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ON GEOMETRIC PROPERTIES OF THE MITTAG-LEFFLER AND WRIGHT FUNCTIONS

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Abstract. The main focus of the present paper is to present new set of sufficient conditions so that the normalized form of the Mittag-Leffler and Wright functions have certain geometric properties like close-to-convexity, univalency, convexity and starlikeness inside the unit disk. Interesting consequences and examples are derived to support that these results are better than the existing ones and improve several results available in the literature.

1. Introduction

1.1. Background and motivation

Mittag-Leffler and Wright functions are important functions of fractional calculus. These functions occur in the solution of fractional order differential equations or fractional order integral equations. These functions play vital role in many branches of science and engineering. For example, these functions have been successfully applied in fractional modeling [23].

Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z,\alpha,\beta\in\mathbb{C},\ \Re(\alpha)>0, \tag{1.1}$$

which was introduced by Mittag-Leffler [13, 14] in 1903 for the case $\beta=1$. Later in 1933, Wright [25] introduced the Wright function, defined as

$$W_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)},\ \beta,z\in\mathbb{C},\ \alpha>-1. \tag{1.2}$$

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In particular, when $\alpha = 1$ and $\beta = \mu+1$, we have the Bessel function [2] defined as follows:

$$J_\mu(z) = \left(\frac{z}{2}\right)^\mu W_{1,\mu+1}\left(-\frac{z^2}{4}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\mu}}{k!\Gamma(k+\mu+1)}. \tag{1.3}$$

In 1935, Wright introduced the Fox-Wright function ${}_p\Psi_q[z]$ with p numerator and q denominator parameters, defined by [6]

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\Big|z\right] = {}_p\Psi_q\left[\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\Big|z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+kA_i)}{\prod_{j=1}^{q}\Gamma(b_j+kB_j)}\frac{z^k}{k!}, \tag{1.4}$$

$$\left(a_i, b_j \in \mathbb{C}, \text{ and } A_i, B_j \in \mathbb{R}^+ \ (i = 1,\ldots,p,\ j = 1,\ldots,q)\right).$$

The series (1.4) converges absolutely and uniformly for all bounded $|z|$, $z \in \mathbb{C}$ when

$$\epsilon = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0.$$

It can be noted from (1.1), (1.2) and (1.4) that Fox-Wright function ${}_p\Psi_q[z]$ generalizes both $E_{\alpha,\beta}(z)$ and $W_{\alpha,\beta}(z)$ and other special functions like Bessel function, hypergeometric function etc. In [19, Theorem 4], Pogány and Srivastava established the following inequality:

$$\psi_0 e^{\psi_1\psi_0^{-1}|z|} \le {}_p\Psi_q\left[\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\Big|z\right] \le \psi_0 - (1-e^{|z|})\psi_1 \tag{1.5}$$

for all $z \in \mathbb{R}$ and for all ${}_p\Psi_q[z]$ satisfying $\psi_1 > \psi_2$ and $\psi_1^2 < \psi_0\psi_2$, where

$$\psi_k = \frac{\prod_{j=1}^{p}\Gamma(a_j+kA_j)}{\prod_{j=1}^{q}\Gamma(b_j+kB_j)},\ k = 0,1,2.$$

Problems for investing geometric properties including starlikeness, closed-to-convexity, convexity or univalency of family of analytic functions in the unit disk $\mathcal{D} = \{z : |z| < 1\}$, involving special functions have always been attracted by several researchers [1, 5, 7, 8, 11, 12, 20].

Geometric properties of normalized form of $W_{\alpha,\beta}(z)$ were discussed by Prajapat in [20]. Geometric properties of normalized form of $E_{\alpha,\beta}(z)$ were studied by Bansal and Prajapat in [1]. Recently, in [17], geometric properties of normalized form of $E_{\alpha,\beta}(z)$ were studied, which improve the results of [1]. Baricz, Toklu and Kadioğlu [4] derived the radii of starlikeness and convexity of normalized form of $W_{\alpha,\beta}(z)$. Radius of starlikeness and Hardy space of $E_{\alpha,\beta}(z)$ were discussed in [21]. Recently, Baricz and Prajapati [3] found the radii of starlikeness and convexity of generalized Mittag-Leffler functions. The above results inspire us to study the geometric properties of the Mittag-Leffler and Wright functions and improve the results available in the literature.

1.2. Main contributions

The main contributions of this paper are listed in the following points.

- Obtain sufficient conditions so that normalized form of $E_{\alpha,\beta}(z)$ has some geometric properties like starlikeness, close-to-convexity (univalency) and convexity.
- Derive sufficient conditions so that normalized form of $W_{\alpha,\beta}(z)$ has certain geometric properties like starlikeness, close-to-convexity (univalency) and convexity.
- Show that obtained results improve the results available in the literature.

1.3. Outlines

The paper is organized as follows. In Section 2, we recall some well-known definitions and results, which will be useful to derive the main results. Section 3 is devoted to discuss the geometric properties of the normalized form of $E_{\alpha,\beta}(z)$. In this section we also show that obtained results improve several results for $E_{\alpha,\beta}(z)$ available in the literature. In Section 4, geometric properties of the normalized form of $W_{\alpha,\beta}(z)$ are studied. It is also verified that obtained results improve several results for $W_{\alpha,\beta}(z)$ available in the literature.

2. Preliminaries

Let $\mathcal{H}$ denote the class of all analytic functions inside the unit disk $\mathcal{D} = \{z : |z| < 1\}$. Suppose that $\mathcal{A}$ is the class of all functions $f \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$ such that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for all $z \in \mathcal{D}$.

