universal* if L represents almost all binary $\mathbb{Z}$-lattices.

For a binary quadratic form $f(x, y) = ax^2 + 2bxy + cy^2$, we will use the notation $f = [a, 2b, c]$. To present a $\mathbb{Z}$-lattice with rank greater than 2, we adopt the notation that is given by Conway and Sloane in [4] (see also [5]).

Any unexplained notation and terminology can be found in [15] or [18].

2. Even 2-universal even $\mathbb{Z}$-lattices of rank 5

The aim of this section is to find all candidates of even 2-universal even $\mathbb{Z}$-lattices of rank 5. Throughout this section, quinary $\mathbb{Z}$-lattices with $*$-mark are not yet determined to be even 2-universal and quinary $\mathbb{Z}$-lattices with $\dagger$-mark are of class number bigger than 1. Let $L = \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_5$ be an even 2-universal $\mathbb{Z}$-lattice of rank 5, which is not necessarily even. If we define

$$L(e) = \{\mathbf{v} \in L : Q(\mathbf{v}) \equiv 0 \pmod 2\},$$

then one may easily show that $L(e)$ is an even $\mathbb{Z}$-sublattice of L. Furthermore, any even $\mathbb{Z}$-lattice that is represented by L is also represented by $L(e)$. Hence $L(e)$ is also even 2-universal. Therefore, in some sense, it suffices to find all candidates of even 2-universal even quinary $\mathbb{Z}$-lattices.

A $\mathbb{Z}$-sublattice of L generated by vectors $\mathbf{v} \in L$ such that $Q(\mathbf{v}) = 2$ is denoted by R_L. Note that R_L is isometric to an orthogonal direct sum of root lattices A_n and D_m for some integers n and m less than or equal to 5.

To find all candidates of even 2-universal even quinary $\mathbb{Z}$-lattices, we will use, so called, the escalation method. We assume that $\{\mathbf{x}_i\}_{i=1}^5$ is a Minkowski reduced basis for L such that $Q(\mathbf{x}_1) \le Q(\mathbf{x}_2) \le \cdots \le Q(\mathbf{x}_5)$. For $k \le 4$, we find an even binary $\mathbb{Z}$-lattice that is not represented by a $k \times k$ section $\mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_k$ of L, though it is represented by L itself by assumption. The following lemma is very useful to give an upper bound of the $(k + 1)$-th successive minimum $m_{k+1}(L)$ of L. For the definition of the successive minimum and its basic property, see Chapter 12 of [3].

Lemma 2.1. *Let ℓ be a $\mathbb{Z}$-lattice of rank n and let $M = \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_m$ be a $\mathbb{Z}$-lattice of rank m greater than n, where $\{\mathbf{x}_i\}_{i=1}^m$ is a Minkowski reduced basis such that $Q(\mathbf{x}_1) \le Q(\mathbf{x}_2) \le \cdots \le Q(\mathbf{x}_m)$. If ℓ is represented by M, but is not represented by the $k \times k$ section $\mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_k$ of M, then*

$$m_{k+1}(M) \le \begin{cases} m_n(\ell) & \text{if } n \ge k+1, \\ C_4(k)C_4(k-1)\cdots C_4(n)m_n(\ell) & \text{otherwise,} \end{cases}$$

where the constant $C_4(k)$, which is defined in [3], depends only on k.

Proof. Assume that $n \ge k+1$. Since $\ell \longrightarrow M$, $m_{k+1}(M) \le m_{k+1}(\ell) \le m_n(\ell)$.

Now, assume that $n \le k$. Let $\phi : \ell \to M$ be a representation and let $\mathbb{Z}\mathbf{y}_1 + \mathbb{Z}\mathbf{y}_2 + \cdots + \mathbb{Z}\mathbf{y}_n$ be a sublattice of $\phi(\ell)$ such that $Q(\mathbf{y}_i) = m_i(\ell)$ for any $i = 1, 2, \ldots, n$. From the assumption, there is an integer j_0 such that $\mathbf{y}_{j_0} \notin \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_k$. Hence for any j such that $n \le j \le k$,

$$m_{j+1}(M) \le \max\{Q(\mathbf{x}_j), Q(\mathbf{y}_{j_0})\} \le \max\{Q(\mathbf{x}_j), m_n(\ell)\}.$$

Note that there is a constant depending only on j such that

$$Q(\mathbf{x}_j) \le C_4(j)m_j(M).$$

Since $m_n(M) \le m_n(\ell)$ and $C_4(n) \ge 1$,

$$m_{n+1}(M) \le \max\{C_4(n)m_n(M), m_n(\ell)\} \le C_4(n)m_n(\ell).$$

Now the lemma follows from the induction. □

Remark 2.2. Note that $C_4(k) = 1$ for any k less than or equal to 4 and $C_4(5) = \frac{5}{4}$ (for this, see [20]). Therefore, if $n = 2 < k \le 4$, then we have $m_{k+1}(M) \le m_2(\ell)$.

Lemma 2.3. *Let $L = \mathbb{Z}\mathbf{x}_1 + \mathbb{Z}\mathbf{x}_2 + \cdots + \mathbb{Z}\mathbf{x}_5$ be an even $\mathbb{Z}$-lattice of rank 5, where $\{\mathbf{x}_i\}_{i=1}^5$ is a Minkowski reduced basis such that $Q(\mathbf{x}_1) \le Q(\mathbf{x}_2) \le \cdots \le Q(\mathbf{x}_5)$. If $m_5(L) \le 6$, then $Q(\mathbf{x}_i) = m_i(L)$ for any $i = 1, 2, \ldots, 5$.*

Proof. Note that $m_i(L) \le Q(\mathbf{x}_i) \le C_4(i)m_i(L)$ (see Theorem 3.1 of Chapter 12 in [3]). Since we are assuming that L is even, and $C_4(i) = 1$ for any $i \le 4$, $C_4(5) = \frac{5}{4}$, we have $Q(\mathbf{x}_i) = m_i(L)$ for any $i = 1, 2, \ldots, 5$. □

Theorem 2.4. *For any even 2-universal even $\mathbb{Z}$-lattice L of rank 5, we have*

$$m_1(L) = m_2(L) = m_3(L) = 2, \ 2 \le m_4(L) \le 4, \quad \text{and} \quad 2 \le m_5(L) \le 6.$$

Furthermore, there are at most 55 even 2-universal even $\mathbb{Z}$-lattices of rank 5, which are listed in Tables 4 and 5.

Proof. Let L be an even 2-universal even $\mathbb{Z}$-lattice of rank 5. Since $A_1 \perp A_1 \longrightarrow R_L$ and $A_2 \longrightarrow R_L$, the rank of R_L should be greater than 2. If the rank of R_L is 3, then R_L must be isometric to either A_3 or $A_1 \perp A_2$.

First, assume that $R_L \simeq A_3$. Since $[2,2,4] \not\to A_3$, $m_4(L) = 4$ by Lemma 2.1. Note that any $\mathbb{Z}$-lattice M of rank 4 containing A_3 with $m_4(M) = 4$ is isometric to one of

$$A_3 \perp \langle 4\rangle, \quad A_3 52[1\tfrac{1}{4}], \quad \text{and} \quad A_3 12[2\tfrac{1}{2}].$$

For each quaternary $\mathbb{Z}$-lattice given above, since

$$[2,2,4] \not\to A_3 \perp \langle 4\rangle, \quad [4,4,4] \not\to A_3 52[1\tfrac{1}{4}], \quad \text{and} \quad [4,2,4] \not\to A_3 12[2\tfrac{1}{2}],$$

we may conclude that $m_5(L) = 4$ by Lemma 2.1. Therefore, after a suitable base change, one may easily show, by Lemma 2.3, that all possible candidates of L in this case are of the form

$$(ij,kl,a) := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & i & k \\ 0 & 1 & 2 & j & l \\ 0 & i & j & 4 & a \\ 0 & k & l & a & 4 \end{pmatrix},$$

where $(i,j),(k,l) = (0,0),(0,1),(1,0)$ and $a = 0,\pm1,\pm2$. Since $(ij,kl,a) \simeq (kl,ij,a)$, there are only 30 possible candidates in this case, which is listed in Table 1. Each binary $\mathbb{Z}$-lattice in the right hand side of Table 1 is not

Table 1. The case when $R_L \simeq A_3$

$L = (ij,kl,a)$	
$(00,00,a)$, $a = 0,\pm1,\pm2$	$[2,2,4]$
$(00,01,a)$, $a = 0,\pm1$, $(01,01,b)$, $b = 0,1$	$[4,4,4]$
$(00,10,0)$	$[4,2,4]$
$(10,10,1)$	$[6,6,10]$
$(01,01,-2)$	$R_L \simeq D_4$
$(01,10,2)$	$R_L \simeq A_4$
$(10,10,-1)$	$R_L \simeq A_1 \perp A_3$
$(10,10,-2)$	$dL = 0$
$A_3(4^0 8)[2\frac{1}{2}\frac{1}{2}] \simeq (00,10,\pm2) \simeq (10,10,a)$, $a = 0,2$	Strongly even 2-universal
$A_3(10^0 12)[1\frac{1}{2}\frac{1}{4}] \simeq (01,01,-1) \simeq (01,10,a)$, $a = 1,-2$	
$A_3(4^0 36)[1\frac{1}{2}\frac{1}{4}]^\dagger \simeq (00,01,\pm2) \simeq (01,01,2)$	Even 2-universal
$A_3(4^2 12)[20\frac{1}{2}]^* \simeq (00,10,\pm1)$	Candidates
$A_3(12^{-4}14)[1\frac{1}{4}\frac{1}{2}]^* \simeq (01,10,0) \simeq (01,10,-1)$	

represented by all of quinary $\mathbb{Z}$-lattices in the corresponding left hand side.

