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A NEW CLASSIFICATION OF REAL HYPERSURFACES WITH REEB PARALLEL STRUCTURE JACOBI OPERATOR IN THE COMPLEX QUADRIC

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Abstract. In this paper, first we introduce the full expression of the Riemannian curvature tensor of a real hypersurface M in the complex quadric Q^m from the equation of Gauss and some important formulas for the structure Jacobi operator R_ξ and its derivatives ∇R_ξ under the Levi-Civita connection ∇ of M. Next we give a complete classification of Hopf real hypersurfaces with Reeb parallel structure Jacobi operator, $\nabla_\xi R_\xi = 0$, in the complex quadric Q^m for $m \geq 3$. In addition, we also consider a new notion of $\mathcal{C}$-parallel structure Jacobi operator of M and give a nonexistence theorem for Hopf real hypersurfaces with $\mathcal{C}$-parallel structure Jacobi operator in Q^m, for $m \geq 3$.

1. Introduction

We consider the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$: it is a complex hypersurface in the complex projective space $\mathbb{C}P^{m+1}$ (see Lee and Suh [14], Romero [22], [23], Smyth [24], Suh [27], [28]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Besse [4], Helgason [5], and Knap [11]). Accordingly, the complex quadric Q^m admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then, for $m \geq 2$, the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Kobayashi and Nomizu [12], Reckziegel [21]).

In addition to the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which

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contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. The set is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^m$, and it is the set of all complex conjugations defined on Q^m. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2-subbundle of End $T_{[z]}Q^m$, $[z] \in Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ respectively (see Smyth [24]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex hyperbolic quadric Q^m:

(a) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) = \{X \in T_{[z]}Q^m | AX = X\}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal.*

(b) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors Z_1, $Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic*, where $V(A) = \{X \in T_{[z]}Q^m | AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^m | AX = -X\}$ are the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^m$, $[z] \in Q^m$.

On the other hand, Okumura [15] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ a classification was obtained by Berndt and Suh [1]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, Berndt and Suh [2] have obtained the following result:

Theorem A. *Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.*

Motivated by this result, recently, some new results have been published regarding the real hypersurface with various geometric tools in the complex quadric (see [3], [13], [14], [16], [17], [20], [25] and so on). In this paper, we want to study Reeb parallelism or $\mathcal{C}$-parallelism of the structure Jacobi operator for a real hypersurface in the complex quadric Q^m with new geometric ideas.

It is known that Jacobi fields along geodesics of a given Riemannian manifold $(\bar{M}, g)$ satisfy a well known differential equation. That is, if R denotes the curvature operator of $\bar{M}$, and X is a tangent vector field to $\bar{M}$, then the Jacobi operator $R_X \in \mathrm{End}(T_p\bar{M})$ with respect to X at $p \in \bar{M}$, defined by $(R_XY)(p) =$

$(\bar{R}(Y,X)X)(p)$ for any $Y \in T_p\bar{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. Thus, each tangent vector field X to $\bar{M}$ provides a Jacobi operator R_X with respect to X. In particular, for the Reeb vector field ξ, the Jacobi operator R_ξ is said to be the *structure Jacobi operator*.

Indeed, many geometers have considered the fact that a real hypersurface M in Kähler manifolds has *parallel structure Jacobi operator* (or *Reeb parallel structure Jacobi operator*, respectively), that is, $\nabla_X R_\xi = 0$ (or $\nabla_\xi R_\xi = 0$, respectively) for any tangent vector field X on M. Recently Ki, Pérez, Santos and Suh [9] have investigated Reeb parallel structure Jacobi operator in the complex space form $M^m(c)$, $c \neq 0$, and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [7] have investigated real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M. Jeong, Suh and Woo [8] and Pérez and Santos [18] have generalized such a notion to recurrency of the structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form β and any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$ or $\mathbb{C}P^m$. In [6], Jeong, Lee, and Suh have considered a Hopf real hypersurface with structure Jacobi operator of Codazzi type, $(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X$, in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez, Santos and Suh [19] have further investigated the property of Lie ξ-parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

Motivated by these results, in this paper we want to give a classification of Hopf real hypersurfaces in Q^m with non-vanishing geodesic Reeb flow and Reeb parallel structure Jacobi operator, that is, $\nabla_\xi R_\xi = 0$. Here a real hypersurface M is said to be *Hopf* if the Reeb vector field ξ of M is principal for the shape operator S, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$. In particular, if the Reeb curvature function $\alpha = g(S\xi, \xi)$ identically vanishes, we say that M has *vanishing geodesic Reeb flow*. Otherwise, M has *non-vanishing geodesic Reeb flow*.

Under these background and motivations, first we prove the following:

Theorem 1.1. *There does not exist a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Reeb parallel structure Jacobi operator and $\mathfrak{A}$-principal normal vector field, provided it has non-vanishing geodesic Reeb flow.*

Next, let us consider a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field N in Q^m. Then by virtue of Theorem A we can give a complete classification of Hopf real hypersurfaces in Q^m with Reeb parallel structure Jacobi operator as follows:

Theorem 1.2. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Reeb parallel structure Jacobi operator and non-vanishing geodesic Reeb flow. Then, M has an $\mathfrak{A}$-isotropic normal vector field in Q^m if and only if M is locally congruent to a tube around the totally geodesic $\mathbb{C}P^k$ in Q^{2k}, where $m = 2k$, and $r \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$.*

By virtue of two Theorems 1.1 and 1.2, we obtained a classification of Hopf real hypersurfaces with singular normal vector field and Reeb parallel structure Jacobi operator in the complex quadric Q^m for $m \geq 3$. Motivated by such a geometric condition of Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$, we want to consider another parallelism related to the structure Jacobi operator R_ξ. Namely, it is said to be $\mathcal{C}$-parallel structure Jacobi operator. That is, the structure Jacobi operator R_ξ of M satisfies

$$(\nabla_X R_\xi)Y = 0 \quad \text{for any } X \in \mathcal{C} \text{ and } Y \in TM$$

where $\mathcal{C}$ denotes a distribution defined by $\mathcal{C} = \{X \in TM \,|\, X \perp \xi\}$. Then in this paper we give a non-existence result for real hypersurfaces in Q^m, $m \geq 3$, with $\mathcal{C}$-parallel structure Jacobi operator as follows:

Theorem 1.3. *There does not exist a Hopf real hypersurface with $\mathcal{C}$-parallel structure Jacobi operator in the complex quadric Q^m for $m \geq 3$.*

As a corollary of Theorem 1.3, we want to introduce the following due to Suh [29].

Corollary A. *There does not exist a Hopf real hypersurface in Q^m, $m \geq 3$, with parallel structure Jacobi operator.*

2. The complex quadric

For more background to this section we refer to [10], [12], [14], [21], [25], [26], [28], and [30]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z, that is, $[z] = \mathbb{C}z = \{\lambda z \,|\, \lambda \in S^1 \subset \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. For each $[z] \in Q^m \subset \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [12]). The tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the

Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector ρ of Q^m at a point $[z] \in Q^m$ we denote by $A = A_\rho$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ρ. The shape operator is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the (-1)-eigenspace of A_ρ. Geometrically this means that the shape operator A_ρ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations. There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A)$ in $T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ & - 2g(JX,Y)JZ + g(AY,Z)AX \\ & - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned} \tag{2.1}$$

It is well known that for every unit tangent vector $U \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors Z_1, $Z_2 \in V(A)$ such that

$$U = \cos(t)Z_1 + \sin(t)JZ_2$$

for some $t \in [0, \pi/4]$ (see [21]). The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$, then the unique maximal flat containing U is $\mathbb{R}Z_1 \oplus \mathbb{R}JZ_2$.

3. Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. By using the Gauss and Weingarten formulas the left-hand side of (2.1) becomes

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - g(SY,Z)SX + g(SX,Z)SY \\ &\quad + \big\{g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z)\big\}N,\end{aligned}$$

where R and S denote the Riemannian curvature tensor and the shape operator of M in Q^m, respectively. Taking tangent and normal components of (2.1) respectively, we obtain

$$\begin{aligned}g(R(X,Y)Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) \\ &\quad - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W) \\ &\quad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \\ &\quad + g(JAY,Z)g(JAX,W) - g(JAX,Z)g(JAY,W) \\ &\quad + g(SY,Z)g(SX,W) - g(SX,Z)g(SY,W)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z) \\ &= \eta(X)g(JY,Z) - \eta(Y)g(JX,Z) - 2\eta(Z)g(JX,Y) \\ &\quad + g(AY,Z)g(AX,N) - g(AX,Z)g(AY,N) \\ &\quad + \eta(AX)g(JAY,Z) - \eta(AY)g(JAX,Z),\end{aligned} \tag{3.2}$$

where X, Y, Z and W are tangent vector fields to M.