A function $f \in \mathcal{A}$ is said to be a starlike function (with respect to the origin 0) in $\mathcal{D}$, if f is univalent in $\mathcal{D}$ and $f(\mathcal{D})$ is a star-like domain with respect to 0 in $\mathbb{C}$. This class of starlike functions is denoted by $\mathcal{S}^*$. The analytic characterization of $\mathcal{S}^*$ is given [5] below:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \ \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{S}^*.$$

If $f(z)$ is a univalent function in $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then $f \in \mathcal{A}$ is said to be a convex function in $\mathcal{D}$. We denote this class of convex functions by $\mathcal{K}$. This class can be analytically characterized as follows:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{K}.$$

It is well-known that zf' is starlike if and only if $f \in \mathcal{A}$ is convex. A function $f(z) \in \mathcal{A}$ is said to be close-to-convex in $\mathcal{D}$ if $\exists$ a starlike function $g(z)$ in $\mathcal{D}$ such that $\Re\left(\frac{zf'(z)}{g(z)}\right) > 0$ for all $z \in \mathcal{D}$. The class of all close-to-convex functions is denoted by $\mathcal{C}$. It can be easily verified that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. It is well-known that every close-to-convex function in $\mathcal{D}$ is also univalent in $\mathcal{D}$.

A function $f \in \mathcal{A}$ is said to be uniformly convex (starlike) if for every circular arc γ contained in $\mathcal{D}$ with center $\zeta \in \mathcal{D}$ the image arc $f(\gamma)$ is convex (starlike with respect to the image $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by UCV (UST) [24]. In [9,10], A. W. Goodman introduced these classes. Later, F. Rønning [24] introduced a new class of starlike functions $\mathcal{S}_p$ defined by

$$\mathcal{S}_p := \{f : f(z) = zF'(z),\ F \in UCV\}.$$

For further details on geometric properties of analytic functions we refer to [2, 5, 7, 8, 11, 12] and references cited therein. Now, we recall some well-known lemmas, which will be helpful to prove the main results.

Lemma 2.1 ([11]). *Let $f \in \mathcal{A}$ and $|(f(z)/z) - 1| < 1$ for each $z \in \mathcal{D}$. Then f is univalent and starlike in $\mathcal{D}_{\frac{1}{2}} = \{z : |z| < \frac{1}{2}\}$.*

Lemma 2.2 ([11]). *If $f \in \mathcal{A}$ and $|f'(z) - 1| < 1$ for each $z \in \mathcal{D}$, then f is convex in $\mathcal{D}_{\frac{1}{2}} = \left\{z \in C,\ |z| < \frac{1}{2}\right\}$.*

Lemma 2.3 ([15]). *Let $f(z) \in \mathcal{A}$ and $|f'(z) - 1| < 2/\sqrt{5}\ \ \forall\, z \in \mathcal{D}$. Then $f(z)$ is a starlike function in $\mathcal{D}$.*

Lemma 2.4 ([18]). *Let $f(z) = z + \sum_{k=2}^{\infty} A_k z^k$. If $1 \leq 2A_2 \leq \cdots \leq nA_n \leq (n+1)A_{n+1} \leq \cdots \leq 2$, or $1 \geq 2A_2 \geq \cdots \geq nA_n \geq (n+1)A_{n+1} \geq \cdots \geq 0$, then f is close-to-convex with respect to $-\log(1-z)$.*

Lemma 2.5 ([18]). *Let $f(z) = z + \sum_{k=2}^{\infty} A_{2k-1} z^{2k-1}$ be analytic in $\mathcal{D}$. If $1 \geq 3A_3 \geq \cdots \geq (2k-1)A_{2k-1} \geq \cdots \geq 0$ or $1 \leq 3A_3 \leq \cdots \leq (2k-1)A_{2k-1} \leq \cdots \leq 2$, then f is univalent in $\mathcal{D}$.*

Lemma 2.6 ([22]). *Let $f(z) \in \mathcal{A}$.*

(i) *If $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$, then $f(z) \in UCV$.*

(ii) *If $\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2}$, then $f(z) \in \mathcal{S}_p$.*

3. Geometric properties of the Mittag-Leffler function

In this section, we discuss the geometric properties of the normalized Mittag-Leffler function defined as

$$\mathbb{E}_{\alpha,\beta}(z) = \Gamma(\beta) z E_{\alpha,\beta}(z),\ z \in \mathcal{D}. \tag{3.1}$$

Theorem 3.1. *Let $\alpha, \beta > 0$ be real numbers satisfying the following conditions*:

$$(H_1) : \begin{cases} \text{(i)} & 3\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta), \\ \text{(ii)} & 2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 3\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} < 1. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$.

Proof. Let

$$\mathcal{E}_{\alpha,\beta}(z) = \frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{\alpha,\beta}(z)},\ z \in \mathcal{D}.$$

Then clearly the function $\mathcal{E}_{\alpha,\beta}(z)$ is analytic in $\mathcal{D}$ and $\mathcal{E}_{\alpha,\beta}(0) = 1$. To show the desired result, it suffices to prove that $\Re\left(\mathcal{E}_{\alpha,\beta}(z)\right) > 0$ for all $z \in \mathcal{D}$. For this, it is enough to show that

$$|\mathcal{E}_{\alpha,\beta}(z) - 1| = \left|\frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{\alpha,\beta}(z)} - 1\right| = \frac{\left|\mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right|}{\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right|} < 1 \text{ for all } z \in \mathcal{D}.$$

From (3.1), we get

$$\begin{aligned}\mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}(z)}{z} &= \sum_{k=1}^{\infty} \frac{k\Gamma(\beta)z^k}{\Gamma(\alpha k+\beta)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\beta)\Gamma(k+2)}{\Gamma(\alpha k+\alpha+\beta)} \frac{z^{k+1}}{k!} \\ &= z\Gamma(\beta)\,{}_1\Psi_1\left[\begin{smallmatrix}(2,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| z\right].\end{aligned}$$

Hence,

$$\left|\mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right| < \Gamma(\beta)\,{}_1\Psi_1\left[\begin{smallmatrix}(2,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right],\ z \in \mathcal{D}. \tag{3.2}$$