Note that the root sublattices of some candidates are not isometric to A_3. The proof of even 2-universality of $A_3(4^0 36)[1\frac{1}{2}\frac{1}{4}]$ is given in Corollary 3.5.

Now, assume that $R_L \simeq A_1 \perp A_2$. Since $[2,0,4] \not\to A_1 \perp A_2$, we have $m_4(L) = 4$ by Lemma 2.1. Note that any $\mathbb{Z}$-lattice M of rank 4 containing $A_1 \perp A_2$ with $m_4(M) = 4$ is isometric to one of

$$A_1 \perp A_2 \perp \langle 4\rangle,\quad A_1 \perp A_2 30[1\tfrac{1}{3}],\quad A_1 14[1\tfrac{1}{2}] \perp A_2,\quad \text{and}\quad A_1A_2 102[11\tfrac{1}{6}].$$

One may easily check that

$$[4,4,4] \not\to A_1 \perp A_2 \perp \langle 4\rangle,\quad [4,4,4] \not\to A_1 \perp A_2 30[1\tfrac{1}{3}],$$

$$[4,4,4] \not\to A_1 14[1\tfrac{1}{2}] \perp A_2,\quad \text{and}\quad [6,0,6] \not\to A_1A_2 102[11\tfrac{1}{6}].$$

Therefore, for the first three cases, we have $m_5(L) = 4$, and in the last case, we have $m_5(L) \le 6$. Since each case can be done in a similar manner, we only consider the last case. Note that there are only 3 new candidates of even 2-universal $\mathbb{Z}$-lattices in the first three cases, which are $A_1 \perp A_2(4^2 22)[1\frac{1}{3}\frac{1}{3}]$, $A_1 \perp A_2(10^5 10)[1\frac{1}{3}\frac{1}{3}]$, and $A_2 \perp A_1(4^0 10)[1\frac{1}{2}\frac{1}{2}]$. The first two quinary $\mathbb{Z}$-lattices are, in fact, even 2-universal. The proof of even 2-universality of the first (second) one is given in Corollary 3.14 (Theorem 3.8, respectively). The third one is a candidate.

In the last case, one may easily show that all possible candidates are the followings:

$$(ij,a,4) := \begin{pmatrix} 2 & 0 & 0 & 1 & i \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & j \\ 1 & 0 & 1 & 4 & a \\ i & 0 & j & a & 4 \end{pmatrix} \quad \text{or} \quad (kl,b,6) := \begin{pmatrix} 2 & 0 & 0 & 1 & k \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & l \\ 1 & 0 & 1 & 4 & b \\ k & 0 & l & b & 6 \end{pmatrix},$$

where $a, b = 0, \pm 1, \pm 2$ and $i, j, k, l = 0, 1$. As given in Table 2, there are exactly two strongly even 2-universal $\mathbb{Z}$-lattices and 13 candidates up to isometry in this case.

Now, assume that the rank of R_L is 4. Then R_L is isometric to one of

$$A_4,\quad D_4,\quad A_1 \perp A_3,\quad A_2 \perp A_2,\quad \text{and}\quad A_1 \perp A_1 \perp A_2.$$

One may easily check that

$$[4,4,4] \not\to A_4,\quad [2,2,4] \not\to D_4,\quad [6,0,6] \not\to A_1 \perp A_3,$$

$$[2,0,4] \not\to A_2 \perp A_2,\quad \text{and}\quad [4,4,6] \not\to A_1 \perp A_1 \perp A_2.$$

Therefore, $m_5(L) \le 6$ in all cases. If $R_L \simeq A_4$, then one may easily show that all possible candidates are

$$A_4 \perp \langle 4\rangle,\quad A_4 80[1\tfrac{1}{5}],\quad \text{and}\quad A_4 70[2\tfrac{1}{5}].$$

The first two quinary $\mathbb{Z}$-lattices do not represent $[4,4,4]$ and the third one is strongly even 2-universal. If $R_L \simeq D_4$, then all possible candidates are $D_4 12[2\frac{1}{2}]$ and $D_4 \perp \langle 4\rangle$. Note that the former is strongly even 2-universal and

TABLE 2. The case when $R_L \simeq A_1 \perp A_2$

$L=(ij,a,4)$ or $(kl,b,6)$	
$(01,0,4)\simeq(11,-1,6),\ (01,1,6)$	$[6,0,6]$
$(01,2,6)\simeq(10,2,6)\simeq(10,-1,6)$	$[10,10,10]$
$(10,-2,4)\simeq(11,-2,6),\ (11,-1,4)$	$R_L\simeq A_1\perp A_3$
$(01,-2,4)$	$R_L\simeq A_4$
$(11,-2,4)$	$dL<0$
$A_1A_2(16^422)[11\frac{1}{3}\frac{1}{6}]\simeq(01,1,4)\simeq(10,-2,6)$, $A_1A_2(8^030)[11\frac{1}{2}\frac{1}{6}]\simeq(11,0,4)\simeq(01,-1,4)$	Strongly even 2-universal
$A_1A_2102[11\frac{1}{6}]\perp\langle 4\rangle^*\simeq(00,0,4)$, $A_1A_2(4^{-2}94)[11\frac{1}{3}\frac{1}{6}]^*\simeq(00,1,4)\simeq(00,-1,4)$, $A_1A_2(4^066)[11\frac{1}{2}\frac{1}{6}]^*\simeq(00,2,4)\simeq(00,-2,4)\simeq(11,2,4)$, $A_1A_2(14^020)[11\frac{1}{6}\frac{1}{3}]^*\simeq(01,2,4)\simeq(10,2,4)\simeq(10,-1,4)$, $A_1A_2(14^{-4}26)[11\frac{1}{6}\frac{1}{3}]^*\simeq(10,0,4)\simeq(10,1,4)$, $A_1A_2(6^048)[11\frac{1}{2}\frac{1}{6}]^*\simeq(11,1,4)\simeq(01,-2,6)$, $A_1A_2102[11\frac{1}{6}]\perp\langle 6\rangle^*\simeq(00,0,6)$, $A_1A_2(6^096)[11\frac{1}{6}\frac{1}{6}]^*\simeq(00,1,6)\simeq(00,-1,6)$, $A_1A_2(6^078)[11\frac{1}{3}\frac{1}{6}]^*\simeq(00,2,6)\simeq(00,-2,6)\simeq(11,2,6)$, $A_1A_2(14^238)[11\frac{1}{3}\frac{1}{6}]^*\simeq(01,0,6)$, $A_1A_2(10^446)[11\frac{1}{3}\frac{1}{6}]^*\simeq(01,-1,6)\simeq(11,0,6)$, $A_1A_2(22^{-8}28)[11\frac{1}{3}\frac{1}{6}]^*\simeq(10,0,6)\simeq(10,1,6)$, $A_1A_2(8^262)[11\frac{1}{3}\frac{1}{6}]^*\simeq(11,1,6)$	Candidates

the latter does not represent $[2,2,4]$. Assume that $R_L\simeq A_1\perp A_3$. In this case, all possible candidates are

$$A_1\perp A_3\perp\langle 4\rangle,\quad A_1\perp A_352[1\tfrac{1}{4}]^\dagger,\quad A_1\perp A_3\perp\langle 6\rangle^\dagger,\quad A_1\perp A_384[1\tfrac{1}{4}]^*,$$
$$A_1\perp A_312[2\tfrac{1}{2}],\quad A_3\perp A_114[1\tfrac{1}{2}]^\dagger,\quad A_1\perp A_320[2\tfrac{1}{2}]^\dagger,\quad A_3\perp A_122[1\tfrac{1}{2}]^\dagger,$$
$$A_1A_344[11\tfrac{1}{4}],\quad A_1A_310[12\tfrac{1}{2}],\quad A_1A_376[11\tfrac{1}{4}]^*,\quad \text{and}\quad A_1A_318[12\tfrac{1}{2}].$$

Among them, one may easily check that

$$A_1\perp A_3\perp\langle 4\rangle,\quad A_1\perp A_312[2\tfrac{1}{2}],\quad A_1A_344[11\tfrac{1}{4}],\quad \text{and}\quad A_1A_310[12\tfrac{1}{2}]$$

are strongly even 2-universal $\mathbb{Z}$-lattices. It is known that $L = I_3 \perp A_1 \perp \langle 5\rangle$ is almost 2-universal, that is, L represents all binary $\mathbb{Z}$-lattices except $[3,0,3]$ (see [7]). Hence the $\mathbb{Z}$-lattice $L(e) = A_1 \perp A_3 20[2\frac{1}{2}]$ is even 2-universal. Note that $[6,0,6] \not\to A_1A_3 18[12\frac{1}{2}]$. The even 2-universalities of the $\mathbb{Z}$-lattices $A_3 \perp A_1 14[1\frac{1}{2}]$ and $A_1 \perp A_3 \perp \langle 6\rangle$ are proved in [12]. the even 2-universality of $A_1 \perp A_3 52[1\frac{1}{4}]$ ($A_3 \perp A_1 22[1\frac{1}{2}]$) will be proved in Theorem 3.17 (Corollary 3.12, respectively). If $R_L \simeq A_2 \perp A_2$, then all possible candidates are

$$A_2 \perp A_2 \perp \langle 4\rangle, \quad A_2 \perp A_2 30[1\tfrac{1}{3}]^{\dagger}, \quad \text{and} \quad A_2A_2 24[11\tfrac{1}{3}].$$