Note that $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where ϕX is the tangential component of JX and N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$. Moreover, since the complex quadric Q^m has also a real structure A, we decompose AX into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$:

$$AX = BX + \rho(X)N, \tag{3.3}$$

where BX denotes the tangential component of AX and

$$\rho(X) = g(AX,N) = g(X,AN) = g(X,AJ\xi) = g(JX,A\xi).$$

From these notations, the equations (3.1) and (3.2) can be written as

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\quad - 2g(\phi X,Y)\phi Z + g(BY,Z)BX - g(BX,Z)BY \\ &\quad + g(\phi BY,Z)\phi BX + g(\phi A\xi,Y)\eta(Z)\phi BX \\ &\quad - g(\phi BY,Z)\rho(X)\xi - g(\phi A\xi,Y)\eta(Z)\rho(X)\xi \\ &\quad - g(\phi BX,Z)\phi BY - g(\phi A\xi,X)\eta(Z)\phi BY \\ &\quad + g(\phi BX,Z)\rho(Y)\xi + g(\phi A\xi,X)\eta(Z)\rho(Y)\xi \\ &\quad + g(SY,Z)SX - g(SX,Z)SY\end{aligned}$$

and

$$
\begin{aligned}
&(\nabla_X S)Y - (\nabla_Y S)X \\
&= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + g(AX, N)BY - g(AY, N)BX \\
&\quad + g(A\xi, X)\phi BY - g(A\xi, X)\rho(Y)\xi - g(A\xi, Y)\phi BX + g(A\xi, Y)\rho(X)\xi,
\end{aligned}
$$

which are called the equations of Gauss and Codazzi, respectively. Moreover, from (3.1) the Ricci tensor $\mathfrak{Ric}$ of M is given by

$$
\begin{aligned}
\mathfrak{Ric}X = (2m-1)X - 3\eta(X)\xi + g(A\xi, \xi)BX - g(AX, N)\phi A\xi \\
+ g(AX, \xi)A\xi + hSX - S^2X,
\end{aligned} \tag{3.4}
$$

where $h = \mathrm{Tr}S$ denotes the trace of the shape operator S of M in Q^m.

As mentioned in Section 2, since the normal vector field N belongs to $T_{[z]}Q^m$, $[z] \in M$, we can choose $A \in \mathfrak{A}_{[z]}$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [21]). Note that t is a function on M. If $t = 0$, then $N = Z_1 \in V(A)$, therefore we see that N becomes an $\mathfrak{A}$-principal tangent vector field. On the other hand, if $t = \frac{\pi}{4}$, then $N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$. That is, N is an $\mathfrak{A}$-isotropic tangent vector field of Q^m. In addition, since $\xi = -JN$, we have

$$
\begin{cases}
\xi = \sin(t)Z_2 - \cos(t)JZ_1, \\
AN = \cos(t)Z_1 - \sin(t)JZ_2, \\
A\xi = \sin(t)Z_2 + \cos(t)JZ_1.
\end{cases} \tag{3.5}
$$

This implies $g(\xi, AN) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on M. At each point $[z] \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of $T_{[z]}M$, $[z] \in M$ as follows:

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}.$$

It is known that if $N_{[z]}$ is $\mathfrak{A}$-principal, then $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$ (see [25]).

We now assume that M is a Hopf hypersurface in the complex quadric Q^m. Then the shape operator S of M in Q^m satisfies $S\xi = \alpha\xi$ with the Reeb function $\alpha = g(S\xi, \xi)$ on M. By virtue of the Codazzi equation, we obtain the following lemma.

Lemma A ([29]). *Let M be a Hopf hypersurface in Q^m for $m \ge 3$. Then we obtain*

$$X\alpha = (\xi\alpha)\eta(X) + 2g(A\xi, \xi)g(X, AN) \tag{3.6}$$

and

$$
\begin{aligned}
&2S\phi SX - \alpha\phi SX - \alpha S\phi X - 2\phi X - 2g(X, \phi A\xi)A\xi \\
&+ 2g(X, A\xi)\phi A\xi + 2g(X, \phi A\xi)g(\xi, A\xi)\xi - 2g(\xi, A\xi)\eta(X)\phi A\xi = 0
\end{aligned} \tag{3.7}
$$

for any tangent vector fields X and Y on M.

Remark 3.1. From (3.6) we know that *if M has vanishing geodesic Reeb flow* (*or constant Reeb curvature*)*, then the normal vector N is singular.* In fact, under this assumption (3.6) becomes $g(A\xi,\xi)g(X,AN)=0$ for any tangent vector field X on M. Since $g(A\xi,\xi)=-\cos(2t)$, the case of $g(A\xi,\xi)=0$ implies that N is $\mathfrak{A}$-isotropic. Besides, if $g(A\xi,\xi)\neq 0$, that is, $g(AN,X)=0$ for all $X\in TM$, then

$$AN=\sum_{i=1}^{2m} g(AN,e_i)e_i+g(AN,N)N=g(AN,N)N$$

for any basis $\{e_1,e_2,\dots,e_{2m-1},e_{2m}:=N\}$ of $T_{[z]}Q^m$, $[z]\in Q^m$. Applying the real structure A to the above formula and using the property $A^2=I$, we get $N=A^2N=g(AN,N)AN$. Taking the inner product of the above equation with the unit normal N, it follows that $g(AN,N)=\pm 1$. Since $g(AN,N)=\cos(2t)$ where $t\in[0,\frac{\pi}{4})$, we obtain $AN=N$. Hence N should be $\mathfrak{A}$-principal.

Lemma B ([25])**.** *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then α is constant. Moreover, if $X\in\mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda\neq\alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Lemma C ([25])**.** *Let M be a Hopf hypersurface in Q^m, $m\geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then α is constant.*

If the normal vector N is $\mathfrak{A}$-isotropic, then we obtain

$$g(A\xi,N)=g(A\xi,\xi)=g(AN,N)=0$$

from (3.5). Taking the covariant derivative of $g(AN,N)=0$ along the direction of any $X\in T_{[z]}M$, $[z]\in M$, we obtain

$$\begin{aligned}0=X\big(g(AN,N)\big)&=g\big(\bar\nabla_X(AN),N\big)+g\big(AN,\bar\nabla_XN\big)\\&=g\big((\bar\nabla_XA)N+A(\bar\nabla_XN),N\big)+g\big(AN,\bar\nabla_XN\big)\\&=g\big(q(X)JAN-ASX,N\big)-g\big(AN,SX\big)\\&=-2g\big(ASX,N\big),\end{aligned}$$

where we have used the covariant derivative of the complex structure A, that is, $(\bar\nabla_XA)Y=q(X)JAY$ and the formula of Weingarten. Then the above formula gives $SAN=0$, because AN becomes a tangent vector field on M for an $\mathfrak{A}$-isotropic unit normal vector field N.

On the other hand, by differentiating $g(A\xi,N)=0$ and using the formula of Gauss, we have:

$$\begin{aligned}0&=g\big(\bar\nabla_X(A\xi),N\big)+g\big(A\xi,\bar\nabla_XN\big)\\&=g\big((\bar\nabla_XA)\xi+A(\bar\nabla_X\xi),N\big)+g\big(A\xi,\bar\nabla_XN\big)\\&=g\big((\bar\nabla_XA)\xi,N\big)+g\big(\nabla_X\xi+\sigma(X,\xi),AN\big)+g\big(A\xi,\bar\nabla_XN\big)\end{aligned}$$

$$= g\big(q(X)JA\xi, N\big) + g\big(\phi SX + g(SX,\xi)N, AN\big) - g\big(A\xi, SX\big)$$
$$= -2g(A\xi, SX),$$

where σ is the second fundamental form of M and $\phi AN = JAN = -AJN = A\xi$. Since $g(A\xi, N) = 0$, the vector field $A\xi$ becomes a tangent vector field on M when the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the above formula gives $SA\xi = 0$.

