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NORMAL COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON THE HARDY SPACE

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Abstract. In this paper, we investigate the normal and complex symmetric weighted composition operators $W_{\psi,\varphi}$ on the Hardy space $H^2(\mathbb{D})$. Firstly, we give the explicit conditions of weighted composition operators to be normal and complex symmetric with respect to conjugations $\mathcal{C}_1$ and $\mathcal{C}_2$ on $H^2(\mathbb{D})$, respectively. Moreover, we particularly investigate the weighted composition operators $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ which are normal and complex symmetric with respect to conjugations $\mathcal{J}$, $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, when φ has an interior fixed point, φ is of hyperbolic type or parabolic type.

1. Introduction

The study of complex symmetry has its origin that could be traced back to both operator theory and complex analysis, and was initiated by Garcia and Putinar in [7] and [8]. Thereafter, a number of papers were contributed to the study of complex symmetry (see, e.g. [2], [5, 6, 9–12], [15–17]). The class of complex symmetric operators includes a large number of examples, including all normal operators, Hankel operators, finite Toeplitz matrices, compressed shift operators and the Volterra integral operator.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. A conjugation on $\mathcal{H}$ is an anti-linear operator $C : \mathcal{H} \to \mathcal{H}$ satisfying $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is complex symmetric with the specific conjugation C.

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the collection of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the collection of all holomorphic self-maps of $\mathbb{D}$, respectively. We denote by $\mathrm{Aut}(\mathbb{D})$ the collection of one-to-one

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holomorphic functions which map $\mathbb{D}$ onto itself, called automorphisms of $\mathbb{D}$. Every automorphism of $\mathbb{D}$ has the form of $\lambda\frac{a-z}{1-\bar{a}z}, z \in \mathbb{D}$, where $|\lambda| = 1, a, z \in \mathbb{D}$.

Every disk automorphism is an automorphism of the Riemann sphere and has two fixed points on the sphere, counting multiplicity. The automorphisms of the unit disk could be classified according to the location of the fixed points:

- An automorphism φ of the unit disk is called an elliptic automorphism if it has one fixed point in the disk and the another one in the complement of the closed disk.
- An automorphism φ of the unit disk is called a hyperbolic automorphism if it has two fixed points on the unit circle.
- An automorphism φ of the unit disk is called a parabolic automorphism if there is one fixed point on the unit circle of multiplicity 2.

Moreover, we introduce the Denjoy-Wolff theorem (see, e.g., [4], Theorem 2.51).

If φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the disk into itself, then there is a point a in $\overline{\mathbb{D}}$ so that the iterated φ_n of φ converge to a uniformly on compact subsets of $\mathbb{D}$. The limit point is called the Denjoy-Wolff point of φ.

The Hardy space $H^2(\mathbb{D})$ is the collection of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty, \ z \in \mathbb{D}.$$

For each $w \in \mathbb{D}$, the reproducing kernel is defined by

$$K_w(z) = \frac{1}{1 - \overline{w}z}, \ z \in \mathbb{D}.$$

The norm of $K_w(z)$ is given by $\|K_w\| = (1 - |w|^2)^{-\frac{1}{2}}$. Hence the normalized reproducing kernel can be written as

$$k_w(z) = \frac{(1 - |w|^2)^{\frac{1}{2}}}{1 - \overline{w}z}.$$

It is easy to check that for each $f \in H^2(\mathbb{D}), \ \langle f, K_w \rangle = f(w)$.

The composition operator C_φ induced by $\varphi \in S(\mathbb{D})$ is defined as $C_\varphi f(z) = f \circ \varphi(z), \ f \in H(\mathbb{D}), \ z \in \mathbb{D}$. For $\psi \in H(\mathbb{D})$, the multiplication operator M_ψ can be defined by $M_\psi(f) = \psi \cdot f$ for all $f \in H(\mathbb{D})$. Combining the composition operator C_φ and the multiplication operator M_ψ, the weighted composition operator $W_{\psi,\varphi}$ is defined by

$$W_{\psi,\varphi} f(z) = \psi(z) f(\varphi(z)), \ z \in \mathbb{D}, \ f \in H(\mathbb{D}).$$

Weighted composition operators on various spaces of analytic functions have been studied extensively during the past few decades. The book [4] contains comprehensive treatments of these weighted composition operators.