In our case,

$$\psi_0 = \frac{1}{\Gamma(\alpha+\beta)},\ \psi_1 = \frac{2}{\Gamma(2\alpha+\beta)},\ \text{and } \psi_2 = \frac{6}{\Gamma(3\alpha+\beta)}.$$

It is easy to see that the conditions "(H_1) : (i), (ii)" are equivalent to $\psi_2 < \psi_1$ and $\psi_1^2 < \psi_0\psi_2$. Therefore, by (1.5), we obtain

$${}_1\Psi_1\left[\begin{smallmatrix}(2,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right] \leq \frac{1}{\Gamma(\alpha+\beta)} + \frac{2(e-1)}{\Gamma(2\alpha+\beta)}. \tag{3.3}$$

In addition, using the triangle inequality $|z_1+z_2| \geq ||z_1| - |z_2||$, we have

$$\begin{aligned}\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right| &\geq 1 - \left|\sum_{k=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} z^k\right| \\ &\geq 1 - \sum_{k=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} = 1 - \Gamma(\beta)\ {}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right].\end{aligned}$$

However, by applying the inequality (1.5), we have

$${}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right] \leq \frac{1}{\Gamma(\alpha+\beta)} + \frac{(e-1)}{\Gamma(2\alpha+\beta)},$$

where $2\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta)$ and $\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 2\Gamma^2(2\alpha+\beta)$. Therefore,

$$\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right| \geq 1 - \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} > 0 \text{ for all } z \in \mathcal{D}. \tag{3.4}$$

Using (3.2), (3.3) and (3.4), we have

$$\left|\frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{\alpha,\beta}(z)} - 1\right| = \frac{\left|\mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right|}{\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right|}$$

$$< \left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)\left(1 - \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^{-1} < 1$$

for all $z \in \mathcal{D}$, under the given hypothesis. □

Putting $\alpha = 1$ in Theorem 3.1, we have the following result:

Corollary 3.2. *Let $\beta > 3.22118$. Then the function $\mathbb{E}_{1,\beta}$ is starlike in $\mathcal{D}$.*

Remark 3.3. Setting $\beta = 5$ in Theorem 3.1, we obtain $\alpha \in [0.769, 1.80]$. Moreover, we can verify that for each positive integer $\beta = n \geq 3$, there exists $\alpha_n \in (0, 1]$ such that $\mathbb{E}_{\alpha_n,\beta}(z)$ is starlike in $\mathcal{D}$. In literature, various results related to starlikeness of $\mathbb{E}_{\alpha,\beta}(z)$ (see [1, 17]) is available with the condition that $\alpha \geq 1$. Hence, Theorem 3.1 improves the results available in [1, 17].

Using the inequality [17, Equation 5]

$$\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z}\right| \geq \frac{\beta^2 - \beta - 1}{\beta^2} \text{ for } \alpha \geq 1 \text{ and } \beta > (1+\sqrt{5})/2,$$

and proceeding similarly as previous theorem, following result can be established.

Theorem 3.4. *Suppose that $\alpha \geq 1$ and $\beta > (1+\sqrt{5})/2$ are real numbers and the following conditions hold*:

$$(H'_1): \begin{cases} \text{(i)} & 3\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta), \\ \text{(ii)} & 2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 3\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{\beta^2}{(\beta^2-\beta-1)}\left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right) < 1. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}$.

Setting $\alpha = 1$ in Theorem 3.4, we have the following result:

Corollary 3.5. *Let $\beta > 3.14658$. Then the function $\mathbb{E}_{1,\beta}$ is starlike in $\mathcal{D}$.*

Remark 3.6. From [1, Example 2.1], we can see that $\mathbb{E}_{1,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq 4$. Further, according to [1, Theorem 2.2], $\mathbb{E}_{1,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq \frac{3+\sqrt{17}}{2} \approx 3.56155$. Moreover, [17, Theorem 6] indicates that $\mathbb{E}_{1,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq 3.214319744$. Hence, Corollary 3.5 provides results for $\mathbb{E}_{1,\beta}(z)$, better than the results available in [1, Theorem 2.1, Theorem 2.2] and [17, Theorem 6].

Theorem 3.7. *Suppose that the following conditions hold*:

$$(H_2): \begin{cases} \text{(i)} & 2\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta), \\ \text{(ii)} & \Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 2\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} < 1. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Proof. A simple computation leads to

$$\frac{\mathbb{E}_{\alpha,\beta}(z)}{z} - 1 = z\Gamma(\beta)\ {}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| z\right].$$

Therefore,

$$\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z} - 1\right| < \Gamma(\beta)\ {}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right] \tag{3.5}$$

for all $z \in \mathcal{D}$. In this cases, we have

$$\psi_0 = \frac{1}{\Gamma(\alpha+\beta)},\ \psi_1 = \frac{1}{\Gamma(2\alpha+\beta)},\ \text{ and }\ \psi_2 = \frac{2}{\Gamma(3\alpha+\beta)}.$$

We observe that the conditions on the parameters "(H_2) : (i), (ii)" is equivalent to $\psi_2 < \psi_1$ and $\psi_1^2 < \psi_0\psi_2$. Therefore, by (1.5) we have

$${}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle| 1\right] \leq \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\beta)(e-1)}{\Gamma(2\alpha+\beta)}. \tag{3.6}$$

Now, keeping (3.5), (3.6) and "(H_2) : (iii)" in mind, we obtain

$$\left|\frac{\mathbb{E}_{\alpha,\beta}(z)}{z} - 1\right| < 1,\ z \in \mathcal{D}.$$