The $\mathbb{Z}$-lattice $A_2A_2 24[11\frac{1}{3}]$ is strongly even 2-universal and one may easily check that $[6,2,6] \not\to A_2 \perp A_2 \perp \langle 4\rangle$. The even 2-universality of the $\mathbb{Z}$-lattice $A_2 \perp A_2 30[1\frac{1}{3}]$, which has class number two, will be proved in Theorem 3.2. Finally, if $R_L \simeq A_1 \perp A_1 \perp A_2$, then one may easily check that all possible candidates are

$$\begin{gathered} A_1 \perp A_1 \perp A_2 \perp \langle 4\rangle^{\dagger}, \quad A_1 \perp A_1 \perp A_2 30[1\tfrac{1}{3}]^{\dagger}, \quad A_1 \perp A_2 \perp A_1 14[1\tfrac{1}{2}]^{\dagger}, \\ A_1 \perp A_1A_2 102[11\tfrac{1}{6}]^{\dagger}, \quad A_1A_1A_2 84[111\tfrac{1}{6}], \quad A_2 \perp A_1A_1 12[11\tfrac{1}{2}], \\ A_1 \perp A_1 \perp A_2 \perp \langle 6\rangle, \quad A_1 \perp A_1 \perp A_2 48[1\tfrac{1}{3}], \quad A_1 \perp A_2 \perp A_1 22[1\tfrac{1}{2}], \\ A_1 \perp A_1A_2 174[11\tfrac{1}{6}]^{*}, \quad A_1A_1A_2 156[111\tfrac{1}{6}]^{*}, \quad \text{and} \quad A_2 \perp A_1A_1 20[11\tfrac{1}{2}]^{\dagger}. \end{gathered}$$

Among them, both $A_1A_1A_2 84[111\frac{1}{6}]$ and $A_2 \perp A_1A_1 12[11\frac{1}{2}]$ are strongly even 2-universal. One may easily check that none of the $\mathbb{Z}$-lattices

$$A_1 \perp A_1 \perp A_2 \perp \langle 6\rangle, \quad A_1 \perp A_1 \perp A_2 48[1\tfrac{1}{3}], \quad \text{and} \quad A_1 \perp A_2 \perp A_1 22[1\tfrac{1}{2}]$$

represent $[4,4,6]$. The proof of the even 2-universality of the $\mathbb{Z}$-lattice $A_1 \perp A_1 \perp A_2 \perp \langle 4\rangle$ is given in [12]. The even 2-universalities of the $\mathbb{Z}$-lattices

$$\begin{gathered} A_1 \perp A_1 \perp A_2 30[1\frac{1}{3}], \quad A_1 \perp A_2 \perp A_1 14[1\frac{1}{2}], \\ A_1 \perp A_1A_2 102[11\frac{1}{6}], \quad \text{and} \quad A_2 \perp A_1A_1 20[11\frac{1}{2}] \end{gathered}$$

will be proved in Theorems 3.15, 3.16, 3.18 and Corollary 3.10, respectively.

Finally, if the rank of R_L is 5, then R_L is isometric to one of

$$\begin{gathered} A_5, \quad D_5, \quad A_1 \perp D_4, \quad A_1 \perp A_4, \quad A_2 \perp A_3, \\ A_1 \perp A_2 \perp A_2, \quad A_1 \perp A_1 \perp A_1 \perp A_2^{\dagger}, \quad \text{and} \quad A_1 \perp A_1 \perp A_3. \end{gathered}$$

All $\mathbb{Z}$-lattices except $A_1 \perp A_1 \perp A_1 \perp A_2$ are strongly even 2-universal. The proof of the even 2-universality of the $\mathbb{Z}$-lattice $A_1 \perp A_1 \perp A_1 \perp A_2$ is given in [12]. This completes the proof. □

3. The proofs

In this section, we prove the even 2-universalities of some candidates which are given in the previous section. To do this, we introduce various method on the representations of binary $\mathbb{Z}$-lattices. In particular, we modify the method mainly developed in [9], [11], and [12].

Lemma 3.1. *Let ℓ be a $\mathbb{Z}$-sublattice of E_8 with rank 6 such that ℓ_2 is isometric to none of the followings*:

$$[2,2,2] \perp [2,2,2] \perp [4,4,4],$$
$$[0,2,0] \perp [4,4,4] \perp [0,4,0],$$
$$[4,4,4] \perp [4,4,4] \perp [4,4,4].$$

Then ℓ is represented by $E_7 \perp A_1$.

Proof. Since the class number of $E_7 \perp A_1$ is one, and $(E_7 \perp A_1)_p \simeq (E_8)_p$ for any odd prime p, it is sufficient to show that $\ell_2 \to (E_7 \perp A_1)_2$. One may easily check by using Theorem 3 of [17] that if ℓ_2 is isometric to none of the $\mathbb{Z}_2$-lattices given above, ℓ_2 is represented by $(E_7 \perp A_1)_2$. □

Theorem 3.2. *The quinary $\mathbb{Z}$-lattice $A_2 \perp A_2 30[1\frac{1}{3}]$ is even 2-universal.*

Proof. Let ℓ be any even $\mathbb{Z}$-lattice of rank 2. Since any orthogonal complement of A_4 in E_7 is isometric to $A_2 30[1\frac{1}{3}]$, it suffices to show that $L = A_4 \perp \ell \longrightarrow E_7 \perp A_2$. We know

$$\text{gen}(E_7 \perp A_2)/\sim = \{E_7 \perp A_2,\ E_8 \perp \langle 6\rangle\}.$$

Since any $\mathbb{Z}$-lattice of rank 6 is locally represented by $E_7 \perp A_2$, $L = A_4 \perp \ell$ is represented by $E_7 \perp A_2$ or $E_8 \perp \langle 6\rangle$. Assume that there is a representation $\phi : L = A_4 \perp \ell \mapsto E_8 \perp \langle 6\rangle$. Then $\phi(L) \cap E_8 \simeq A_4 \perp \phi(\ell) \cap E_8 \longrightarrow E_8$. Since $(A_4)_2 \simeq [0,2,0] \perp [2,2,2]$, we have $\phi(L) \cap E_8 \longrightarrow E_7 \perp A_1$ by Lemma 3.1. Therefore, we have $L \longrightarrow E_7 \perp A_1 \perp \langle 6\rangle \longrightarrow E_7 \perp A_2$, as desired. □

Let I_n be the $\mathbb{Z}$-lattice of rank n whose corresponding symmetric matrix is the identity matrix.

Lemma 3.3. *Let ℓ be a $\mathbb{Z}$-lattice of rank 1 or 2 that is represented by I_3. Then for any odd prime p,*

$$r(p\ell, I_3) - r(\ell, I_3) > 0.$$

Proof. Since the class number of I_3 is one, one may easily check by using the Minkowski-Siegel formula that

$$\frac{r(p\ell, I_3)}{r(\ell, I_3)} = \frac{\alpha_p(p\ell, I_3)}{\alpha_p(\ell, I_3)},$$

where $\alpha_p(\cdot,\cdot)$ is the local density over $\mathbb{Z}_p$. Hence it suffices to show that the right hand side is greater than 1. For the proof of the case when ℓ is unary, see [13]. The proof of the binary case is quite similar to that of the unary case. For the computation of the local density $\alpha_p(\ell, I_3)$ in the case when ℓ is a binary $\mathbb{Z}$-lattice, see [14]. □

Theorem 3.4. *The quinary $\mathbb{Z}$-lattice*

$$L = I_1 \perp A_3 36[1\tfrac{1}{4}] = \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

represents all binary $\mathbb{Z}$-lattices except $[1,0,1]$.

Proof. Since

$$\{(x_1,x_2,x_3,x_4)\in I_4 \ : \ \sum_{i=1}^{4} x_i \equiv 0 \ (\mathrm{mod}\ 3)\} \simeq A_3 36[1\tfrac{1}{4}] = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

we may assume that

$$L = \left\{\mathbf{x} = (x_1,x_2,\dots,x_5)\in I_5 : \sum_{i=2}^{5} x_i \equiv 0 \ (\mathrm{mod}\ 3)\right\}.$$

Let $\ell = [a,2b,c]$ be a binary $\mathbb{Z}$-lattice. Since I_5 is 2-universal, there are two vectors $\mathbf{x}=(x_1,x_2,\dots,x_5), \mathbf{y}=(y_1,y_2,\dots,y_5)\in I_5$ such that $\mathbb{Z}\mathbf{x}+\mathbb{Z}\mathbf{y}\simeq [a,2b,c]$. Let E be the set of vectors in $\mathbb{Z}^5$ whose coordinates are either 1 or -1. Note that for any $\mathbf{e}=(e_1,\dots,e_5)\in E$, $\ell_{\mathbf{e}} = \mathbb{Z}(e_1x_1,\dots,e_5x_5)+\mathbb{Z}(e_1y_1,\dots,e_5y_5)\simeq \ell$. If there are a subset $\{i_1,\dots,i_4\}\subset\{1,2,3,4,5\}$ and a vector $\mathbf{e}=(e_1\dots,e_5)\in E$ such that

$$\sum_{k=1}^{4} e_{i_k}x_{i_k} \equiv \sum_{k=1}^{4} e_{i_k}y_{i_k} \equiv 0 \ (\mathrm{mod}\ 3), \tag{3.1}$$

then ℓ is represented by L from the above observation. Note that for any $(x,y)\in\mathbb{Z}^2$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \begin{bmatrix} -x \\ -y \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ (\mathrm{mod}\ 3). \tag{3.2}$$