Moreover, if the normal vector N is $\mathfrak{A}$-isotropic, then the tangent vector space $T_{[z]}M$, $[z] \in M$, is decomposed as

$$T_{[z]}M = [\xi] \oplus \mathrm{Span}[A\xi, AN] \oplus \mathcal{Q},$$

where $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^{\perp} = \mathrm{Span}[A\xi, AN]$. From the equation (3.7), we obtain:

$$(2\lambda - \alpha)S\phi X = (\alpha\lambda + 2)\phi X$$

for some principal curvature vector $X \in \mathcal{Q} \subset T_{[z]}M$ such that $SX = \lambda X$. If $2\lambda - \alpha = 0$ (i.e., $\lambda = \frac{\alpha}{2}$), then $\alpha\lambda + 2 = \frac{\alpha^2+4}{2} = 0$, which makes a contradiction. Hence we obtain:

Lemma 3.2. *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-isotropic. Then $SA\xi = 0$ and $SAN = 0$. Moreover, if $X \in \mathcal{Q}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and its corresponding vector ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

On the other hand, from the property of $g(A\xi, N) = 0$ on a real hypersurface M in Q^m we see that the non-zero vector field $A\xi$ is tangent to M. Hence by Gauss formula it yields

$$\nabla_X(A\xi) = \bar{\nabla}_X(A\xi) - \sigma(X, A\xi)$$
$$= q(X)JA\xi + A(\nabla_X\xi) + g(SX,\xi)AN - g(SX, A\xi)N$$

for any $X \in TM$. By using $AN = AJ\xi = -JA\xi$ and $JA\xi = \phi A\xi + \eta(A\xi)N$, the tangential part and normal part of this formula give us, respectively,

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX,\xi)\phi A\xi \tag{3.8}$$

and

$$\begin{aligned} q(X)g(A\xi,\xi) &= -g(AN, \nabla_X\xi) + g(SX,\xi)g(A\xi,\xi) + g(SX, A\xi) \\ &= 2g(SX, A\xi). \end{aligned} \tag{3.9}$$

In particular, if M is Hopf, then (3.9) becomes

$$q(\xi)g(A\xi,\xi) = 2\alpha g(A\xi,\xi). \tag{3.10}$$

Now, the fact that a real hypersurface M has $\mathfrak{A}$-principal normal vector field N in Q^m implies $A\xi = -\xi$ and $AN = N$. Therefore, we obtain:

Lemma 3.3. *Let M be a real hypersurface with $\mathfrak{A}$-principal normal vector field N in the complex quadric Q^m for $m \geq 3$. Then we obtain:*

(a) $AX = BX$,
(b) $A\phi X = -\phi AX$,
(c) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX, \xi)$,
(d) $ASX = SX - 2g(SX, \xi)\xi$ *and* $SAX = SX - 2\eta(X)S\xi$

for all $X \in T_{[z]}M$, $[z] \in M$.

Proof. Since the normal vector field N is $\mathfrak{A}$-principal, we see that $AN = N$. By virtue of the symmetric property of the real structure A, it yields that for any $X \in TM$

$$g(AX, N) = g(X, AN) = g(X, N) = 0.$$

From this and (3.3), we can assert that $AX = BX$ for any tangent vector field X of M. In addition, on the complex quadric Q^m the complex structure J anti-commutes with the real structure A. From this property and the formula $JX = \phi X + \eta(X)N$, we obtain (b).

By using the assumption of $\mathfrak{A}$-principal normal N, (3.9) gives us $q(X) = 2g(SX, \xi)$. In addition, from (a) in Lemma 3.3 and $A\xi = -\xi$, (3.8) can be arranged as

$$\nabla_X(A\xi) = B\phi SX = A\phi SX.$$

Therefore, differentiating the equation $A\xi = -\xi$ with respect to the Levi-Civita connection ∇ of M, we get

$$-\phi SX = A\phi SX.$$

Finally, taking the covariant derivatives of $AN = N$ with respect to the Levi-Civita connection $\bar{\nabla}$ of Q^m and using the formulas $(\bar{\nabla}_U A)V = q(U)JAV$ for any U, $V \in TQ^m$, it follows

$$\begin{aligned} -q(X)\xi - ASX &= q(X)JAN - ASX \\ &= (\bar{\nabla}_X A)N + A(\bar{\nabla}_X N) \\ &= \bar{\nabla}_X N = -SX, \end{aligned}$$

where we have used the Weingarten formula, $\bar{\nabla}_X N = -SX$ and $JN = -\xi$. By virtue of (c) in Lemma 3.3, that is, $q(X) = 2g(SX, \xi)$, we consequently obtain $ASX = SX - 2g(SX, \xi)\xi$. Since the shape operator S of M and the real structure A are both symmetric, we also obtain $SAX = SX - 2\eta(X)S\xi$. $\square$

Remark 3.4. If M is Hopf in Q^m, the statements (c) and (d) in Lemma 3.3 can be rewritten as follows.

(c′) $A\phi SX = -\phi SX$ and $q(X) = 0$,
(d′) $ASX = SX = SAX$

for any $X \in \mathcal{C} = \{X \in TM \mid X \perp \xi\}$.

From now on, let us consider the Hessian tensor of the Reeb curvature function $\alpha = g(S\xi, \xi)$ which is defined by

$$(\mathrm{Hess}\,\alpha)(X, Y) = g(\nabla_X \mathrm{grad}\,\alpha, Y)$$

for any X and Y tangent to M. Then, it satisfies

$$(\mathrm{Hess}\,\alpha)(X,Y) = (\mathrm{Hess}\,\alpha)(Y,X),$$

that is, $g(\nabla_X \mathrm{grad}\alpha, Y) = g(\nabla_Y \mathrm{grad}\alpha, X)$. From this property we obtain the following lemma which plays a key role in the proof of our Theorem 1.3.

Lemma 3.5. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then we obtain*:

$$\begin{aligned}
&- 2\beta g(SA\xi, X)\eta(Y) + 2\alpha\beta g(A\xi, X)\eta(Y) + (\xi\alpha)g(\phi SX, Y)\\
&\quad + 4g(S\phi A\xi, X)g(\phi A\xi, Y) + 4g(SA\xi, X)g(A\xi, Y) - 2\beta g(BSX, Y)\\
&= -2\beta g(SA\xi, Y)\eta(X) + 2\alpha\beta g(A\xi, Y)\eta(X) + (\xi\alpha)g(\phi SY, X)\\
&\quad + 4g(S\phi A\xi, Y)g(\phi A\xi, X) + 4g(SA\xi, Y)g(A\xi, X) - 2\beta g(BSY, X),
\end{aligned} \tag{3.11}$$

where $\beta = g(A\xi, \xi)$ and X, $Y \in TM$.

Proof. From (3.6) the gradient of the Reeb curvature function α is given by

$$\mathrm{grad}\alpha = (\xi\alpha)\xi - 2\beta\phi A\xi$$

together with $AN = -\phi A\xi - g(A\xi, \xi)N$ and $\beta = g(A\xi, \xi)$. Taking the covariant derivative of gradα and using the formula $\nabla_X \xi = \phi SX$, it follows

$$\begin{aligned}
\nabla_X \mathrm{grad}\alpha &= X(\xi\alpha)\xi + (\xi\alpha)\nabla_X\xi - 2(X\beta)\phi A\xi - 2\beta(\nabla_X\phi)A\xi - 2\beta\phi(\nabla_X A\xi)\\
&= X(\xi\alpha)\xi + (\xi\alpha)\phi SX - 2(X\beta)\phi A\xi - 2\beta(\nabla_X\phi)A\xi - 2\beta\phi(\nabla_X A\xi)
\end{aligned} \tag{3.12}$$

for any tangent vector field X to M. By using $(\nabla_X\phi)Y = \eta(Y)SX - g(SX, Y)\xi$ and $A\xi \in T_{[z]}M$ for any $[z] \in M$, we get

$$- 2\beta(\nabla_X\phi)A\xi = -2\beta^2 SX + 2\beta g(SA\xi, X)\xi. \tag{3.13}$$

In addition, by using (3.8) and the property of $\phi^2 X = -X + \eta(X)\xi$, we obtain:

$$\begin{aligned}
\phi(\nabla_X A\xi) &= q(X)\phi^2 A\xi + \phi B\phi SX - g(SX, \xi)\phi^2 A\xi\\
&= -q(X)A\xi + \beta q(X)\xi + \phi B\phi SX + \alpha\eta(X)A\xi - \alpha\beta\eta(X)\xi.
\end{aligned} \tag{3.14}$$

On the other hand, from the anti-commuting property of $JA = -AJ$ we get

$$\phi BX + g(X, \phi A\xi)\xi = -B\phi X + \eta(X)\phi A\xi \tag{3.15}$$

for any $X \in T_{[z]}M$, $[z] \in M$. By virtue of this formula, we have

$$\begin{aligned}
\phi B\phi SX &= -B\phi^2 SX + \eta(\phi SX)\phi A\xi - g(\phi SX, \phi A\xi)\xi\\
&= BSX - \alpha\eta(X)B\xi - g(SA\xi, X)\xi + \alpha\beta\eta(X)\xi.
\end{aligned}$$