By applying Lemma 2.1, we get the required result. □

Corollary 3.8. *If* $\beta > \sqrt{e} \approx 1.64872$*, then the function* $\mathbb{E}_{1,\beta}(z)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Example 3.9. The following functions are starlike in $\mathcal{D}_{1/2}$:

$$\begin{aligned}
\mathbb{E}_{1,2}(z) &= e^z - 1, \\
\mathbb{E}_{1,\frac{5}{2}}(z) &= \frac{3}{4}\left(\frac{\sqrt{\pi}e^z \mathrm{erf}(\sqrt{z}) - 2\sqrt{z}}{\sqrt{z}}\right), \\
\mathbb{E}_{1,3}(z) &= \frac{2\left(e^z - z - 1\right)}{z}, \\
\mathbb{E}_{1,\frac{7}{2}}(z) &= \frac{5}{8}\left(\frac{3\sqrt{\pi}e^z \mathrm{erf}(\sqrt{z})}{z^{3/2}} - \frac{6}{z} - 4\right), \\
\mathbb{E}_{1,4}(z) &= \frac{6\left(e^z - 1 - z\right) - 3z^2}{z^2}.
\end{aligned}$$

Remark 3.10. It can be noted that Corollary 3.8 provides results sharper than Corollary 3.5. Let us now consider the following functions $f_1(x), g_1(x)$ and $h_1(x)$, defined by

$$f_1(x) = \Gamma(3x+2) - 2\Gamma(2x+2),$$
$$g_1(x) = 2\Gamma^2(2x+2) - \Gamma(x+2)\Gamma(3x+2),$$
$$h_1(x) = 1 - \frac{1}{\Gamma(x+2)} - \frac{(e-1)}{\Gamma(2x+2)}.$$

The above functions are positive for all $x \in [0.915, 1.897]$. Figure 1 verifies our claim. Theorem 3.7 indicates that the function $\mathbb{E}_{\alpha,2}(z)$ is starlike in $\mathcal{D}_{1/2}$, if $\alpha \in [0.915, 1.897]$. Similarly, using Theorem 3.7, we can verify that $\mathbb{E}_{\alpha_3,3}(z)$ and $\mathbb{E}_{\alpha_4,4}(z)$ are starlike in $\mathcal{D}_{1/2}$ if $\alpha_3 \in [0.59, 2.156]$ and $\alpha_4 \in [0.482, 2.376]$ respectively. Further, one can verify that for each positive integer $\beta = n \geq 3$, there exists $\alpha_n \in (0, 1]$ such that $\mathbb{E}_{\alpha_n,\beta}(z)$ is starlike in $\mathcal{D}_{1/2}$. In literature, various results related to starlikeness of $\mathbb{E}_{\alpha,\beta}(z)$ in $\mathcal{D}_{1/2}$ is available with the condition that $\alpha \geq 1$. For example, see [1, Theorem 2.4]. It is important to note that Theorem 3.7 discusses the cases when $0 < \alpha \leq 1$. Hence, Theorem 3.7 improves the results available in [1].

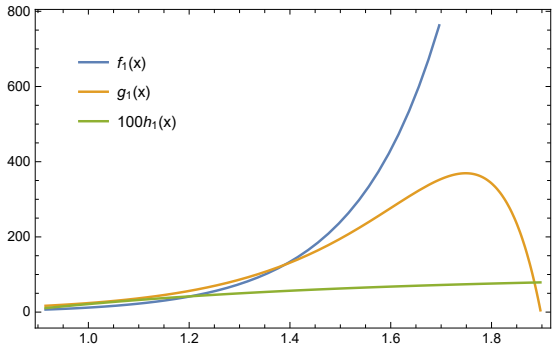

FIGURE 1. Graphs of $f_1(x)$, $g_1(x)$ and $100h_1(x)$ for $x \in [0.915, 1.897]$.

Theorem 3.11. *Suppose that the following conditions hold*:

$$(H_3): \begin{cases} \text{(i)} & 8\Gamma(2\alpha+\beta) < 3\Gamma(3\alpha+\beta), \\ \text{(ii)} & 9\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 16\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} < M. \end{cases}$$

(a) *If* $M = 1$, *then the function* $\mathbb{E}_{\alpha,\beta}(z)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$.
(b) *If* $M = \frac{2}{\sqrt{5}}$, *then the function* $\mathbb{E}_{\alpha,\beta}(z)$ *is starlike in* $\mathcal{D}$.

Proof. By using (3.1), we have

$$\begin{aligned}\mathbb{E}'_{\alpha,\beta}(z)-1 &= \sum_{k=0}^{\infty}\frac{(k+2)\Gamma(\beta)z^{k+1}}{\Gamma(\alpha k+\alpha+\beta)}\\ &= \sum_{k=0}^{\infty}\frac{\Gamma(\beta)\Gamma(k+1)\Gamma(k+3)}{\Gamma(k+2)\Gamma(\alpha k+\alpha+\beta)}\frac{z^{k+1}}{k!}\\ &= z\,\Gamma(\beta)\,{}_2\Psi_2\left[\begin{matrix}(1,1),(3,1)\\(2,1),(\alpha+\beta,\alpha)\end{matrix}\Big|z\right].\end{aligned} \tag{3.7}$$

This implies that

$$\left|\mathbb{E}'_{\alpha,\beta}(z)-1\right| < \Gamma(\beta)\,{}_2\Psi_2\left[\begin{matrix}(1,1),(3,1)\\(2,1),(\alpha+\beta,\alpha)\end{matrix}\Big|1\right],\quad z\in\mathcal{D}. \tag{3.8}$$

Moreover, we observe that the conditions "(H_3) : (i), (ii)" are equivalent to the inequalities $\psi_2<\psi_1$ and $\psi_1^2<\psi_0\psi_2$, where

$$\psi_0=\frac{2}{\Gamma(\alpha+\beta)},\ \psi_1=\frac{3}{\Gamma(2\alpha+\beta)},\ \text{ and }\ \psi_2=\frac{8}{\Gamma(3\alpha+\beta)}.$$

Using (1.5), we obtain

$${}_2\Psi_2\left[\begin{matrix}(1,1),(3,1)\\(2,1),(\alpha+\beta,\alpha)\end{matrix}\Big|1\right] < \frac{2}{\Gamma(\alpha+\beta)}+\frac{3(e-1)}{\Gamma(2\alpha+\beta)}.$$

Combining the above inequality, (3.8) and "(H_3) : (iii)", we have

$$\left|\mathbb{E}'_{\alpha,\beta}(z)-1\right| < M,\ z\in\mathcal{D}.$$

Finally, using Lemma 2.2 and Lemma 2.3, the desired result can be established. □

Corollary 3.12. *If* $\beta>\frac{1}{2}+\sqrt{3e-3/4}\approx 3.22118457393$*, the function* $\mathbb{E}_{1,\beta}(z)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$.