Let a_i be the number of vectors (x_k,y_k) for $k=1,2,\dots,5$ satisfying the i-th congruence condition in Equation (3.2). If $a_1=4$, then clearly, $\mathbf{e}=(1,1,\dots,1)\in E$ satisfies Equation (3.1). If $a_2+a_3+a_4+a_5\ge 3$, then one may easily show that there is a vector $\mathbf{e}\in E$ satisfying Equation (3.1). For example, if $a_1=a_2=a_3=a_4=1$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ (\mathrm{mod}\ 3),$$

and if $a_2=1, a_3=2, a_4=1$, then

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ (\mathrm{mod}\ 3).$$

Hence we may assume that $a_1 = 3$. Without loss of generality, assume that $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) satisfy the first congruence condition in Equation (3.2). If at least one vector among (x_i, y_i) for $i = 1, 2, 3$ is a nonzero vector, then by Lemma 3.3, there are integers $\tilde{x}_i$'s and $\tilde{y}_i$'s such that

$$\mathbb{Z}(x_1, x_2, x_3) + \mathbb{Z}(y_1, y_2, y_3) \simeq \mathbb{Z}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) + \mathbb{Z}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3),$$

and at least one among $\tilde{x}_i$'s and $\tilde{y}_i$'s is not divisible by 3. Therefore, there is a vector $\mathbf{e} \in E$ satisfying Equation (3.1) if we choose a basis for ℓ such that

$$\ell = \mathbb{Z}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, x_4, x_5) + \mathbb{Z}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, y_4, y_5).$$

Summing up, if ℓ is a sum of at least 3 nonzero squares in the sense of [8], then ℓ is represented by L. Note that any binary $\mathbb{Z}$-lattice except $[1, 0, 1]$ is a sum of at least 3 nonzero squares by [8]. This completes the proof. □

Corollary 3.5. *The $\mathbb{Z}$-lattice $L = A_3(4^0 36)[1\frac{1}{2}\frac{1}{4}]$ is even 2-universal.*

Proof. Note that

$$L \simeq \{\mathbf{x} \in \langle 1 \rangle \perp A_3 36[1\tfrac{1}{4}] : Q(\mathbf{x}) \equiv 0 \pmod 2\}.$$

The corollary follows directly from this. □

As far as the authors know, there is no known general method on finding all binary $\mathbb{Z}$-lattices that are represented by an arbitrary quinary $\mathbb{Z}$-lattice. However, in some very special case, there is a method to do this, which is developed in [11] and [12]. To apply this method to find some even 2-universal quniary $\mathbb{Z}$-lattices, we explain this method a little bit more precisely. Let $\ell = [a, 2b, c]$ be a binary $\mathbb{Z}$-lattice. For any integers s, t, u, we define

$$\ell_{s,t}(u) := \begin{pmatrix} a - us^2 & b - ust \\ b - ust & c - ut^2 \end{pmatrix}.$$

Let M be a quaternary $\mathbb{Z}$-lattice with class number one and $L = M \perp \langle u \rangle$ for some positive integer u. To determine whether or not a binary $\mathbb{Z}$-lattice ℓ is represented by L, we try to find integers s, t such that $\ell_{s,t}(u) \longrightarrow M$. Since we are assuming the class number of M is one, it suffices to find integers s, t such that $(\ell_{s,t}(u))_p \longrightarrow M_p$ for any prime p, and $\ell_{s,t}(u)$ is positive definite by the local-global principle. If p is odd and dM is a square unit in $\mathbb{Z}_p$, then M_p is 2-universal over $\mathbb{Z}_p$. Assume that p is odd and dM is a nonsquare unit in $\mathbb{Z}_p$. Then M_p represents all binary $\mathbb{Z}_p$-lattices that represent a unit in $\mathbb{Z}_p$. Hence if we choose integers s, t such that $\gcd(a - us^2, b - ust)$ has no odd prime factors p such that dM is a nonsquare unit in $\mathbb{Z}_p$, then $(\ell_{s,t}(u))_p \longrightarrow M_p$ for any prime p not dividing $2dM$. Note that if the discriminant of a quaternary $\mathbb{Z}$-lattice M is not a square of an integer, then such primes exist infinitely many. The following lemma will be used to choose suitable integers s, t in this situation.

Lemma 3.6. *For $k \geq 2$, let $p_1 < p_2 < \cdots < p_k$ be primes and let d be an integer satisfying $\gcd(d, p_1 p_2 \cdots p_k) = 1$. If $n \geq \frac{p_1+k-1}{p_1-1} 2^k$, then there is a* *number in the set* $\{a, a+d, \dots, a+(n-1)d\}$ *that is relatively prime to* $p_1p_2\cdots p_k$ *for any integer* a.

Proof. See [11]. □

If p is a prime dividing u and $\mathfrak{s}(\ell) \subseteq p\mathbb{Z}$, then p divides $\gcd(a-us^2, b-ust)$ for any integers s,t. For this difficulty, we consider this case separately.

For each prime p dividing $2dM$, we find a suitable condition on s,t such that $(\ell_{s,t}(u))_p \longrightarrow M_p$. Then we may choose integers s,t suitably so that $(\ell_{s,t}(u))_p \longrightarrow M_p$ for any prime $p \mid 2dM$ by using Chinese Remainder Theorem. The following lemma shows that $\ell_{s,t}(u)$ is positive definite if a is sufficiently large.

Lemma 3.7. *Let* $\ell = [a, 2b, c]$ *be a Minkowski reduced binary* $\mathbb{Z}$*-lattice, that is,* $2|b| \le a \le c$. *If* $a > \frac{4}{3}u(s^2+|st|+t^2)$, *then* $\ell_{s,t}(u)$ *is positive definite.*

Proof. Since $a-us^2 > 0$ by assumption, it suffices to show that the discriminant of $\ell_{s,t}(u)$ is positive. Note that

$$\begin{aligned} d(\ell_{s,t}(u)) &= ac - b^2 - us^2c + 2ustb - ut^2a \\ &= \frac{1}{4}ac - b^2 + \frac{3}{4}ac - u(s^2c - 2stb + t^2a) \\ &\ge \frac{3}{4}ac - u(s^2c + |st|c + t^2c) \\ &= \frac{3}{4}c\big(a - \frac{4}{3}u(s^2+|st|+t^2)\big) > 0. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *The quinary* $\mathbb{Z}$*-lattice* $L = A_1 \perp A_2(10^510)[1\frac{1}{3}\frac{1}{3}]$ *is even* 2*-universal.*

Proof. Note that the quaternary $\mathbb{Z}$-sublattice

$$M = A_2(10^510)[1\tfrac{1}{3}\tfrac{1}{3}] \simeq \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 1 & -1 & 4 \end{pmatrix}$$

of L has class number one, and $dM = 5^2$. Let $\ell = [a, 2b, c]$ be any even binary $\mathbb{Z}$-lattice such that $0 \le 2b \le a \le c$. We further assume that ℓ_5 is a primitive $\mathbb{Z}_5$-lattice. Note that $\ell \longrightarrow M \perp \langle 2\rangle = L$ if and only if $\ell_{s,t}(2) \longrightarrow M$ for some integers s,t.

Since $M_2 \simeq [0,2,0] \perp [0,2,0]$ and $dM = 5^2$, M_p is even 2-universal over $\mathbb{Z}_p$ for any prime $p \ne 5$. Furthermore, since the class number of M is one, we have $\ell \longrightarrow M$ if and only if $\ell_5 \longrightarrow M_5$.

First, assume that ℓ_5 represents a unit in $\mathbb{Z}_5$. Since we are assuming that ℓ_5 is primitive, $\mathrm{ord}_5(d\ell) = 0$ or 1. We first consider the case when $a \ge 12$. From the fact that $M_5 \simeq \langle 1, 2, 5, 10\rangle$ over $\mathbb{Z}_5$, we may easily verify the followings:

- If $\mathrm{ord}_5(d\ell) = 1$, then $\ell \longrightarrow M$.
- If $d\ell \equiv 2, 3 \pmod 5$, then $\ell \longrightarrow M$.
- If $d\ell \equiv 1, 4 \pmod 5$ and $5 \nmid a$, then $\ell_{0,1}(2) \longrightarrow M$ or $\ell_{0,2}(2) \longrightarrow M$.
- If $d\ell \equiv 1, 4 \pmod 5$ and $5 \nmid c$, then $\ell_{1,0}(2) \longrightarrow M$ or $\ell_{2,0}(2) \longrightarrow M$.
- If $5 \mid a, c$ and $5 \nmid b$, then $\ell_{1,1}(2) \longrightarrow M$ or $\ell_{1,-1}(2) \longrightarrow M$.