Traditional choice of study with respect to weighted composition operators includes: boundedness, compactness, compact difference, essential norm, closed range, etc. With the advances in the study of complex symmetry, it makes sense to ask the question: are there any complex symmetric weighted composition operators? The above question was studied in [13] by Jung as well as the work by Garcia and Hammond in [6].

One the one hand, since normal operators are complex symmetric, there are examples of normal and complex symmetric weighted composition operators on $H^2(\mathbb{D})$ (see, e.g., Theorem 10 and Proposition 12 in [1]). On the other hand, as it was proved in [13], all $\mathcal{J}$-symmetric composition operators are normal (see, Corollary 3.10 in [13]), where $\mathcal{J}$ is the standard conjugation defined by $\mathcal{J}f(z) = \overline{f(\overline{z})}$, $f \in H(\mathbb{D})$. And Jung etc. in [13] found the explicit conditions for weighted composition operators to be normal and complex symmetric with respect to conjugation $\mathcal{J}$ (see, Corollary 3.7 in [13]).

Recently, Lim and Khoi in [15] classified the weighted anti-linear conjugation $\mathcal{A}_{u,v}$ into two cases $\mathcal{C}_1$ and $\mathcal{C}_2$ and found the explicit conditions for $W_{\psi,\varphi}$ to be complex symmetric with respect to conjugations $\mathcal{C}_1$ and $\mathcal{C}_2$ on $H_\gamma(\mathbb{D})$. In [1], Bourdon and Narayan investigated the normality of $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ when φ has an interior fixed point or a Denjoy-Wolff point on $\overline{\mathbb{D}}$. Thus we try to *characterize all weighted composition operators that are normal and complex symmetric with respect to conjugations* $\mathcal{C}_1$ *and* $\mathcal{C}_2$.

Furthermore, arose by the open question raised by Noor in [18] that "Does there exist a non-constant and non-automorphic symbol φ for which C_φ is complex symmetric but not normal on $H^2(\mathbb{D})$?", we raise another question: *What are the explicit conditions for the weighted composition operators* $W_{\psi,\varphi}$ *on* $H^2(\mathbb{D})$ *to be normal and complex symmetric when* $\varphi \in \mathrm{Aut}(\mathbb{D})$*?*

Raised by the two questions above, this paper is arranged as follows. Firstly, we find the explicit conditions of $W_{\psi,\varphi}$ to be normal and complex symmetric with the conjugations $\mathcal{C}_1$ and $\mathcal{C}_2$ on $H^2(\mathbb{D})$. Moreover, we particularly investigate the weighted composition operators $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ which are normal and complex symmetric with respect to conjugations $\mathcal{J}, \mathcal{C}_1$ and $\mathcal{C}_2$, respectively, when φ has an interior fixed point, φ is of hyperbolic type or parabolic type.

It is noted that although some treatments in this paper contain trivial but tedious calculations, our work in this paper give some examples of normal and complex symmetric with respect to new conjugations ($\mathcal{C}_1$ and $\mathcal{C}_2$) and some counterexamples for the question raised by Noor in [18], which is mentioned before. Moreover, we only show some essential details and omit the tedious part of some proofs for the convenience of readers.