Remark 3.13. It can be noted from [1, Theorem 2.4] and [17, Theorem 7] that $\mathbb{E}_{1,\beta}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$ if $\beta\geq\frac{3+\sqrt{17}}{2}\approx 3.561552813$. Hence, the above result improves the results for $\mathbb{E}_{1,\beta}(z)$, available in [1, 17].

Example 3.14. The following functions are convex in $\mathcal{D}_{\frac{1}{2}}$:

$$\begin{aligned}\mathbb{E}_{1,\frac{7}{2}}(z) &= \frac{5}{8}\left(\frac{3\sqrt{\pi}e^z\mathrm{erf}(\sqrt{z})}{z^{3/2}}-\frac{6}{z}-4\right),\\ \mathbb{E}_{1,4}(z) &= \frac{(6\,(e^z-1)-3z)}{z},\\ \mathbb{E}_{1,\frac{9}{2}}(z) &= \frac{105\sqrt{\pi}e^z\mathrm{erf}(\sqrt{z})}{16z^{5/2}}-\frac{7(4z^2+10z+15)}{8z^2},\\ \mathbb{E}_{1,5}(z) &= -\frac{4\left(z^3+3z^2+6z-6e^z+6\right)}{z^3}.\end{aligned}$$

Remark 3.15. Consider the following functions defined as

$$f_2(x) = 3\Gamma(3x+2) - 8\Gamma(2x+2),$$
$$g_2(x) = 16\Gamma^2(2x+2) - 9\Gamma(x+2)\Gamma(3x+2),$$
$$h_2(x) = 1 - \frac{2}{\Gamma(x+2)} - \frac{3(e-1)}{\Gamma(2x+2)}.$$

The above functions are positive for any $x \in [1.3749, 1.6472]$. Figure 2 verifies our claim. Hence, $\mathbb{E}_{\alpha,2}(z)$ is convex on $\mathcal{D}_{\frac{1}{2}}$ if $\alpha \in [1.3749, 1.6472]$. Similarly, we can prove that $\mathbb{E}_{\alpha_8,8}(z)$ is convex on $\mathcal{D}_{\frac{1}{2}}$ if $\alpha_8 \in [0.61, 2.71]$. It can be verified that for each positive integer $\beta = n \geq 8$, there exists $\alpha_n \in (0,1)$ such that $\mathbb{E}_{\alpha_n,\beta}(z)$ is convex on $\mathcal{D}_{\frac{1}{2}}$. Hence, Theorem 3.11 is useful to discus the convexity of $\mathbb{E}_{\alpha,\beta}(z)$ on $\mathcal{D}_{\frac{1}{2}}$ when $0 < \alpha < 1$. In [1, 17], sufficient condition for convexity of $\mathbb{E}_{\alpha,\beta}(z)$ is given as $\alpha \geq 1$ and $\beta \geq 3.561552813$. Therefore, Theorem 3.11 improves the results in [1, 17].

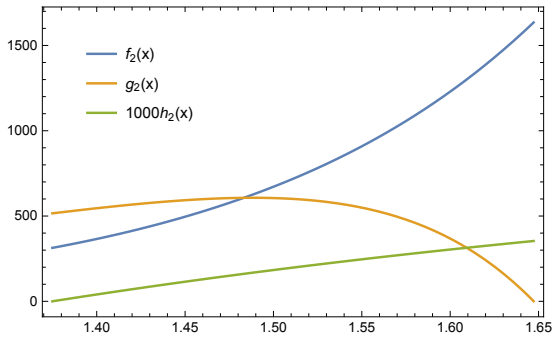

FIGURE 2. Graphs of $f_2(x)$, $g_2(x)$ and $1000h_2(x)$ for $x \in [1.3749, 1.6472]$.

Theorem 3.16. *Suppose that α and β satisfy the hypothesis H_3 (with $M = 1$) of Theorem 3.11. In addition, they also satisfy the following conditions*:

$$(H_4): \begin{cases} \text{(i)} & 4\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta), \\ \text{(ii)} & 3\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 4\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{4\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{9(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} < 1. \end{cases}$$

Then $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}$.

Proof. We set

$$\Lambda_{\alpha,\beta}(z) = \frac{z\mathbb{E}''_{\alpha,\beta}(z)}{\mathbb{E}'_{\alpha,\beta}(z)}, \ z \in \mathcal{D}.$$

To prove that $\mathbb{E}_{\alpha,\beta}(z)$ is convex on $\mathcal{D}$, it is enough to prove that $|\Lambda_{\alpha,\beta}(z)| < 1$ for all $z \in \mathcal{D}$. A simple computation gives us

$$\mathbb{E}''_{\alpha,\beta}(z) = \Gamma(\beta)\ {}_1\Psi_1\left[\begin{smallmatrix}(3,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle|z\right],\ z \in \mathcal{D}, \tag{3.9}$$

and consequently

$$\left|\mathbb{E}''_{\alpha,\beta}(z)\right| < \Gamma(\beta)\ {}_1\Psi_1\left[\begin{smallmatrix}(3,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle|1\right],\ z \in \mathcal{D}. \tag{3.10}$$