From this and (3.9), the equation (3.14) yields

$$\begin{aligned}
-2\beta\phi(\nabla_X A\xi) &= 2\beta q(X)A\xi - 2\beta^2 q(X)\xi - 2\beta\phi B\phi SX\\
&\quad - 2\alpha\beta\eta(X)A\xi + 2\alpha\beta^2\eta(X)\xi\\
&= 2\beta q(X)A\xi - 2\beta^2 q(X)\xi - 2\beta BSX + 2\beta g(SA\xi, X)\xi\\
&= 4g(SA\xi, X)A\xi - 4\beta g(SA\xi, X)\xi - 2\beta BSX + 2\beta g(SA\xi, X)\xi
\end{aligned} \tag{3.16}$$

$$= 4g(SA\xi, X)A\xi - 2\beta g(SA\xi, X)\xi - 2\beta BSX.$$

Substituting (3.13) and (3.15) into (3.12), we get

$$\begin{aligned}\nabla_X \mathrm{grad}\alpha &= X(\xi\alpha)\xi + (\xi\alpha)\phi SX - 2(X\beta)\phi A\xi \\ &\quad - 2\beta^2 SX + 4g(SA\xi, X)A\xi - 2\beta BSX\end{aligned}$$

for any $X \in T_{[z]}M$, $[z] \in M$. Thus, the property of $g(\nabla_X \mathrm{grad}\alpha, Y) = g(\nabla_Y \mathrm{grad}\alpha, X)$ gives us

$$\begin{aligned}&X(\xi\alpha)\eta(Y) + (\xi\alpha)g(\phi SX, Y) - 2(X\beta)g(\phi A\xi, Y) \\ &\quad + 4g(SA\xi, X)g(A\xi, Y) - 2\beta g(BSX, Y) \\ &= Y(\xi\alpha)\eta(X) + (\xi\alpha)g(\phi SY, X) - 2(Y\beta)g(\phi A\xi, X) \\ &\quad + 4g(SA\xi, Y)g(A\xi, X) - 2\beta g(BSY, X)\end{aligned} \tag{3.17}$$

for any tangent vector fields X and Y.

Now, since M is Hopf, the equation (3.8) leads to

$$\begin{aligned}Y\beta &= \nabla_Y\big(g(A\xi, \xi)\big) \\ &= g(\nabla_Y A\xi, \xi) + g(A\xi, \nabla_Y \xi) \\ &= q(Y)g(\phi A\xi, \xi) + g(B\phi SY, \xi) - \alpha\eta(Y)g(\phi A\xi, \xi) + g(A\xi, \phi SY) \\ &= -2g(S\phi A\xi, Y).\end{aligned} \tag{3.18}$$

Furthermore $\xi\beta = 0$. From this and putting $Y = \xi$ in (3.17), it follows

$$\begin{aligned}X(\xi\alpha) &= -2\beta g(SA\xi, X) + \xi(\xi\alpha)\eta(X) - 2(\xi\beta)g(\phi A\xi, X) + 2\alpha\beta g(A\xi, X) \\ &= -2\beta g(SA\xi, X) + \xi(\xi\alpha)\eta(X) + 2\alpha\beta g(A\xi, X).\end{aligned} \tag{3.19}$$

Summing up (3.18) and (3.19) and bearing in mind (3.17), we get a complete proof of our lemma. $\square$

4. Proof of Theorem 1.1
- Reeb parallel structure Jacobi operator with $\mathfrak{A}$-principal normal vector field -

Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with Reeb parallel structure Jacobi operator, that is,

$$(\nabla_\xi R_\xi)Y = 0 \tag{$*$}$$

for any tangent vector field Y on M.

As mentioned in Section 1, the structure Jacobi operator $R_\xi \in \mathrm{End}(TM)$ is induced from the curvature tensor R of M in Q^m as follows: For any tangent vector fields Y, $Z \in TM$

$$\begin{aligned}g(R_\xi Y, Z) &= g(R(Y, \xi)\xi, Z) \\ &= g(Y, Z) - \eta(Y)\eta(Z) + g(A\xi, \xi)g(AY, Z) \\ &\quad - g(Y, A\xi)g(A\xi, Z) - g(AY, N)g(AN, Z) \\ &\quad + \alpha g(SY, Z) - \alpha^2\eta(Y)\eta(Z),\end{aligned} \tag{4.1}$$

where we have used $J\xi = N$, $JA = -AJ$, and $g(A\xi, N) = 0$.

Remark 4.1. For any tangent vector field X on M the vector field AX belongs to TQ^m, that is, $AX = BX + \rho(X)N \in TM \oplus (TM)^\perp = TQ^m$. Therefore, from (4.1) the structure Jacobi operator R_ξ is given by

$$\begin{aligned} R_\xi Y = Y - \eta(Y)\xi + g(A\xi, \xi)BY - g(A\xi, Y)A\xi \\ - g(\phi A\xi, Y)\phi A\xi + \alpha SY - \alpha^2\eta(Y)\xi. \end{aligned} \tag{4.2}$$

Here we have used that $A\xi = B\xi \in TM$ (i.e., $\rho(\xi) = g(AN, \xi) = 0$) and $AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N$.

Taking the covariant derivative of (4.2) along the direction of $X \in TM$ we have

$$\begin{aligned}
&(\nabla_X R_\xi)Y \\
= {}& - g(Y, \nabla_X\xi)\xi - \eta(Y)\nabla_X\xi + g\big(\nabla_X(A\xi), \xi\big)BY + g(A\xi, \nabla_X\xi)BY \\
&+ g(A\xi, \xi)(\nabla_X B)Y - g\big(\nabla_X(A\xi), Y\big)A\xi - g(A\xi, Y)\nabla_X(A\xi) \\
&- g\big((\nabla_X\phi)A\xi, Y\big)\phi A\xi + g\big(\nabla_X(A\xi), \phi Y\big)\phi A\xi \\
&- g(\phi A\xi, Y)(\nabla_X\phi)A\xi - g(\phi A\xi, Y)\phi\big(\nabla_X(A\xi)\big) \\
&+ (X\alpha)SY + \alpha(\nabla_X S)Y - 2\alpha(X\alpha)\eta(Y)\xi \\
&- \alpha^2 g(Y, \nabla_X\xi)\xi - \alpha^2\eta(Y)\nabla_X\xi \\
= {}& - g(Y, \phi SX)\xi - \eta(Y)\phi SX + g(B\phi SX, \xi)BY + g(A\xi, \phi SX)BY \\
&+ g(A\xi, \xi)\Big\{q(X)\phi BY - q(X)g(AN, Y)\xi - g(SX, Y)\phi A\xi + g(AN, Y)SX\Big\} \\
&- \Big\{\big(q(X) - \alpha\eta(X)\big)g(\phi A\xi, Y) + g(B\phi SX, Y)\Big\}A\xi \\
&- g(A\xi, Y)\Big\{\big(q(X) - \alpha\eta(X)\big)\phi A\xi + B\phi SX\Big\} \\
&- \Big\{g(A\xi, \xi)g(SX, Y) - g(SX, A\xi)\eta(Y)\Big\}\phi A\xi \\
&+ \big(q(X) - \alpha\eta(X)\big)g(A\xi, Y)\phi A\xi \\
&- \Big\{\big(q(X) - \alpha\eta(X)\big)g(A\xi, \xi)\eta(Y) - g(B\phi SX, \phi Y)\Big\}\phi A\xi \\
&- g(\phi A\xi, Y)\Big\{g(A\xi, \xi)SX - g(SX, A\xi)\xi\Big\} \\
&+ g(\phi A\xi, Y)\Big\{\big(q(X) - \alpha\eta(X)\big)A\xi - g(A\xi, \xi)\big(q(X) - \alpha\eta(X)\big)\xi - \phi B\phi SX\Big\} \\
&+ (X\alpha)SY + \alpha(\nabla_X S)Y - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \phi SX)\xi - \alpha^2\eta(Y)\phi SX,
\end{aligned} \tag{4.3}$$

where we have used (3.8) and

$$\begin{aligned}
(\nabla_X B)Y &= \nabla_X(BY) - B(\nabla_X Y) \\
&= \bar\nabla_X(BY) - \sigma(X, BY) - B(\nabla_X Y) \\
&= \bar\nabla_X\big(AY - g(AY, N)N\big) - g(SX, BY)N - B(\nabla_X Y) \\
&= (\bar\nabla_X A)Y + A\big(\bar\nabla_X Y\big) - g\big(\bar\nabla_X(AY), N\big)N - g(AY, \bar\nabla_X N)N
\end{aligned}$$

$$
\begin{aligned}
&\quad - g(AY,N)\bar{\nabla}_X N - g(SX,BY)N - B(\nabla_X Y)\\
&= q(X)JAY + A(\nabla_X Y) + g(SX,Y)AN - q(X)g(JAY,N)N\\
&\quad - g(\nabla_X Y, AN)N - g(SX,Y)g(AN,N)N\\
&\quad + g(AY,SX)N + g(AY,N)SX - g(SX,BY)N - B(\nabla_X Y)\\
&= q(X)JAY + g(SX,Y)AN - q(X)g(AY,\xi)N\\
&\quad + g(SX,Y)g(A\xi,\xi)N + g(AY,N)SX\\
&= q(X)\{\phi BY - g(AY,N)\xi\} - g(SX,Y)\phi A\xi + g(AY,N)SX.
\end{aligned}
$$