Therefore, ℓ is represented by $L = M \perp \langle 2 \rangle$. Assume that $a \le 11$. Since other cases can be done in a similar manner, we only consider the case when $a = 3$. Then $\ell = [3, 0, c]$ or $\ell = [3, 2, c]$. In the former case, we have

$$\begin{cases} \ell \longrightarrow M & \text{if } c \not\equiv \pm 2 \pmod 5, \\ \ell_{1,0}(2) \longrightarrow M & \text{otherwise.} \end{cases}$$

Therefore, ℓ is represented by L. In the latter case, we have

$$\begin{cases} \ell \longrightarrow M & \text{if } c \not\equiv 0, 4 \pmod 5, \\ \ell_{0,1}(2) \longrightarrow M & \text{if } c \equiv 0 \pmod 5, \\ \ell_{1,0}(2) \longrightarrow M & \text{if } c \equiv 4 \pmod 5. \end{cases}$$

Now, assume that $\mathfrak{s}(\ell) \subseteq 5\mathbb{Z}$. Since the $\mathbb{Z}$-lattice $A_1 \perp A_4$ is *strongly* even 2-universal, $\ell^{\frac{1}{5}} \longrightarrow A_1 \perp A_4$. Therefore, we have

$$\ell \longrightarrow \big(A_1 \perp A_4\big)^5 \longrightarrow L.$$

This completes the proof. $\square$

Theorem 3.9. *The quinary $\mathbb{Z}$-lattice $L = I_2 \perp A_2 \perp \langle 5 \rangle = [1, 0, 1] \perp [2, 2, 2] \perp \langle 5 \rangle$ represents all binary $\mathbb{Z}$-lattices except $[2, 0, 3]$, $[5, 2, 5]$, $[5, 4, 5]$, and $[5, 2, 11]$.*

Proof. Note that the quaternary $\mathbb{Z}$-sublattice $M = [1, 0, 1] \perp [2, 2, 2]$ of L has class number one. Let $\ell = [a, 2b, c]$ be any binary $\mathbb{Z}$-lattice such that $0 \le 2b \le a \le c$. Note that $\ell \longrightarrow M \perp \langle 5 \rangle$ if and only if $\ell_{s,t}(5) \longrightarrow M$ for some integers s, t.

If $a \le 21$, then one may directly show that $\ell = [a, 2b, c] \longrightarrow M \perp \langle 5 \rangle$. As a sample, we consider the case when $a = 5, b = 1$. For a binary $\mathbb{Z}$-lattice $\ell = [5, 2, c]$, we have

$$\begin{cases} \ell \longrightarrow M & \text{if } c \not\equiv 1, 5, 6 \pmod 8 \text{ and } c \not\equiv 2 \pmod 3, \\ \ell_{0,1}(5) \longrightarrow M \text{ or } \ell_{0,3}(5) \longrightarrow M & \text{if } c \equiv 1 \pmod 4 \text{ and } c > 45, \\ \ell_{0,2}(5) \longrightarrow M \text{ or } \ell_{0,6}(5) \longrightarrow M & \text{if } c \equiv 6 \pmod 8 \text{ and } c > 180, \\ \ell_{0,1}(5) \longrightarrow M \text{ or } \ell_{0,2}(5) \longrightarrow M & \text{if } c \equiv 2 \pmod 3 \text{ and } c > 21. \end{cases}$$

By a direct calculation for any small integer c, one may conclude that

$$[5, 2, c] \longrightarrow M \perp \langle 5 \rangle \quad \text{for any } c \ne 5, 11.$$

For $a \le 21$, we may verify that $\ell \longrightarrow L$ except

$$[2, 0, 3],\ [5, 2, 5],\ [5, 4, 5],\ \text{and } [5, 2, 11]. \tag{3.3}$$

From now on, we assume that $a \geq 22$ and for each prime $p \in \{2,5\}$, ℓ_p is a primitive $\mathbb{Z}_p$-lattice. Note that ℓ_5 is a primitive $\mathbb{Z}_5$-lattice if and only if

$$\mathrm{ord}_5(d\ell_5) \leq 1 \quad \text{or} \quad \ell_5 \simeq \langle 5, -\Delta_5 5\rangle,$$

where Δ_5 is a nonsquare unit in $\mathbb{Z}_5^\times$.

First, assume further that $\mathfrak{s}(\ell) \not\subseteq 5\mathbb{Z}$. Note that

$$M_2 \simeq \langle 1,3,3,3\rangle \quad \text{and} \quad M_3 \simeq \langle 1,1,2,6\rangle.$$

By checking the local structures of $\ell_{s,t}(5)$, M over $\mathbb{Z}_2$ and over $\mathbb{Z}_3$, we obtain the following properties.

- If $a \equiv 7 \pmod 8$ or $c \equiv 7 \pmod 8$, then for any s,t, $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $a \equiv 3 \pmod 8$, $2 \mid s$ or $c \equiv 3 \pmod 8$, $2 \mid t$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $a \equiv 1 \pmod 4$, $2 \mid b$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $c \equiv 1 \pmod 4$, $2 \mid b$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $(a,b,c) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $(a,b,c) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $(a,b,c) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$, then $(\ell_{s,t}(5))_2 \longrightarrow M_2$.
- If $3 \mid ac$ and $3 \nmid st$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (1,0,2),(2,0,1),(2,0,2) \pmod 3$ and $3 \nmid st$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (1,1,2),(1,2,1) \pmod 3$ and $st \equiv 1 \pmod 3$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (2,1,1),(2,1,2) \pmod 3$ and $st \equiv 1 \pmod 3$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (1,1,1),(1,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (2,2,1),(2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.
- If $(a,b,c) \equiv (1,0,1) \pmod 3$ and $st \equiv 0 \pmod 3$, then $(\ell_{s,t}(5))_3 \longrightarrow M_3$.

Since we are assuming that ℓ_2 is primitive, ℓ satisfies one of the first seven cases given above. Note that if

$$a \equiv b \equiv c \equiv 0 \pmod 2 \quad \text{or} \quad a \equiv b \equiv c \equiv 1 \pmod 2,\ a \equiv c \pmod 4,$$

then ℓ_2 is not primitive. For any case, one may easily check that there are $s \in \{1,2\}$ and $t \in \{0,1,\dots,5\}$ such that $\ell_{s,t'}$ is represented by M over $\mathbb{Z}_2$ and over $\mathbb{Z}_3$ simultaneously, for any t' such that $t' \equiv t \pmod 6$. Since other cases can be done in a similar manner, we only consider the case when $\ell_{2,-1}(5) \longrightarrow M$ over $\mathbb{Z}_2$ and over $\mathbb{Z}_3$.

Let $\mathfrak{P} = \{5,7,17,19,29,31,\dots\}$ be the set of primes p such that $\left(\frac{dM}{p}\right) = -1$. From the assumption that $\mathfrak{s}(\ell) \not\subseteq 5\mathbb{Z}$, we have $(\ell_{2,t}(5))_5 \longrightarrow M_5 = \langle 1,1,1,\Delta_5\rangle$

for any integer t. Let

$$\{p \in \mathfrak{P} - \{5\} : a - 20 \equiv 0 \pmod{p}\} = \{p_1, p_2, \dots, p_k\}.$$

If t is an integer such that $(b - 10t, p_1 p_2 \cdots p_k) = 1$ and $t \equiv -1 \pmod{6}$, then we have

$$\ell_{2,t}(5) = [a - 20, 2(b - 10t), c - 5t^2] = \begin{pmatrix} a - 20 & b - 10t \\ b - 10t & c - 5t^2 \end{pmatrix} \longrightarrow M_p$$

for any prims p. If $k = 0$, then $(\ell_{2,-1}(5))_p \longrightarrow M_p$ for any prime p. Furthermore, if $a \geq 47$, then $\ell_{2,-1}(5)$ is positive definite by Lemma 3.7 and hence $\ell \longrightarrow L$. If $22 \leq a \leq 46$, then $\ell_{2,-1}(5)$ is positive definite for any integer c such that $c > \frac{5a+b^2+20b}{a-20}$. In the remaining finite cases, one may directly check that $\ell \longrightarrow L$.

If $1 \leq k \leq 6$, then there is an integer t with

$$t \in \left\{ 6m - 1 : -\left[\frac{k}{2}\right] \leq m \leq \left[\frac{k+1}{2}\right] \right\}$$

such that $(\ell_{2,t}(5))_p \longrightarrow M_p$ for any prime p. If $k = 1, 2$, similarly to the case when $k = 0$, $\ell_{2,t}(5)$ is positive definite for any integer c such that $c > \frac{5t^2a+b^2-20tb}{a-20}$, and hence $\ell \longrightarrow L$. For any integer c such that $c \leq \frac{5t^2a+b^2-20tb}{a-20}$, one may directly check that $\ell \longrightarrow L$. Since $a \geq 20 + p_1 \cdots p_k$, one may easily check that $\ell_{2,t}(5)$ is positive definite by Lemma 3.7 for any $k = 3, 4, 5, 6$.

Finally, assume that $k \geq 7$. Since $k2^{k-2} > \frac{7+k-1}{7-1} 2^k$, there is an integer

$t \in \{-3k2^{k-2}+5, \dots, -1, 5, \dots, 3k2^{k-2}-1\}$ such that $(b-10t, p_1 p_2 \cdots p_k) = 1$

by Lemma 3.6. Hence $(\ell_{2,t}(5))_p \longrightarrow M_p$ for any prime p. Furthermore, since $a \geq 20 + 7 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 41^{k-5}$, $\ell_{2,t}(5)$ is positive definite by Lemma 3.7, and therefore $\ell \longrightarrow L$.

Now, assume that $\mathfrak{s}(\ell) \subseteq 5\mathbb{Z}$, that is, ℓ is of the form of $[5a_1, 10b_1, 5c_1]$. If we let $\tilde{\ell} = [a_1, 2b_1, c_1]$, then $\ell = (\tilde{\ell})^5$. Since we are assuming that ℓ_5 is a primitive $\mathbb{Z}_5$-lattice, $\tilde{\ell}_5 \simeq \langle 1, -\Delta_5 \rangle$. This is equivalent to $d(\tilde{\ell}) \equiv 2, 3 \pmod{5}$. Consider the quaternary $\mathbb{Z}$-lattice

$$K = [1, 0, 3] \perp [2, 2, 3].$$

Note that K has class number one and one may easily check that

$$\big(K \perp \langle 5 \rangle\big)^5 \longrightarrow L.$$

If $\tilde{\ell} \longrightarrow K \perp \langle 5 \rangle$, then $(\tilde{\ell})^5 = \ell \longrightarrow L$. Therefore, it suffices to show that $\tilde{\ell}$ is represented by $K \perp \langle 5 \rangle$.