Under the given conditions and the inequalities, we obtain

$${}_1\Psi_1\left[\begin{smallmatrix}(3,1)\\(\alpha+\beta,\alpha)\end{smallmatrix}\middle|1\right] \leq \frac{2}{\Gamma(\alpha+\beta)} + \frac{6(e-1)}{\Gamma(2\alpha+\beta)}. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\left|\mathbb{E}''_{\alpha,\beta}(z)\right| < \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{6(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)},\ z \in \mathcal{D}. \tag{3.12}$$

Again, we have

$$\begin{aligned}
\left|\mathbb{E}'_{\alpha,\beta}(z)\right| &= \left|1 + \sum_{k=1}^{\infty} \frac{(k+1)\Gamma(\beta)}{\Gamma(\alpha k+\beta)} z^k\right| \\
&> 1 - \sum_{k=1}^{\infty} \frac{(k+1)\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \\
&= 1 - \sum_{k=0}^{\infty} \frac{(k+2)\Gamma(\beta)}{\Gamma(\alpha k+\alpha+\beta)} \\
&= 1 - \Gamma(\beta) \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(k+3)}{\Gamma(k+2)\Gamma(\alpha k+\alpha+\beta)} \frac{(1)^k}{k!} \\
&= 1 - \Gamma(\beta)_2\Psi_2\left[\begin{smallmatrix}(1,1),(3,1)\\(2,1),(\alpha+\beta,\alpha)\end{smallmatrix}\middle|1\right].
\end{aligned}$$

Since α, β satisfy H_3 (with $M = 1$), using the same technique as in Theorem 3.11, we obtain

$$\left|\mathbb{E}'_{\alpha,\beta}(z)\right| > 1 - \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} > 0. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\left|\frac{z\mathbb{E}''_{\alpha,\beta}(z)}{\mathbb{E}'_{\alpha,\beta}(z)}\right| < \frac{\frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{6(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}}{1 - \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} - \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}} < 1,$$

under the given hypothesis. The proof is now completed. □

Corollary 3.17. *Suppose that α and β satisfy the hypothesis H_3 (with $M = 1$) of Theorem 3.11. In addition, they also satisfy the following conditions*:

$$(H_4') : \begin{cases} \text{(i)} & 4\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta), \\ \text{(ii)} & 3\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 4\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{3\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{15}{2}\frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} < 1/2. \end{cases}$$

Then $\mathbb{E}_{\alpha,\beta}(z) \in UCV$ for all $z \in \mathcal{D}$.

Remark 3.18. Consider the following functions defined as

$$\begin{aligned} f_3(x) &= \Gamma(3x+8) - 4\Gamma(2x+8), \\ g_3(x) &= 4\Gamma^2(2x+8) - 3\Gamma(x+8)\Gamma(3x+8), \\ h_3(x) &= 1 - \frac{4\Gamma(8)}{\Gamma(x+8)} - \frac{9(e-1)\Gamma(8)}{\Gamma(2x+8)}. \end{aligned}$$

Using mathematical software, we can verify that the above functions are positive for any $x \in [0.885, 1.78]$. From Remark 3.15, we see that $\mathbb{E}_{\alpha,8}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$ if $\alpha_8 \in [0.61, 2.71]$. Hence, if $\alpha \in [0.885, 1.78]$ and $\beta = 8$, then the conditions (H_3) and (H_4) are satisfied. Consequently, $\mathbb{E}_{\alpha,8}(z)$ is convex in $\mathcal{D}$ if $\alpha \in [0.885, 1.78]$. It can be verified that for each positive integer $\beta = n \geq 8$, there exists $\alpha_n \in (0,1)$ such that $\mathbb{E}_{\alpha_n,\beta}(z)$ is convex in $\mathcal{D}$. Hence, Theorem 3.16 is useful to discuss the convexity of $\mathbb{E}_{\alpha,\beta}(z)$ on $\mathcal{D}$ when $0 < \alpha \leq 1$. Recently, in [17], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}$ if $\alpha \geq 1$ and $\beta \geq 3.56155281$. Therefore, Theorem 3.16 improves the results in [17].

Remark 3.19. Using Corollary 3.17 and proceeding similarly as Remark 3.18, we can verify the following statements:

(i) $\mathbb{E}_{1.13,\beta}(z) \in UCV$ if $\beta \geq 6.67$,
(ii) $\mathbb{E}_{\alpha,11}(z) \in UCV$ if $\alpha \in [0.9877, 2]$,
(iii) for each positive integer $\beta = n \geq 11$, there exists $\alpha_n \in (0,1)$ such that $\mathbb{E}_{\alpha_n,\beta}(z) \in UCV$.

In [16, Theorem 2.6], it is proved that $\mathbb{E}_{\alpha,\beta}(z) \in UCV$ if $\alpha \geq 1$ and $\beta \geq 9.1112597744$. Hence, Corollary 3.17 improves the results (especially Theorem 2.6) of [16].

Theorem 3.20. *Let $\alpha, \beta \geq 1$ be such that the following conditions are satisfied*:

$$(H_5) : \begin{cases} \text{(i)} & 3\Gamma(\beta+2) < \Gamma(\beta+3), \\ \text{(ii)} & 2\Gamma(\beta+1)\Gamma(\beta+3) < 3\Gamma^2(\beta+2), \\ \text{(iii)} & \frac{2}{\beta} + \frac{3(e-1)}{\beta(\beta+1)} < 1. \end{cases}$$

Then the function $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}_{1,\beta}(z)$.