Since M is a Hopf real hypersurface in Q^m with Reeb parallel structure Jacobi operator, it yields

$$
\begin{aligned}
&g(A\xi,\xi)\Big\{q(\xi)\phi BY - q(\xi)g(AN,Y)\xi - \alpha\eta(Y)\phi A\xi + \alpha g(AN,Y)\xi\Big\}\\
&- \big(q(\xi)-\alpha\big)g(A\xi,\xi)\eta(Y)\phi A\xi - g(\phi A\xi,Y)g(\xi,A\xi)\big(q(\xi)-\alpha\big)\xi\\
&+ (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - 2\alpha(\xi\alpha)\eta(Y)\xi = 0.
\end{aligned} \tag{4.4}
$$

From now on, we assume that M is a real hypersurface with non-vanishing geodesic Reeb flow and Reeb parallel structure Jacobi operator in the complex quadric Q^m, $m \geq 3$. In addition, we suppose that the unit normal vector field N of M is $\mathfrak{A}$-principal. Then from (3.5) it follows that

$$AN = N \quad \text{and} \quad A\xi = -\xi.$$

So it implies that $AY \in TM$ for all $Y \in TM$, that is, $g(AY,N) = g(Y,AN) = 0$. Moreover, taking the derivative of $AN = N$ with respect to the Levi-Civita connection $\bar{\nabla}$ of Q^m and using (3.8), we get

$$ASY = SY - 2\alpha(Y)\xi, \tag{4.5}$$

together with $(\bar{\nabla}_Y A)X = q(Y)JAX$ and $\bar{\nabla}_Y N = -SY$.

From these properties, (4.4) can be rearranged as follows:

$$
\begin{aligned}
0 &= (\nabla_\xi R_\xi)Y\\
&= -q(\xi)JAY - q(\xi)\eta(Y)N + (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - 2\alpha(\xi\alpha)\eta(Y)\xi.
\end{aligned} \tag{4.6}
$$

In addition, from (3.10) we know $q(\xi) = 2\alpha$. By Lemma B in Section 3 and our assumption that M has non-vanishing geodesic Reeb flow, the Reeb curvature function α is a non-zero constant on M. So (4.6) reduces to the following

$$(\nabla_\xi S)Y = 2\phi AY, \tag{4.7}$$

together with $JAY = \phi AY + \eta(AY)N = \phi AY - \eta(Y)N$.

On the other hand, by using the equation of Codazzi (3.2), we have

$$
\begin{aligned}
g\big((\nabla_\xi S)Y - (\nabla_Y S)\xi, Z\big) &= g(\phi Y,Z) - g(AY,N)g(A\xi,Z)\\
&\quad + g(\xi,A\xi)g(JAY,Z) + g(\xi,AY)g(AN,Z)\\
&= g(\phi Y,Z) - g(\phi AY,Z).
\end{aligned}
$$

Since M is a Hopf real hypersurface in Q^m with $\mathfrak{A}$-principal normal vector field N, Lemma B in Section 3 gives

$$(\nabla_\xi S)Y = (\nabla_Y S)\xi + \phi Y - \phi AY = \alpha\phi SY - S\phi SY + \phi Y - \phi AY.$$

From this, together with (4.7), it follows that

$$\alpha\phi SY - S\phi SY + \phi Y = 3\phi AY. \tag{4.8}$$

From Lemma A, as N is $\mathfrak{A}$-principal we obtain:

$$2S\phi SY = \alpha(S\phi + \phi S)Y + 2\phi Y. \tag{4.9}$$

Therefore, (4.8) can be written as

$$\alpha(\phi S - S\phi)Y = 6\phi AY. \tag{4.10}$$

Inserting $Y = SX$ for $X \in \mathcal{C}$ into (4.10) and applying the structure tensor ϕ leads to

$$\alpha S^2 X + \alpha\phi S\phi SX = 6ASX,$$

where $\mathcal{C} = \ker\eta$ denotes the maximal complex subbundle of TM. From this, together with (4.5) and (4.9), it follows that

$$\alpha^2\phi S\phi X = -2\alpha S^2 X + \alpha^2 SX + 2\alpha X + 12SX \tag{4.11}$$

for any $X \in \mathcal{C}$.

In this section, we have assumed that the normal vector field N of M is $\mathfrak{A}$-principal. So, it follows that $AY \in TM$ for all $Y \in TM$. From this, the anti-commuting property between J and A implies $\phi AX = -A\phi X$. Hence (4.10) can be expressed as

$$\alpha(\phi S - S\phi)Y = -6A\phi Y. \tag{4.12}$$

Putting $Y = \phi X$ into (4.12), it follows

$$\alpha\phi S\phi X = -\alpha SX + 6AX$$

for all $X \in \mathcal{C}$. Inserting this into (4.11) gives

$$3\alpha AX + \alpha S^2 X - \alpha^2 SX - \alpha X - 6SX = 0. \tag{4.13}$$

Applying the real structure A to (4.13) and using (4.5) again, we get

$$3\alpha X + \alpha S^2 X - \alpha^2 SX - \alpha AX - 6SX = 0 \tag{4.14}$$

for any $X \in \mathcal{C}$. Summing up (4.13) and (4.14), we obtain $AX = X$ for all $X \in \mathcal{C}$. This gives a contradiction. In fact, it is well known that the trace of the real structure A vanishes, that is, $\mathrm{Tr}A = 0$ (see Lemma 1 in [24]). For an orthonormal basis $\{ e_1, e_2, \ldots, e_{2m-2}, e_{2m-1} = \xi, e_{2m} = N \}$ for TQ^m, where $e_j \in \mathcal{C}$ $(j = 1, 2, \ldots, 2m-2)$, the trace of A is given by

$$\begin{aligned} \mathrm{Tr}A &= \sum_{i=1}^{2m} g(Ae_i, e_i) \\ &= g(AN, N) + g(A\xi, \xi) + \sum_{i=1}^{2m-2} g(Ae_i, e_i) \\ &= 2m-2. \end{aligned}$$

It implies $m = 1$. But we have considered that $m \geq 3$.

Consequently, this completes the proof that *there does not exist a Hopf real hypersurface* $(\alpha \neq 0)$ *in the complex quadric* Q^m, $m \geq 3$, *with Reeb parallel structure Jacobi operator and* $\mathfrak{A}$*-principal normal vector field.*

5. Proof of Theorem 1.2 - Reeb parallel structure Jacobi operator with $\mathfrak{A}$-isotropic normal vector field -

In this section, we assume that the unit normal vector field N is $\mathfrak{A}$-isotropic and M is a Hopf real hypersurface in complex quadric Q^m with non-vanishing geodesic Reeb flow and Reeb parallel structure Jacobi operator. Then the normal vector field N can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for some orthonormal vectors Z_1, $Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \; AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2) \text{ and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

These formulas imply the following

$$g(\xi, A\xi) = g(JN, AJN) = 0, \; g(\xi, AN) = 0 \text{ and } g(AN, N) = 0,$$

which means that both vector fields AN and $A\xi$ are tangent to M. From this and Lemma C, the equation (4.4) gives us that the shape operator S of M becomes to be Reeb parallel, that is, $(\nabla_\xi S)Y = 0$ for all tangent vector field Y on M.

On the other hand, from the Codazzi equation (3.2) we obtain:

$$\begin{aligned} (\nabla_\xi S)Y &= (\nabla_Y S)\xi + \phi Y - g(AY, N)A\xi + g(A\xi, Y)AN \\ &= (Y\alpha)\xi + \alpha\phi SY - S\phi SY + \phi Y + g(A\xi, Y)AN - g(AN, Y)A\xi \\ &= \frac{\alpha}{2}(\phi S - S\phi)Y, \end{aligned}$$

where the third equality holds from Lemmas A and C. From this and as M has non-vanishing geodesic Reeb flow, we see that M has isometric Reeb flow, that is, $S\phi = \phi S$.

Consequently, we obtain:

Proposition 5.1. *Let M be a real hypersurface with non-vanishing geodesic Reeb flow in the complex quadric Q^m, $m \geq 3$. If the unit normal vector field N of M is $\mathfrak{A}$-isotropic and the structure Jacobi operator R_ξ of M is Reeb parallel, then the shape operator S of M is Reeb parallel. Moreover, it means that the Reeb flow on M is isometric.*

Then by virtue of Theorem A, we assert: *if M is a real hypersurface in the complex quadric Q^m, $m \geq 3$, with the assumptions given in Proposition 5.1, then M is locally congruent to an open part of a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$.*

From now on, let us check the converse problem, that is, such a tube satisfies all assumptions stated in Proposition 5.1. In order to do this, we first introduce a proposition given in [25].