Consider $\tilde{\ell}_{s,t}(5) = [a_1 - 5s^2, 2(b_1 - 5st), c_1 - 5t^2]$. From the fact that $5 \nmid d(\tilde{\ell})$, we have $5 \nmid d(\tilde{\ell}_{s,t}(5))$, and $\tilde{\ell}_{s,t}(5) \longrightarrow K_5$ for any integers s, t. By checking the local structures of $\tilde{\ell}_{s,t}(5)$, M over $\mathbb{Z}_2$ and over $\mathbb{Z}_3$, we obtain the following properties.

- If $a_1 \equiv 3 \pmod{8}$ or $c_1 \equiv 3 \pmod{8}$, then for all s, t, $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.

- If $a_1 \equiv 7 \pmod 8$, $2 \mid s$ or $c_1 \equiv 7 \pmod 8$, $2 \mid t$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $a_1 \equiv 1 \pmod 4$, $2 \mid b_1$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $c_1 \equiv 1 \pmod 4$, $2 \mid b_1$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $(a_1,b_1,c_1) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $(a_1,b_1,c_1) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $(a_1,b_1,c_1) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(5))_2 \longrightarrow K_2$.
- If $3 \mid a_1c_1$ and $3 \nmid st$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1) \equiv (1,0,2),(2,0,1),(2,0,2) \pmod 3$ and $3 \nmid st$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1) \equiv (1,1,2),(1,2,1) \pmod 3$ and $st \equiv 1 \pmod 3$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1) \equiv (2,1,1),(2,1,2) \pmod 3$ and $st \equiv 1 \pmod 3$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1) \equiv (1,1,1),(1,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1)) \equiv (2,2,1),(2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.
- If $(a_1,b_1,c_1) \equiv (1,0,1) \pmod 3$ and $st \equiv 0 \pmod 3$, then $(\tilde{\ell}_{s,t}(5))_3 \longrightarrow K_3$.

Using the same method to the above, we may show that $\tilde{\ell} \longrightarrow K \perp \langle 5\rangle$ except the cases when

$$\tilde{\ell} \simeq [2,2,2], [2,2,14], \text{ and } [6,6,6].$$

Even in the exceptional cases, one may directly check that $(\tilde{\ell})^5 \longrightarrow L$. Therefore, we may conclude that any binary $\mathbb{Z}$-lattice ℓ whose scale is contained in $5\mathbb{Z}$ is represented by L.

Finally, one may easily check that any binary $\mathbb{Z}$-sublattices of each $\mathbb{Z}$-lattice in (3.3) with index 2 or 5 are represented by L. This completes the proof. □

Corollary 3.10. *The $\mathbb{Z}$-lattice $A_2 \perp A_1A_1 20[11\frac{1}{2}]$ is even 2-universal.*

Proof. Let $L = I_2 \perp A_2 \perp \langle 5\rangle$. By Theorem 3.9, L represents all even binary $\mathbb{Z}$-lattices. Hence its even $\mathbb{Z}$-sublattice

$$L(e) = A_2 \perp A_1A_1 20[11\tfrac{1}{2}] = [2,2,2] \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

also represents all even binary $\mathbb{Z}$-lattices. This completes the proof. □

The proof of the almost 2-universality or even 2-universality of each $\mathbb{Z}$-lattice L given below is quite similar to the above. So, we only provide the following data:

(1) quaternary $\mathbb{Z}$-sublattice M of L which has class number one,
(2) the integer u such that $M \perp \langle u\rangle$ is a $\mathbb{Z}$-sublattice of L,

(3) conditions for integers s, t such that $(\ell_{s,t}(u))_p \longrightarrow M_p$ for each prime $p \mid 2dM$,

(4) some data for the case when $\mathfrak{s}(\ell) \subseteq q\mathbb{Z}$ for a prime $q \mid u$ and $\left(\frac{dM}{q}\right) = -1$.

Theorem 3.11. *The quinary $\mathbb{Z}$-lattice*

$$L = I_3 \perp A_1 22[1\tfrac{1}{2}] = \langle 1,1,1\rangle \perp [2,2,6]$$

represents all binary $\mathbb{Z}$-lattices except $[3,0,3]$.

Proof. Let $M = \langle 1,1,1,2\rangle$ and $u = 22$. Clearly, $M \perp \langle 22\rangle$ is a $\mathbb{Z}$-sublattice of L. Let $\ell = [a, 2b, c]$ be any binary $\mathbb{Z}$-lattice such that ℓ_p is a primitive $\mathbb{Z}_p$-lattice for any $p \in \{2, 11\}$. Then one may easily check the followings:

- If $(a,b,c) \equiv (0,1,0), (0,1,1) \pmod 2$, then for any s, t, $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $(a,b,c) \equiv (1,0,1), (1,1,0) \pmod 2$, then for any s, t, $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 3 \pmod 8$ or $c \equiv 3 \pmod 8$, then for any s, t, $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 1 \pmod 8$, $2 \nmid s$ or $c \equiv 1 \pmod 8$, $2 \nmid t$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 5 \pmod 8$, $2 \mid s$ or $c \equiv 5 \pmod 8$, $2 \mid t$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 2 \pmod 8$, $b \equiv 2 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 2 \pmod 8$, $b \equiv 0 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,0) \pmod 2$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 6 \pmod 8$, $b \equiv 0 \pmod 4$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.
- If $a \equiv 6 \pmod 8$, $b \equiv 2 \pmod 4$ and $(s,t) \equiv (1,0) \pmod 2$, then $(\ell_{s,t}(22))_2 \longrightarrow M_2$.

Using this information, one may prove that similarly to Theorem 3.9, ℓ is represented by L under the assumption that $\mathfrak{s}(\ell) \nsubseteq 11\mathbb{Z}$. When $\mathfrak{s}(\ell) \subseteq 11\mathbb{Z}$, we consider the quaternary $\mathbb{Z}$-lattice $K = \langle 1,1,1,11\rangle$. Note that

$$\mathrm{gen}(K)/\sim = \{\langle 1,1,1,11\rangle,\ [1,0,1] \perp [3,2,4]\}.$$

It can easily be verified that $\left(K' \perp \langle 11\rangle\right)^{11} \longrightarrow L$ for any $K' \in \mathrm{gen}(K)$. Let $\tilde{\ell} = [a_1, 2b_1, c_1]$ be a binary $\mathbb{Z}$-lattice such that $(\tilde{\ell})^{11} = \ell$. Since we are assuming that ℓ_p is primitive over $\mathbb{Z}_p$ for any prime $p \in \{2, 11\}$, $d(\tilde{\ell}) \equiv 1, 3, 4, 5,$ or $9 \pmod{11}$, and $\tilde{\ell}_{11}$ is represented by K_{11}. If there exist integers s, t such that $(\tilde{\ell}_{s,t}(11))_p \longrightarrow K_p$ for any prime p and $\tilde{\ell}_{s,t}(11)$ is positive definite, then $\tilde{\ell}_{s,t}(11) \longrightarrow K'$ and $\tilde{\ell} \longrightarrow K' \perp \langle 11\rangle$ for some $K' \in \mathrm{gen}(K)$. Hence we have

$$\ell = (\tilde{\ell})^{11} \longrightarrow \left(K' \perp \langle 11\rangle\right)^{11} \longrightarrow L.$$

To prove the existence of such integers s, t, one may use

- If $(a_1, b_1, c_1) \equiv (1,0,1) \pmod 2$ and $(s,t) \equiv (1,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $(a_1, b_1, c_1) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.

- If $(a_1,b_1,c_1)\equiv(1,1,0) \pmod 2$ and $(s,t)\equiv(1,0) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $(a_1,b_1,c_1)\equiv(0,1,1) \pmod 2$ and $(s,t)\equiv(0,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $a_1\equiv 1 \pmod 8$ or $c_1\equiv 1 \pmod 8$, then for any s,t, $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $a_1\equiv 5 \pmod 8$, $2\mid s$ or $c_1\equiv 5 \pmod 8$, $2\mid t$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $a_1\equiv 0 \pmod 4$, $2\nmid s$ or $c_1\equiv 0 \pmod 4$, $2\nmid t$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $a_1\equiv 3 \pmod 4$, $2\mid b_1$ and $(s,t)\equiv(1,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.
- If $c_1\equiv 3 \pmod 4$, $2\mid b_1$ and $(s,t)\equiv(1,1) \pmod 2$, then $(\tilde{\ell}_{s,t}(11))_2 \longrightarrow K_2$.