Proof. From Theorem 3.1, it can be easily verified that $\mathbb{E}_{1,\beta}(z)$ is a starlike function in $\mathcal{D}$ under the given hypothesis. A simple computation gives

$$\begin{aligned}\left|\mathbb{E}'_{\alpha,\beta}(z)-\frac{\mathbb{E}_{1,\beta}(z)}{z}\right| &< \sum_{k=1}^{\infty}\left|\frac{(k+1)\Gamma(\beta)}{\Gamma(k\alpha+\beta)}-\frac{\Gamma(\beta)}{\Gamma(k+\beta)}\right| \\ &\leq \sum_{k=1}^{\infty}\frac{k\Gamma(\beta)}{\Gamma(k+\beta)}=\sum_{k=0}^{\infty}\frac{(k+1)\Gamma(\beta)}{\Gamma(k+\beta+1)} \\ &= \Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(k+2)\Gamma(\beta)}{k!\Gamma(k+\beta+1)} \\ &= \Gamma(\beta)\,{}_1\Psi_1\left[\begin{smallmatrix}(2,1)\\(\beta+1,1)\end{smallmatrix}\middle|1\right].\end{aligned} \tag{3.14}$$

In our case,

$$\psi_0=\frac{1}{\Gamma(\beta+1)},\ \psi_1=\frac{2}{\Gamma(\beta+2)}\ \text{ and }\ \psi_2=\frac{6}{\Gamma(\beta+3)}.$$

It can be easily verified that the conditions "(H_5) : (i), (ii)" are equivalent to $\psi_2<\psi_1$ and $\psi_1^2<\psi_0\psi_2$. Therefore, using (1.5) and (3.14), we obtain

$$\left|\mathbb{E}'_{\alpha,\beta}(z)-\frac{\mathbb{E}_{1,\beta}(z)}{z}\right|<\frac{1}{\beta}+\frac{2(e-1)}{\beta(\beta+1)}. \tag{3.15}$$

Again, we have

$$\begin{aligned}\left|\frac{\mathbb{E}_{1,\beta}(z)}{z}\right| &\geq 1-\sum_{k=1}^{\infty}\frac{\Gamma(\beta)}{\Gamma(k+\beta)}=1-\Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{k!\Gamma(k+\beta)} \\ &= 1-\Gamma(\beta)\,{}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\beta+1,1)\end{smallmatrix}\middle|1\right].\end{aligned}$$

In this case,

$$\psi'_0=\frac{1}{\Gamma(\beta+1)},\ \psi'_1=\frac{2}{\Gamma(\beta+2)}\ \text{ and }\ \psi'_2=\frac{6}{\Gamma(\beta+3)},$$

which satisfy $\psi'_2<\psi'_1$ and ${\psi'_1}^2<\psi'_0\psi'_2$ under the hypothesis "(H_5) : (i), (ii)". Therefore, using (1.5), we have

$$\left|\frac{\mathbb{E}_{1,\beta}(z)}{z}\right|>1-\left(\frac{1}{\beta}+\frac{(e-1)}{\beta(\beta+1)}\right). \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\left|\frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{1,\beta}(z)}-1\right|=\left|\frac{\mathbb{E}'_{\alpha,\beta}(z)-\frac{\mathbb{E}_{1,\beta}(z)}{z}}{\frac{\mathbb{E}_{1,\beta}(z)}{z}}\right|<\frac{\frac{1}{\beta}+\frac{2(e-1)}{\beta(\beta+1)}}{1-\left(\frac{1}{\beta}+\frac{(e-1)}{\beta(\beta+1)}\right)}<1,$$

under the hypothesis "(H_5) : (iii)". Hence, $\Re\left(\frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{1,\beta}(z)}\right)>0$, which completes the proof. □

The above theorem can be simplified as follows:

Corollary 3.21. *If $\alpha \geq 1$ and $\beta \geq 3.23$, then the function $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}_{1,\beta}(z)$.*

Remark 3.22. In [1], it is proved that $\mathbb{E}_{\alpha,\beta}(z)$ is close-to-convex with respect to $\mathbb{E}_{1,\beta}(z)$ if $\alpha \geq 1$ and $\beta \geq (3+\sqrt{17})/2 \approx 3.56155$. Hence, Theorem 3.20 improves the results in [1].

4. Geometric properties of the Wright function

In this section, we discuss the geometric properties of the normalized Wright function defined as

$$\mathbb{W}_{\alpha,\beta}(z) = z\Gamma(\beta)W_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta)z^{k+1}}{k!\Gamma(\alpha k+\beta)}. \tag{4.1}$$

Similarly, using (1.3) and (4.1), we have the normalized Bessel function defined as follows:

$$\mathbb{J}_{\mu}(z) = \mathbb{W}_{1,\mu+1}(-z) = \Gamma(\mu+1)z^{1-\mu/2}J_{\mu}(2\sqrt{z}). \tag{4.2}$$

Using the similar technique as in Section 3, following results can be established.

Theorem 4.1. *Suppose that the following conditions hold*:

$$(H_6): \begin{cases} \text{(i)} & 2\Gamma(2\alpha+\beta) < 3\Gamma(3\alpha+\beta), \\ \text{(ii)} & 3\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 4\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{(e-1)\Gamma(\beta)}{2\Gamma(2\alpha+\beta)} < 1. \end{cases}$$

Then the function $\mathbb{W}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$.

Corollary 4.2. *$\mathbb{W}_{1,\beta}(z)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$ if $\beta > 2$.*

Example 4.3. $\mathbb{J}_{\mu}(z)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$ if $\mu > 1$.

Remark 4.4. Setting $\beta = 2$ in H_6, we have $\alpha \in [0.645, 0.999]$. Therefore, $\mathbb{W}_{\alpha,2}(z)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$ if $\alpha \in [0.645, 0.999]$. Hence, Theorem 4.1 includes the case for $0 < \alpha < 1$. In [20], it is proved that $\mathbb{W}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$ if $\alpha \geq 1$ and $\beta \geq (1+\sqrt{5})/2$. Therefore, Theorem 4.1 improves the results of [20].

Theorem 4.5. *Under the following conditions*

$$(H_7): \begin{cases} \text{(i)} & 8\Gamma(2\alpha+\beta) < 9\Gamma(3\alpha+\beta), \\ \text{(ii)} & 27\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 32\Gamma^2(2\alpha+\beta), \\ \text{(iii)} & \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{3(e-1)\Gamma(\beta)}{2\Gamma(2\alpha+\beta)} < 1, \end{cases}$$

the function $\mathbb{W}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$.