Proposition A. *Let $(\mathcal{T}_A)$ be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$ in Q^{2k}. Then the following statements hold:*

(i) *$(\mathcal{T}_A)$ is a Hopf hypersurface.*
(ii) *Every unit normal vector N of a real hypersurfaces of type $(\mathcal{T}_A)$ is $\mathfrak{A}$-isotropic and therefore can be written in the form $N = (Z_1 + JZ_2)/\sqrt{2}$ with some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $A \in \mathfrak{A}$.*
(iii) *$(\mathcal{T}_A)$ has four distinct constant principal curvatures and the property that the shape operator leaves invariant the maximal complex subbundle $\mathcal{C}$ of $T(\mathcal{T}_A)$. The principal curvatures and corresponding principal curvature spaces of type $(\mathcal{T}_A)$ are as follows.*

principal curvature	eigenspace	multiplicity
$\alpha = -2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1
$\beta = 0$	$T_\beta = \mathbb{C}(JZ_1 + Z_2)$	2
$\lambda = \tan(r)$	$T_\lambda = T\mathbb{C}P^k \ominus \mathbb{C}(JZ_1 + Z_2)$	$2k - 2$
$\mu = -\cot(r)$	$T_\mu = \nu\mathbb{C}P^k \ominus \mathbb{C}N$	$2k - 2$

Here, $T\mathbb{C}P^k$ and $\nu\mathbb{C}P^k$ denote the tangent and normal bundles of $\mathbb{C}P^k$, respectively. Moreover, we have $A(T\mathbb{C}P^k \ominus \mathbb{C}(JZ_1 + Z_2)) = \nu\mathbb{C}P^k \ominus \mathbb{C}N$.
(iv) *Each of the two focal sets of $(\mathcal{T}_A)$ is a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*
(v) *$S\phi = \phi S$ (isometric Reeb flow).*
(vi) *$(\mathcal{T}_A)$ is a homogeneous hypersurface of Q^{2k}. More precisely, it is an orbit of the U_{k+1}-action on Q^{2k} isomorphic to $U_{k+1}/U_{k-1}U_1$, an S^{2k-1}-bundle over $\mathbb{C}P^k$.*

By virtue of (i) and (ii) in Proposition A, $(\mathcal{T}_A)$ is a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field N in Q^m. Moreover, the structure Jacobi operator R_ξ of a type $(\mathcal{T}_A)$ real hypersurface should be Reeb parallel. In fact, from (4.2) its structure Jacobi operator is given as follows.

$$R_\xi Y = \begin{cases} 0 & \text{if } \ Y \in T_\alpha \oplus T_\beta, \\ (1+\lambda)Y & \text{if } \ Y \in T_\lambda, \\ (1+\mu)Y & \text{if } \ Y \in T_\mu. \end{cases}$$

On the other hand, from (4.3) and the equation of Codazzi (3.2), the covariant derivative of R_ξ along the Reeb direction becomes

$$\begin{aligned}(\nabla_\xi R_\xi)Y &= \alpha(\nabla_\xi S)Y \\ &= \alpha\big\{(\nabla_Y S)\xi + \phi Y - g(AN,Y)A\xi + g(A\xi,Y)AN\big\}.\end{aligned}$$

Since $(\mathcal{T}_A)$ is a Hopf real hypersurface with constant principal curvature α, it implies

$$(\nabla_\xi R_\xi)Y = \alpha\big\{\alpha\phi SY - S\phi SY + \phi Y - g(AN,Y)A\xi + g(A\xi,Y)AN\big\},$$

which vanishes identically on $(\mathcal{T}_A)$. That is, we can assert that *the structure Jacobi operator R_ξ of a type $(\mathcal{T}_A)$ real hypersurface is Reeb parallel.*

6. Proof of Theorem 1.3
- $\mathcal{C}$-parallel structure Jacobi operator -

In this section we define the new notion of $\mathcal{C}$-parallel structure Jacobi operator of a Hopf real hypersurface M in the complex quadric Q^m for $m \geq 3$ as follows: the structure Jacobi operator $R_\xi \in \mathrm{End}(TM)$ of M satisfies

$$(\nabla_X R_\xi)Y = 0 \tag{$**$}$$

for any $X \in \mathcal{C} = \{X \in TM \mid X \perp \xi\}$ and $Y \in TM$, then it is said to be *$\mathcal{C}$-parallel.* In fact, by virtue of (4.3) this condition is equivalent to

$$\begin{aligned}&-(1+\alpha^2)g(Y,\phi SX)\xi - \eta(Y)\phi SX + 2g(A\xi,\phi SX)BY \\ &+ 2g(SX,A\xi)\phi BY + 3g(SX,A\xi)g(\phi A\xi,Y)\xi \\ &+ 2g(SX,A\xi)g(\phi A\xi,Y)\xi - 2g(A\xi,\xi)g(SX,Y)\phi A\xi \\ &- g(SX,A\xi)\eta(Y)\phi A\xi + g(B\phi SX,\phi Y)\phi A\xi \\ &- 2g(A\xi,\xi)g(\phi A\xi,Y)SX - g(B\phi SX,Y)A\xi - \alpha^2\eta(Y)\phi SX \\ &- g(A\xi,Y)B\phi SX - g(\phi A\xi,Y)\phi B\phi SX + (X\alpha)SY \\ &+ \alpha(\nabla_X S)Y - 2\alpha(X\alpha)\eta(Y)\xi = 0\end{aligned} \tag{6.1}$$

for any $X \in \mathcal{C}$, $Y \in TM$. By using this equation, we obtain:

Proposition 6.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. If the structure Jacobi operator R_ξ of M is $\mathcal{C}$-parallel, then the unit normal vector field N is singular.*

Proof. From Remark 3.1 and the definition of $\mathfrak{A}$-isotropic normal vector field, if either the Reeb curvature function $\alpha = g(S\xi,\xi)$ or $\beta = g(A\xi,\xi)$ vanish, then the normal vector field N should be singular. So, hereafter let us assume that two functions α and β are non-vanishing.

Taking the inner product of (6.1) with ξ, we get

$$\begin{aligned} &g(S\phi Y, X) - 2g(A\xi, Y)g(S\phi A\xi, X) + 5g(\phi A\xi, Y)g(SA\xi, X) \\ &+ \beta g(S\phi BY, X) + g(A\xi, Y)g(S\phi A\xi, X) + \alpha g(S\phi SY, X) = 0 \end{aligned} \tag{6.2}$$

for any $X \in \mathcal{C}$ and $Y \in TM$. This equation yields

$$-\phi SX - g(S\phi A\xi, X)A\xi + 5g(SA\xi, X)\phi A\xi - \beta B\phi SX - \alpha S\phi SX = 0$$

for all $X \in \mathcal{C}$. Taking its inner product with $A\xi$ and using $BA\xi = \xi$, it gives us

$$\alpha g(S\phi SA\xi, X) = 0$$

for any $X \in \mathcal{C}$. Since $\alpha \neq 0$, we have $g(S\phi SA\xi, X) = 0$ for any $X \in \mathcal{C}$, which implies that

$$S\phi SA\xi = g(S\phi SA\xi, \xi)\xi = 0.$$

From this and taking $X = A\xi$ in (3.6), we get

$$\alpha S\phi A\xi = -\alpha\phi SA\xi - 2\beta^2\phi A\xi. \tag{6.3}$$

In addition, we put

$$\begin{aligned} W_Y = &S\phi Y - 2g(A\xi, Y)S\phi A\xi + 5g(\phi A\xi, Y)SA\xi \\ &+ \beta S\phi BY + g(A\xi, Y)S\phi A\xi + \alpha S\phi SY \end{aligned}$$

for any $Y \in TM$. Then, from (6.2) and using that M is Hopf, we see that

$$W_Y = g(W_Y, \xi)\xi = 5\alpha\beta g(\phi A\xi, Y)\xi.$$

That is,

$$\begin{aligned} 5\alpha\beta g(\phi A\xi, Y)\xi = &S\phi Y - 2g(A\xi, Y)S\phi A\xi + 5g(\phi A\xi, Y)SA\xi \\ &+ \beta S\phi BY + g(A\xi, Y)S\phi A\xi + \alpha S\phi SY \end{aligned} \tag{6.4}$$

for any $Y \in TM$. Taking ϕY instead of Y in (6.4) and using $\phi^2 Y = -Y + \eta(Y)\xi$, it yields

$$\begin{aligned} &5\alpha\beta g(A\xi, Y)\xi - 5\alpha\beta^2\eta(Y)\xi \\ = &-SY + \alpha\eta(Y)\xi + g(\phi A\xi, Y)S\phi A\xi + 5g(A\xi, Y)SA\xi - 5\beta\eta(Y)SA\xi \\ &+ \beta SBY - \alpha\beta g(A\xi, Y)\xi - \beta\eta(Y)SA\xi + \alpha\beta^2\eta(Y)\xi + \alpha S\phi S\phi Y \end{aligned} \tag{6.5}$$

for any $Y \in TM$. Putting $Y = A\xi$ in (6.5), we get

$$\alpha S\phi S\phi A\xi = 2(2 - 3\beta^2)\{\alpha\beta\xi - SA\xi\}. \tag{6.6}$$