Using this information, one may show that $\tilde{\ell}$ is represented by $K'\perp\langle 11\rangle$ for some $K'\in \mathrm{gen}(K)$ except the binary $\mathbb{Z}$-lattices $\tilde{\ell}$ such that

$$\tilde{\ell}\simeq[3,0,3],[3,0,71],[2,2,3],[3,2,18],[7,2,10],[7,4,7],[7,4,23], \text{ or } [19,4,19].$$

Even in these exceptional cases, one may directly check that $\ell=(\tilde{\ell})^{11}\longrightarrow L$. Note that any binary $\mathbb{Z}$-sublattice of $[3,0,3]$ with index 2 or 11 is represented by L. This completes the proof. □

Corollary 3.12. *The $\mathbb{Z}$-lattice $A_3\perp A_1 22[1\frac{1}{2}]$ is even 2-universal.*

Proof. Note that if $L=I_3\perp A_1 22[1\frac{1}{2}]$, then $L(e)\simeq A_3\perp A_1 22[1\frac{1}{2}]$. Therefore, the proof follows directly from Theorem 3.11. □

Theorem 3.13. *The $\mathbb{Z}$-lattice $L=I_1\perp A_1\perp A_2 21[1\frac{1}{3}]$ represents all binary $\mathbb{Z}$-lattices except $I_2=[1,0,1]$.*

Proof. In this case, we let

$$M=I_1\perp A_2 21[1\tfrac{1}{3}]=\langle 1\rangle\perp\begin{pmatrix}2&1&0\\1&2&1\\0&1&3\end{pmatrix} \text{ and } u=2.$$

Let $\ell=[a,2b,c]$ be any binary $\mathbb{Z}$-lattice such that ℓ_p is primitive over $\mathbb{Z}_p$ for any $p\in\{2,7\}$. Then we may easily check the followings:

- If $(a,b,c)\equiv(0,1,0) \pmod 2$, then for any integer s,t, $\ell_{s,t}\longrightarrow M_2$.
- If $a\equiv 1 \pmod 4$ and $2\nmid s$, then $(\ell_{s,t}(2))_2\longrightarrow M_2$.
- If $c\equiv 1 \pmod 4$ and $2\nmid t$, then $(\ell_{s,t}(2))_2\longrightarrow M_2$.
- If $a\equiv 3 \pmod 4$ and $2\mid s$, then $(\ell_{s,t}(2))_2\longrightarrow M_2$.
- If $c\equiv 3 \pmod 4$ and $2\mid t$, then $(\ell_{s,t}(2))_2\longrightarrow M_2$.
- If $a\equiv 3,4,6 \pmod 7$ and $s\equiv\pm1 \pmod 7$, then $(\ell_{s,t}(2))_7\longrightarrow M_7$.
- If $a\equiv 2,3,5 \pmod 7$ and $s\equiv\pm2 \pmod 7$, then $(\ell_{s,t}(2))_7\longrightarrow M_7$.
- If $a\equiv 1,5,6 \pmod 7$ and $s\equiv\pm3 \pmod 7$, then $(\ell_{s,t}(2))_7\longrightarrow M_7$.
- If $a\equiv 0 \pmod 7$, $b\not\equiv 0 \pmod 7$ and $7\nmid s$, then $(\ell_{s,t}(2))_7\longrightarrow M_7$.
- If $a,b\equiv 0 \pmod 7$, $c\not\equiv 0 \pmod 7$ and $7\nmid s$, then $(\ell_{s,t}(2))_7\longrightarrow M_7$.
- If $a,b,c\equiv 0 \pmod 7$, then there is an integer t_0 such that for any integer t with $t\equiv t_0 \pmod 7$, $(\ell_{1,t}(2))_7\longrightarrow M_7$.

In the last paragraph, the existence of t_0 can be proved as follows: Note that for a nonsquare unit $\Delta_7 \in \mathbb{Z}_7^\times$, $\langle 1, 7\rangle, \langle \Delta_7, 7\Delta_7\rangle \longrightarrow M_7$. Let $a = 7a_0, b = 7b_0$, and $c = 7c_0$, then

$$d(\ell_{s,t}(2)) \equiv -14(a_0t^2 - 2b_0st + c_0s^2) \pmod{7^2}.$$

Since ℓ_7 is $\mathbb{Z}_7$-primitive, $a_0, c_0 \not\equiv 0 \pmod 7$ and $d\ell = 7^2(a_0c_0-b_0^2) \not\equiv 0 \pmod{7^3}$. Hence there is an integer t_0 such that $a_0t_0^2 - 2b_0t_0 + c_0$ is a nonsquare modulo 7 (see Theorem 8.2 of Chapter 7 in [6]). Therefore, $(\ell_{1,t_0}(2))_7 \simeq \langle 1, 7\rangle$ or $\langle \Delta_7, 7\Delta_7\rangle$, which is represented by M_7. Note that any binary $\mathbb{Z}$-sublattice of $[1, 0, 1]$ with index 2 or 7 is represented by L. This completes the proof. $\square$

Corollary 3.14. *The $\mathbb{Z}$-lattice $L = A_1 \perp A_2(4^222)[1\frac{1}{3}\frac{1}{3}]$ is even 2-universal.*

Proof. Note that $L = K(e)$, where $K = I_1 \perp A_1 \perp A_221[1\frac{1}{3}]$. Hence the corollary follows directly from Theorem 3.13. $\square$

Theorem 3.15. *The $\mathbb{Z}$-lattice $L = A_1 \perp A_1 \perp A_230[1\frac{1}{3}]$ is even 2-universal.*

Proof. Let $M = A_1 \perp A_230[1\frac{1}{3}] = \langle 2\rangle \perp \begin{pmatrix} 2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 4\end{pmatrix}$ and $u = 2$. Note that $M \perp \langle 2\rangle \longrightarrow L$. Let $\ell = [a, 2b, c]$ be any even binary $\mathbb{Z}$-lattice such that ℓ_5 is a primitive $\mathbb{Z}_5$-lattice. Then one may easily check the followings:

- If b is odd, then for any integers s, t, $(\ell_{s,t}(2))_2 \longrightarrow M_2$.
- If $a \equiv 1, 3 \pmod 5$, then $(\ell_{1,t}(2))_5 \longrightarrow M_5$.
- If $a \equiv 2, 4 \pmod 5$, then $(\ell_{2,t}(2))_5 \longrightarrow M_5$.
- If $a \equiv 0$, $b \not\equiv 0 \pmod 5$, then $(\ell_{5,t}(2))_5 \longrightarrow M_5$.
- If $a, b \equiv 0$, $c \not\equiv 0 \pmod 5$, then for any $5 \nmid s$, $(\ell_{s,t}(2))_5 \longrightarrow M_5$.
- If $a, b, c \equiv 0 \pmod 5$, then there is an integer t_0 such that for any integer t with $t \equiv t_0 \pmod 5$, $(\ell_{1,t}(2))_5 \longrightarrow M_5$.

Using this information, one may show that ℓ is represented by $M \perp \langle 2\rangle$ for any ℓ such that $\mathfrak{s}(\ell) \not\subseteq 2\mathbb{Z}$.

It is known that $K = I_3 \perp A_110[1\frac{1}{2}] = \langle 1, 1, 1\rangle \perp [2, 2, 3]$ is 2-universal (see [11]). Since $K^2 \longrightarrow L$, L represents any binary $\mathbb{Z}$-lattice ℓ such that $\mathfrak{s}(\ell) \subseteq 2\mathbb{Z}$. This completes the proof. $\square$

Theorem 3.16. *The $\mathbb{Z}$-lattice $L = A_1 \perp A_2 \perp A_114[1\frac{1}{2}]$ is even 2-universal.*

Proof. Let $M = A_2 \perp A_114[1\frac{1}{2}] = [2, 2, 2] \perp [2, 2, 4]$ and $u = 2$. Let $\ell = [a, 2b, c]$ be any even binary $\mathbb{Z}$-lattice such that ℓ_7 is a primitive $\mathbb{Z}_7$-lattice. Then one may easily check the followings:

- If b is odd, then for any integers s, t, $(\ell_{s,t}(2))_2 \longrightarrow M_2$.
- If $a \equiv 3, 4, 6 \pmod 7$ and $s \equiv \pm 1 \pmod 7$, then $(\ell_{s,t}(2))_7 \longrightarrow M_7$.
- If $a \equiv 2, 3, 5 \pmod 7$ and $s \equiv \pm 2 \pmod 7$, then $(\ell_{s,t}(2))_7 \longrightarrow M_7$.
- If $a \equiv 1, 5, 6 \pmod 7$ and $s \equiv \pm 3 \pmod 7$, then $(\ell_{s,t}(2))_7 \longrightarrow M_7$.
- If $a \equiv 0 \pmod 7$ and $b \not\equiv 0 \pmod 7$, then $(\ell_{7,t}(2))_7 \longrightarrow M_7$.

- If $a, b \equiv 0 \pmod 7$ and $c \not\equiv 0 \pmod 7$, then for any $7 \nmid s$, $(\ell_{s,t}(2))_7 \longrightarrow M_7$.
- If $a, b, c \equiv 0 \pmod 7$, then there is an integer t_0 such that for any integer t with $t \equiv t_0 \pmod 7$, $(\ell_{1,t}(2))_7 \longrightarrow M_7$.

Using this information, one may show that ℓ is represented by $M \perp \langle 2 \rangle$ for any ℓ such that $\mathfrak{s}(\ell) \nsubseteq 2\mathbb{Z}$.