Remark 4.6. In [20], it is proved that $\mathbb{W}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$ if $\alpha \geq 1$ and $\beta \geq 1+\sqrt{3}$. Setting $\beta = 4$ in (H_7) of Theorem 4.5, we get $\alpha \in [0.76, 0.95]$. Hence, Theorem 4.5 discusses the case for $0 < \alpha < 1$. Therefore, Theorem 4.5 improves the results of [20].

Theorem 4.7. *Let $\alpha \geq 1$, $\beta \geq 1$ and $\Gamma(\alpha+\beta) \geq 2\Gamma(\beta)$. Then $\mathbb{W}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$.*

Proof. Let $\mathbb{W}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} a_{k-1} z^k$, where $a_{k-1} = \frac{1}{(k-1)!\Gamma(\alpha(k-1)+\beta)} > 0$ for all $k \geq 2$. Clearly, $2a_1 = \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} \leq 1$, under the given hypothesis. Now we will show that $\{ka_{k-1}\}_{k=2}^{\infty}$ is decreasing. For any $k \geq 2$, we obtain

$$\begin{aligned} k^2\Gamma(\alpha k+\beta) &= k^2\Gamma(\alpha(k-1)+\alpha+\beta) \\ &\geq k^2\Gamma(\alpha(k-1)+1+\beta) \\ &= k^2(\alpha(k-1)+\beta)\Gamma(\alpha(k-1)+\beta) \\ &> (k+1)\Gamma(\alpha(k-1)+\beta). \end{aligned}$$

Using the above inequality, for any $k \geq 2$, we have

$$\begin{aligned} ka_{k-1} - (k+1)a_k &= \Gamma(\beta)\left[\frac{k}{(k-1)!\Gamma(\alpha(k-1)+\beta)} - \frac{k+1}{k!\Gamma(\alpha k+\beta)}\right] \\ &= \Gamma(\beta)\left[\frac{k^2\Gamma(\alpha k+\beta) - (k+1)\Gamma(\alpha(k-1)+\beta)}{k!\Gamma(\alpha(k-1)+\beta)\Gamma(\alpha k+\beta)}\right] > 0. \end{aligned}$$

Hence, $\{ka_{k-1}\}_{k\geq 2}$ is a decreasing sequence. Using Lemma 2.4, we conclude that $\mathbb{W}_{\alpha,\beta}(z)$ is close-to-convex with respect to the function $-\log(1-z)$. □

Theorem 4.8. *If $\alpha \geq 1$, $\beta \geq 1$, then $\mathcal{W}_{\alpha,\beta}(z) = z\Gamma(\beta)W_{\alpha,\beta}(z^2)$ is close-to-convex with respect to the function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.*

Proof. Let $\mathcal{W}_{\alpha,\beta}(z) = z\Gamma(\beta)W_{\alpha,\beta}(z^2) = z + \sum_{k=2}^{\infty} B_{2k-1} z^{2k-1}$, where $B_{2k-1} = \frac{\Gamma(\beta)}{(k-1)!\Gamma(\alpha(k-1)+\beta)} > 0$ for all $k \geq 2$. Clearly, $3B_3 = \frac{3\Gamma(\beta)}{2\Gamma(2\alpha+\beta)} < 1$, under the given hypothesis. We will prove that $\{(2k-1)B_{2k-1}\}_{k=2}^{\infty}$ is a decreasing sequence. For all $\alpha, \beta \geq 1$ and $k \geq 2$, we have

$$\begin{aligned} k(2k-1)\Gamma(\alpha k+\beta) &= k(2k-1)\Gamma(\alpha(k-1)+\alpha+\beta) \\ &\geq k(2k-1)\Gamma(\alpha(k-1)+1+\beta) \\ &= k(2k-1)(\alpha(k-1)+\beta)\Gamma(\alpha(k-1)+\beta) \\ &> (2k+1)\Gamma(\alpha(k-1)+\beta). \end{aligned}$$

Using the above inequality, for all $\alpha, \beta \geq 1$ and $k \geq 2$, we obtain

$$\begin{aligned} &(2k-1)B_{2k-1} - (2k+1)B_{2k+1} \\ &= \Gamma(\beta)\left[\frac{(2k-1)}{(k-1)!\Gamma(\alpha(k-1)+\beta)} - \frac{(2k+1)}{k!\Gamma(\alpha k+\beta)}\right] \\ &= \Gamma(\beta)\left[\frac{k(2k-1)\Gamma(\alpha k+\beta) - (2k+1)\Gamma(\alpha(k-1)+\beta)}{k!\Gamma(\alpha(k-1)+\beta)\Gamma(\alpha k+\beta)}\right] > 0. \end{aligned}$$

Therefore, $\{(2k-1)B_{2k-1}\}_{k\geq 2}$ is a decreasing sequence. Consequently, the hypothesis of Lemma 2.5 is satisfied. It is well-known that (see [2, p. 55]), if a function $f : \mathcal{D} \to C$ satisfies the hypothesis of Lemma 2.5, then it is

close-to-convex with respect to the function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. Hence, the theorem is proved. □

5. Conclusion

In this work, we have considered normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$ and normalized Wright function $\mathbb{W}_{\alpha,\beta}(z)$ and studied certain geometric properties such as close-to-convexity, univalency, convexity and starlikeness for $\alpha, \beta > 0$ and $z \in \mathcal{D}$. In literature, various results related to geometric properties of $\mathbb{E}_{\alpha,\beta}(z)$ and $\mathbb{W}_{\alpha,\beta}(z)$ are available (see [1,16,17,20]) with the condition that $\alpha, \beta \geq 1$. Results obtained in this work, discuss the case $0 < \alpha \leq 1$. Numerical computation shows that the obtained results are better than the existing ones and improve several results available in the literature.

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