On the other hand, putting $X = \phi A\xi$ in (3.7), it gives us

$$2S\phi S\phi A\xi = \alpha\phi S\phi A\xi - \alpha SA\xi + \alpha^2\beta\xi - 2\beta^2 A\xi + 2\beta^3\xi, \tag{6.7}$$

where we have used $AN = AJ\xi = -JA\xi = -\phi A\xi - \beta N$ and $g(\phi A\xi, \phi A\xi) = 1 - \beta^2$. From this and using (6.3), we obtain:

$$S\phi S\phi A\xi = 0.$$

Thus, (6.6) becomes

$$(2-3\beta^2)\{\alpha\beta\xi - SA\xi\} = 0.$$

From this, we have the following two cases.

- **Case I.** $\beta^2 = \frac{2}{3}$

Since $\beta \neq 0$, we see that $\beta = g(A\xi, \xi) = -\cos(2t)$, $t \in [0, \frac{\pi}{4})$. That is, the function β should be constant, therefore $Y\beta = 0$ for any $Y \in TM$. Thus from (3.18), we get $S\phi A\xi = 0$. Moreover, by using (6.7), we also obtain:

$$\alpha SA\xi = \beta(\alpha^2 + 2\beta^2)\xi - 2\beta^2 A\xi. \tag{6.8}$$

On the other hand, under our assumptions and using the fact that $S\phi A\xi = 0$, (3.11) can be rewritten as

$$\begin{aligned} &(\xi\alpha)g(\phi SY, X) + 4g(SA\xi, Y)g(A\xi, X) - 2\beta g(BSY, X) \\ = &-2\beta g(SA\xi, X)\eta(Y) + 2\alpha\beta g(A\xi, X)\eta(Y) + (\xi\alpha)g(\phi SX, Y) \\ &+ 4g(SA\xi, X)g(A\xi, Y) - 2\beta g(BSX, Y) \end{aligned}$$

for any $X \in \mathcal{C}$ and $Y \in TM$. Putting $X = \phi A\xi \in \mathcal{C}$ and $S\phi A\xi = 0$, it yields

$$(\xi\alpha)g(\phi SY, \phi A\xi) - 2\beta g(BSY, \phi A\xi) = 0. \tag{6.9}$$

From (3.15), we have $B\phi A\xi = \beta\phi A\xi$, which implies

$$g(BSY, \phi A\xi) = \beta g(SY, \phi A\xi) = 0.$$

Thus (6.9) becomes $(\xi\alpha)g(\phi SY, \phi A\xi) = 0$ for any tangent vector field Y on M. It implies that

$$(\xi\alpha)\{SA\xi - \alpha\beta\xi\} = 0.$$

$\circ$ Subcase I-1. $SA\xi = \alpha\beta\xi$

Since $\alpha \neq 0$, this assumption becomes $\alpha SA\xi = \alpha^2\beta\xi$. From this and (6.8), we get

$$A\xi = \beta\xi.$$

Taking the inner product with $A\xi$, it implies that $\beta^2 = 1$. It makes a contradiction with our condition $\beta^2 = \frac{2}{3}$.

$\circ$ Subcase I-2. $\xi\alpha = 0$

Under this assumption, (3.19) becomes

$$-2\beta(SA\xi, Y) + 2\alpha\beta g(A\xi, Y) = 0, \qquad \forall Y \in TM.$$

From this and $\beta \neq 0$, we get $SA\xi = \alpha A\xi$. Then, (6.8) gives

$$(\beta\alpha^2 + 2\beta^3)\xi - 2\beta^2 A\xi = \alpha^2 A\xi.$$

Taking the inner product with $A\xi$, it leads to $(\alpha^2 + 2\beta^2)(\beta^2 - 1) = 0$. But $\beta^2 = \frac{2}{3}$ and it makes a contradiction.

Summing up these two subcases, we can assert that the first case of $\beta^2 = \frac{2}{3}$ does not occur.

- **Case II.** $SA\xi = \alpha\beta\xi$

Under this assumption, the equation (3.11) in Lemma 3.5 becomes

$$\begin{aligned}(6.10)\qquad &2\alpha\beta g(A\xi, X)\eta(Y) + (\xi\alpha)g(\phi SX, Y)\\ &\quad + 4g(S\phi A\xi, X)g(\phi A\xi, Y) - 2\beta g(BSX, Y)\\ &= (\xi\alpha)g(\phi SY, X) + 4g(S\phi A\xi, Y)g(\phi A\xi, X)\\ &\quad + 4\alpha\beta\eta(Y)g(A\xi, X) - 2\beta g(BSY, X)\end{aligned}$$

for any $X \in \mathcal{C}$ and $Y \in TM$. Put

$$\begin{aligned}W_Y = {}& 2\alpha\beta\eta(Y)A\xi - (\xi\alpha)S\phi Y + 4g(\phi A\xi, Y)S\phi A\xi - 2\beta SBY\\ &- (\xi\alpha)\phi SY - 4g(S\phi A\xi, Y)\phi A\xi - 4\alpha\beta\eta(Y)A\xi + 2\beta BSY\end{aligned}$$

for any $Y \in TM$. From (6.10), we get $W_Y = g(W_Y, \xi)\xi = -2\alpha\beta g(A\xi, Y)\xi$, which is equivalent to

$$\begin{aligned}(6.11)\qquad &2\alpha\beta\eta(Y)A\xi - (\xi\alpha)S\phi Y + 4g(\phi A\xi, Y)S\phi A\xi - 2\beta SBY - (\xi\alpha)\phi SY\\ &- 4g(S\phi A\xi, Y)\phi A\xi - 4\alpha\beta\eta(Y)A\xi + 2\beta BSY = -2\alpha\beta g(A\xi, Y)\xi\end{aligned}$$

for any $Y \in TM$. Taking the inner product with $A\xi$ and bearing in mind that $\phi SA\xi = 0$ and $BA\xi = \xi$, it yields

$$(6.12)\qquad (\xi\alpha)g(S\phi A\xi, Y) = 0, \qquad \forall Y \in TM.$$

Besides, by virtue of (6.3), our condition, $SA\xi = \alpha\beta\xi$, gives us

$$\alpha S\phi A\xi = -2\beta^2\phi A\xi.$$

From this and $\alpha \neq 0$, the equation (6.12) leads to

$$(6.13)\qquad 0 = \alpha(\xi\alpha)g(S\phi A\xi, Y) = -2\beta^2(\xi\alpha)g(\phi A\xi, Y), \qquad \forall Y \in TM.$$

Substituting $Y = \phi A\xi$ in (6.13) and using $\beta \neq 0$, it becomes

$$(\xi\alpha)(1 - \beta^2) = 0.$$

∘ Subcase II-1. $\beta^2 = 1$

Since $\beta = g(A\xi, \xi) = -\cos 2t$, $t \in [0, \frac{\pi}{4})$, it means that $\beta = -1$ (i.e., $t = 0$). Then, from the definition of $\mathfrak{A}$-principal tangent vector field of Q^m, we see that the normal vector field N of M should be $\mathfrak{A}$-principal.

∘ Subcase II-2. $\xi\alpha = 0$

Since $SA\xi = \alpha\beta\xi$ and $\xi\alpha = 0$, (3.19) becomes

$$(6.14)\qquad 2\alpha\beta g(A\xi, X) = 0$$

for any $X \in \mathcal{C}$. Since $\alpha \neq 0$ and $\beta \neq 0$, it implies that $A\xi = g(A\xi, \xi)\xi = \beta\xi$. Taking its inner product with $A\xi$, we obtain $\beta^2 = 1$, which means that the normal vector field N is $\mathfrak{A}$-principal.

Summing up these all observations, we give a complete proof of our proposition. □

From now on, from Proposition 6.1, let us consider the case of $\mathfrak{A}$-principal normal vector field.