One may easily check that $K = \langle 1, 1, 1, 3, 7 \rangle$ is locally 2-universal and $\left(K'\right)^2 \longrightarrow L$ for any $K' \in \text{gen}(K)$. Hence L represents any binary $\mathbb{Z}$-lattice ℓ such that $\mathfrak{s}(\ell) \subseteq 2\mathbb{Z}$. This completes the proof. $\square$

Theorem 3.17. *The $\mathbb{Z}$-lattice $L = A_1 \perp A_3 52[1\frac{1}{4}]$ is even 2-universal.*

Proof. Let $M = A_3 52[1\frac{1}{4}] = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ and $u = 2$. Let $\ell = [a, 2b, c]$ be any even binary $\mathbb{Z}$-lattice such that ℓ_{13} is a primitive $\mathbb{Z}_{13}$-lattice. Then one may easily check the followings:

- If b is odd, then for any integers s, t, $(\ell_{s,t}(2))_2 \longrightarrow M_2$.
- If $a \equiv 1, 3, 5, 6, 11, 12 \pmod{13}$, then $(\ell_{1,t}(2))_{13} \longrightarrow M_{13}$.
- If $a \equiv 4, 7, 9 \pmod{13}$, then $(\ell_{2,t}(2))_{13} \longrightarrow M_{13}$.
- If $a \equiv 2, 8 \pmod{13}$, then $(\ell_{3,t}(2))_{13} \longrightarrow M_{13}$.
- If $a \equiv 10 \pmod{13}$, then $(\ell_{4,t}(2))_{13} \longrightarrow M_{13}$.
- If $a \equiv 0 \pmod{13}$ and $b \not\equiv 0 \pmod{13}$, then $(\ell_{13,t}(2))_{13} \longrightarrow M_{13}$.
- If $a, b \equiv 0 \pmod{13}$ and $c \not\equiv 0 \pmod{13}$, then $(\ell_{1,t}(2))_{13} \longrightarrow M_{13}$.

Using this information, one may show that ℓ is represented by $M \perp \langle 2 \rangle$ for any ℓ such that $\mathfrak{s}(\ell) \nsubseteq 2\mathbb{Z}$ and $\mathfrak{s}(\ell) \nsubseteq 13\mathbb{Z}$.

Consider the $\mathbb{Z}$-lattice $K = \langle 1, 1, 1 \rangle \perp [2, 2, 7]$. One may easily check that K is locally 2-universal and $\left(K'\right)^2 \longrightarrow L$ for any $K' \in \text{gen}(K)$. Hence L represents any binary $\mathbb{Z}$-lattice ℓ such that $\mathfrak{s}(\ell) \subseteq 2\mathbb{Z}$. Note that if $J = A_3 52[1\frac{1}{4}] \perp \langle 26 \rangle$, then J is locally even 2-universal and $(J')^{13} \longrightarrow L$ for any $J' \in \text{gen}(J)$. Hence L represents any even binary $\mathbb{Z}$-lattice ℓ such that $\mathfrak{s}(\ell) \subseteq 13\mathbb{Z}$. This completes the proof. $\square$

Theorem 3.18. *The $\mathbb{Z}$-lattice $L = A_1 \perp A_1 A_2 102[11\frac{1}{6}]$ is even 2-universal.*

Proof. Let $M = A_1 A_2 102[11\frac{1}{6}] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}$ and $u = 2$. Let $\ell = [a, 2b, c]$ be any even binary $\mathbb{Z}$-lattice such that ℓ_{17} is a primitive $\mathbb{Z}_{17}$-lattice. Then one may easily check the followings:

- For any integers s, t, $(\ell_{s,t}(2))_2 \longrightarrow M_2$ (in fact, M_2 is even 2-universal).
- If $a \equiv 5, 7, 8, 9, 12, 13, 14, 16 \pmod{17}$, then $(\ell_{1,t}(2))_{17} \longrightarrow M_{17}$.
- If $a \equiv 1, 2, 3, 11, 15 \pmod{17}$, then $(\ell_{2,t}(2))_{17} \longrightarrow M_{17}$.

- If $a \equiv 4, 6 \pmod{17}$, then $(\ell_{3,t}(2))_{17} \longrightarrow M_{17}$.
- If $a \equiv 10 \pmod{17}$, then $(\ell_{4,t}(2))_{17} \longrightarrow M_{17}$.
- If $a \equiv 0 \pmod{17}$ and $b \not\equiv 0 \pmod{17}$, then $(\ell_{17,t}(2))_{17} \longrightarrow M_{17}$.
- If $a, b \equiv 0 \pmod{17}$ and $c \not\equiv 0 \pmod{17}$, then $(\ell_{1,t}(2))_{17} \longrightarrow M_{17}$.

Using this information, one may show that ℓ is represented by $M \perp \langle 2\rangle$ for any ℓ such that $\mathfrak{s}(\ell) \not\subseteq 17\mathbb{Z}$. Note that if $K = A_1A_2102[11\frac{1}{6}] \perp \langle 34\rangle$, then K is locally even 2-universal and $(K')^{17} \longrightarrow L$ for any $K' \in \mathrm{gen}(L)$. Hence L represents any even binary $\mathbb{Z}$-lattice ℓ such that $\mathfrak{s}(\ell) \subseteq 17\mathbb{Z}$. This completes the proof. $\square$

Table 3. The number of (candidates of) even 2-universal even $\mathbb{Z}$-lattices of rank 5

R_L	Proved		Candidates
	$h = 1$	$h \geq 2$	
$\mathrm{rank}(R_L) = 5$	7	1	0
A_4, D_4	2	0	0
$A_1 \perp A_3$	4	5	2
$A_2 \perp A_2$	1	1	0
$A_1 \perp A_1 \perp A_2$	2	5	2
A_3	2	1	2
$A_1 \perp A_2$	2	2	14
Total	**20**	**15**	**20**

Table 4. Even 2-universal even $\mathbb{Z}$-lattices of rank 5

$A_5,\ D_5,\ A_1 \perp A_4,\ A_4 70[2\frac{1}{5}],\ A_1 \perp D_4,\ D_4 12[2\frac{1}{2}],$
$A_2 \perp A_3,\ A_1 \perp A_1 \perp A_3,\ A_1 \perp A_3 \perp \langle 4\rangle,\ A_1 \perp A_3 \perp \langle 6\rangle^{\dagger},$
$A_1 \perp A_3 12[2\frac{1}{2}],\ A_1 \perp A_3 20[2\frac{1}{2}]^{\dagger},\ A_1 \perp A_3 52[1\frac{1}{4}]^{\dagger},\ A_3 \perp A_1 14[1\frac{1}{2}]^{\dagger},$
$A_3 \perp A_1 22[1\frac{1}{2}]^{\dagger},\ A_3(4^0 8)[2\frac{1}{2}\frac{1}{2}],\ A_3(10^0 12)[1\frac{1}{2}\frac{1}{4}],\ A_3(4^0 36)[1\frac{1}{2}\frac{1}{4}]^{\dagger},$
$A_1A_3 44[11\frac{1}{4}],\ A_1A_3 10[12\frac{1}{2}],\ A_2A_2 24[11\frac{1}{3}],\ A_2 \perp A_2 30[1\frac{1}{3}]^{\dagger},$
$A_1 \perp A_2 \perp A_2,\ A_1 \perp A_1 \perp A_1 \perp A_2{}^{\dagger},\ A_1 \perp A_1 \perp A_2 \perp \langle 4\rangle^{\dagger},$
$A_1 \perp A_1 \perp A_2 30[1\frac{1}{3}]^{\dagger},\ A_1 \perp A_2 \perp A_1 14[1\frac{1}{2}]^{\dagger},$
$A_2 \perp A_1A_1 12[11\frac{1}{2}],\ A_2 \perp A_1A_1 20[11\frac{1}{2}]^{\dagger},$
$A_1 \perp A_1A_2 102[11\frac{1}{6}]^{\dagger},\ A_1A_1A_2 84[111\frac{1}{6}],\ A_1 \perp A_2(4^2 22)[1\frac{1}{3}\frac{1}{3}]^{\dagger},$
$A_1 \perp A_2(10^5 10)[1\frac{1}{3}\frac{1}{3}]^{\dagger},\ A_1A_2(16^4 22)[11\frac{1}{3}\frac{1}{6}],\ A_1A_2(8^0 30)[11\frac{1}{2}\frac{1}{6}]$

TABLE 5. Candidates of even 2-universal even $\mathbb{Z}$-lattices of rank 5

$A_1 \perp A_3 84[1\frac{1}{4}]$, $A_1A_3 76[11\frac{1}{4}]$, $A_3(4^2 12)[20\frac{1}{2}]$, $A_3(12^{-4}14)[1\frac{1}{4}\frac{1}{2}]$,
$A_1A_1A_2 156[111\frac{1}{6}]$, $A_1 \perp A_1A_2 174[11\frac{1}{6}]$,
$A_1A_2 102[11\frac{1}{6}] \perp \langle 4\rangle$, $A_1A_2 102[11\frac{1}{6}] \perp \langle 6\rangle$, $A_2 \perp A_1(4^0 10)[1\frac{1}{2}\frac{1}{2}]$,
$A_1A_2(4^0 66)[11\frac{1}{2}\frac{1}{6}]$, $A_1A_2(4^{-2}94)[11\frac{1}{3}\frac{1}{6}]$, $A_1A_2(6^0 48)[11\frac{1}{2}\frac{1}{6}]$,
$A_1A_2(6^0 78)[11\frac{1}{3}\frac{1}{6}]$, $A_1A_2(6^0 96)[11\frac{1}{6}\frac{1}{6}]$, $A_1A_2(8^2 62)[11\frac{1}{3}\frac{1}{6}]$,
$A_1A_2(10^4 46)[11\frac{1}{3}\frac{1}{6}]$, $A_1A_2(14^2 20)[11\frac{1}{6}\frac{1}{3}]$, $A_1A_2(14^2 38)[11\frac{1}{3}\frac{1}{6}]$,
$A_1A_2(14^{-4}26)[11\frac{1}{6}\frac{1}{3}]$, $A_1A_2(22^{-8}28)[11\frac{1}{3}\frac{1}{6}]$

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