Lemma 6.2. *Let M be a Hopf real hypersurface with $\mathcal{C}$-parallel structure Jacobi operator in the complex quadric Q^m, $m \geq 3$. If the normal vector field N of M is $\mathfrak{A}$-principal, then M has isometric Reeb flow.*

Proof. As mentioned in Lemma B, if M has an $\mathfrak{A}$-principal normal vector field, then the Reeb curvature function α is constant. So, if a Hopf real hypersurface M has $\mathcal{C}$-parallel structure Jacobi operator and $\mathfrak{A}$-principal normal vector field N, then (6.1) becomes

$$\begin{aligned} &-(1+\alpha^2)g(Y,\phi SX)\xi - \eta(Y)\phi SX + g(B\phi SX, Y)\xi \\ &-\alpha^2\eta(Y)\phi SX + \eta(Y)B\phi SX + \alpha(\nabla_X S)Y = 0 \end{aligned} \tag{6.15}$$

for any $X \in \mathcal{C}$ and $Y \in TM$. Putting $Y = \xi$ in (6.15) and using the formulas (a) and (b) in Lemma 3.3, it follows

$$-2\phi SX - \alpha S\phi SX = 0$$

for any $X \in \mathcal{C}$. Since M is Hopf, we also obtain that $-2\phi S\xi - \alpha S\phi S\xi = 0$. Accordingly, we assert that

$$\alpha S\phi SY = -2\phi SY \tag{6.16}$$

for any $Y \in TM$. Taking the symmetric part of (6.16), we obtain

$$\alpha S\phi SY = -2S\phi Y \tag{6.17}$$

for any $Y \in TM$. From (6.16) and (6.17), we see that the shape operator S commutes with the structure tensor ϕ, that is, $S\phi Y = \phi SY$ for all $Y \in TM$. It means that M has isometric Reeb flow. □

From this lemma and Theorem A, we assert that *if a Hopf real hypersurface M in Q^m with $m \geq 3$ satisfies the conditions in Lemma 6.2, then M is locally congruent to a model space of type $(\mathcal{T}_A)$.* But by virtue of Proposition A in Section 5, a model space of type $(\mathcal{T}_A)$ has an $\mathfrak{A}$-isotropic normal vector field. So, we conclude:

Proposition 6.3. *There does not exist a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with $\mathcal{C}$-parallel structure Jacobi operator and $\mathfrak{A}$-principal unit normal vector field.*

In the remained part, let us consider that a Hopf real hypersurface M in Q^m with $\mathcal{C}$-parallel structure Jacobi operator has $\mathfrak{A}$-isotropic normal vector field N. Using the facts mentioned in Section 3, for example, Lemmas C and 3.2, the equation (6.1) becomes

$$\begin{aligned} &-(1+\alpha^2)g(Y,\phi SX)\xi - \eta(Y)\phi SX + g(B\phi SX,\phi Y)\phi A\xi \\ &-g(B\phi SX, Y)A\xi - \alpha^2\eta(Y)\phi SX - g(A\xi, Y)B\phi SX \\ &-g(\phi A\xi, Y)\phi B\phi SX + \alpha(\nabla_X S)Y = 0 \end{aligned} \tag{6.18}$$

for any $X \in \mathcal{C}$ and $Y \in TM$. By using this equation, we obtain:

Lemma 6.4. *Let M be a Hopf real hypersurface with $\mathcal{C}$-parallel structure Jacobi operator in the complex quadric Q^m, $m \geq 3$. If the normal vector field N of M is $\mathfrak{A}$-isotropic, then M has isometric Reeb flow.*

Proof. Taking the inner product of (6.18) with ξ and using that $(\nabla_Y S)\xi = (Y\alpha)\xi + \alpha\phi SY - S\phi SY$, then we get

$$g(Y, \phi SX) + \alpha g(S\phi SX, Y) = 0, \qquad \forall X \in \mathcal{C},\ Y \in TM.$$

From this, we obtain that

$$\alpha S\phi SY + S\phi Y = g(\alpha S\phi SY + S\phi Y, \xi)\xi = 0 \tag{6.19}$$

for any $Y \in TM$. Moreover, the skew-symmetric part of this equation becomes

$$\alpha S\phi SY + \phi SY = 0. \tag{6.20}$$

Subtracting (6.20) from (6.19), we have $S\phi Y - \phi SY = 0$. Hence we complete the proof of this lemma. □

Bearing in mind Theorem A, this lemma tells us that *if a Hopf real hypersurface M in Q^m, $m \geq 3$, satisfies the conditions in Lemma 6.4, then M is locally congruent to a model space of type $(\mathcal{T}_A)$*. Now, by using the information for $(\mathcal{T}_A)$ given in Proposition A let us check the converse statement, that is, when a model space of type $(\mathcal{T}_A)$ satisfies the conditions in Lemma 6.4. From Proposition A, we already know that $(\mathcal{T}_A)$ is a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field. So, now, we want to check whether $(\mathcal{T}_A)$ has $\mathcal{C}$-parallel structure Jacobi operator or not. In order to do this, we assume that the structure Jacobi operator R_ξ of type $(\mathcal{T}_A)$ is $\mathcal{C}$-parallel.

On the other hand, since the Reeb flow of a type $(\mathcal{T}_A)$ real hypersurface is isometric, we can use the equation for $(\nabla_X S)Y$ given by Berndt and Suh (see page 1350050-14 in [2]). Then the equation with respect to the $\mathcal{C}$-parallelism for the structure Jacobi operator R_ξ of type $(\mathcal{T}_A)$ real hypersurfaces becomes

$$\begin{aligned}
&-(1+\alpha^2)g(Y,\phi SX)\xi - \eta(Y)\phi SX + g(B\phi SX, \phi Y)\phi A\xi \\
&- g(B\phi SX, Y)A\xi - \alpha^2\eta(Y)\phi SX - g(A\xi, Y)B\phi SX \\
&- g(\phi A\xi, Y)\phi B\phi SX + \alpha^2 g(S\phi X, Y)\xi \\
&- \alpha g(S^2\phi X, Y)\xi + \alpha g(A\xi, X)g(AN, Y)\xi \\
&+ \alpha\eta(Y)g(AN, X)A\xi + \alpha g(BX, \phi Y)A\xi + \alpha g(BX, Y)\phi A\xi \\
&- \alpha g(AN, Y)BX - \alpha\eta(Y)\phi X - \alpha g(A\xi, Y)\phi BX = 0
\end{aligned} \tag{6.21}$$

for any $X \in \mathcal{C}$ and $Y \in T_{[z]}(\mathcal{T}_A)$. Since $\mathcal{C} = T_\beta \oplus T_\lambda \oplus T_\mu$, we may take $X \in T_\lambda$. Then we get

$$SX = \lambda X \quad \text{and} \quad S\phi X = \lambda\phi X.$$

From this, (6.21) can be rewritten as

$$\begin{aligned}
&-(1+\alpha^2)\lambda g(Y,\phi X)\xi-\lambda\eta(Y)\phi X+\lambda g(B\phi X,\phi Y)\phi A\xi\\
&-\lambda g(B\phi X,Y)A\xi-\alpha^2\lambda\eta(Y)\phi X-\lambda g(A\xi,Y)B\phi X\\
&-\lambda g(\phi A\xi,Y)\phi B\phi X+\alpha^2\lambda g(\phi X,Y)\xi-\alpha\lambda^2 g(\phi X,Y)\xi\\
&+\alpha g(BX,\phi Y)A\xi+\alpha g(BX,Y)\phi A\xi\\
&-\alpha g(AN,Y)BX-\alpha\eta(Y)\phi X-\alpha g(A\xi,Y)\phi BX=0
\end{aligned}\tag{6.22}$$

for any $X\in T_\lambda$, $Y\in T_{[z]}(\mathcal{T}_A)$. Putting $Y=\xi$ and using $-\phi A\xi=AN$, this equation gives us

$$-(\lambda+\alpha^2\lambda+\alpha)\phi X=0\tag{6.23}$$

for any $X\in T_\lambda$. Moreover, substituting $Y=A\xi\in T_\beta$ in (6.22), it implies

$$-\lambda B\phi X-\alpha\phi BX=0\tag{6.24}$$

for all $X\in T_\lambda$. From (3.15), we see that $B\phi X=-\phi BX$ for any $X\in T_\lambda$. Therefore, (6.24) becomes $(\lambda-\alpha)\phi BX=0$, which yields $\alpha=\lambda$. Combining this formula and (6.23), it makes a contradiction. Hence we conclude:

Proposition 6.5. *There does not exist a Hopf real hypersurface in the complex quadric Q^m, $m\geq 3$, with $\mathcal{C}$-parallel structure Jacobi operator and $\mathfrak{A}$-isotropic unit normal vector field.*

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