2. Preliminaries

2.1. Cowen's formula for the adjoint of a linear-fractional composition operator

For a non-constant linear-fractional self-map $\varphi(z) = \frac{az+b}{cz+d}$ of the unit disk, Cowen in [3] established the important formula $C_\varphi^* = M_g C_\sigma M_h^*$, where the

Cowen auxiliary functions are defined as follows:

$$g(z) = \frac{1}{-\overline{b}z + \overline{d}},\ \sigma(z) = \frac{\overline{a}z + \overline{c}}{-\overline{b}z + \overline{d}},\ h(z) = cz + d. \tag{2.1}$$

2.2. Normality of weighted composition operators

In [1], Bourdon and Narayan described all normal weighted composition operators $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ when φ has an interior fixed point. They also presented the equivalent conditions of weighted composition operators $W_{\psi,\varphi}$ to be normal when φ has a Denjoy-Wolff point on $\overline{\mathbb{D}}$. The results are of vital importance for our work and thus we show them as follows.

Proposition 2.1 ([1, Theorem 10])**.** *Suppose that φ has a fixed point $p \in \mathbb{D}$. Then $W_{\psi,\varphi}$ is normal if and only if $\psi = \psi(p)\frac{K_p}{K_p \circ \varphi}$ and $\varphi = \varphi_p \circ (\delta \varphi_p)$, where $\varphi_p(z) = \frac{p-z}{1-\overline{p}z}$ and $\delta \in \mathbb{C}$ with $|\delta| \leq 1$.*

Proposition 2.2 ([1, Theorem 12])**.** *Suppose that $\varphi(z) = \frac{az+b}{cz+d}$ is a linear-fractional self-map of the unit disk and $\psi = K_{\sigma(0)}$, where $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$. Then $W_{\psi,\varphi}$ is normal if and only if*

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\overline{b}a - \overline{d}c)z} C_{\sigma \circ \varphi} = \frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\varphi \circ \sigma}.$$

Remark 2.3. It is easy to check that the condition

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\overline{b}a - \overline{d}c)z} C_{\sigma \circ \varphi} = \frac{|d|^2}{|d|^2 - |c|^2 - (\overline{b}d - c\overline{a})z} C_{\varphi \circ \sigma}$$

is equivalent with

$$|\varphi(0)| = |\sigma(0)| \quad \text{and} \quad \varphi \circ \sigma = \sigma \circ \varphi \tag{2.2}$$

(see, also, [14, Proposition 4.6]).

Furthermore, Bourdon and Narayan showed that if φ is a linear-fractional self-map of parabolic type and $\psi = K_{\sigma(0)}$, then $W_{\psi,\varphi}$ is normal. They also showed that no hyperbolic non-automorphic linear-fractional map can induce a normal weighted composition operator $W_{\psi,\varphi}$ when $\psi = K_{\sigma(0)}$ (see, [1, Proposition 13] and the remarks below).

3. Complex symmetric weighted composition operators on the Hardy space

In [15], Lim and Khoi found the explicit conditions for weighted composition operators $W_{\psi,\varphi}$ to be complex symmetric with respect to conjugations $\mathcal{J}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ on $H_\gamma(\mathbb{D})$. When $\gamma = 1$, the conditions describe the complex symmetric weighted composition operators on $H^2(\mathbb{D})$. Since our work is based on their results, we show them as follows. The first theorem can be obtained by [15, Proposition 2.3].

Proposition 3.1. *$W_{\psi,\varphi}$ is $\mathcal{J}$-symmetric if and only if for each $z \in \mathbb{D}$, $\psi(z) = \frac{b}{1-a_o z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$, where $a_0 \in \mathbb{D}, a_1 \in \mathbb{D}$ and $b \in \mathbb{C}$.*

Let $u \in H(\mathbb{D})$ and $v \in S(\mathbb{D})$. For each $f \in H^2(\mathbb{D})$ and $z \in \mathbb{D}$, the weighted anti-linear operator $\mathcal{A}_{u,v}$ on $H^2(\mathbb{D})$ is defined by

$$\mathcal{A}_{u,v} = u(z)\overline{f(\overline{v(z)})},$$

which is a generalization of the standard conjugation $\mathcal{J}$. Observe that $W_{u,v} = \mathcal{A}_{u,v}\mathcal{J}$ and $\mathcal{A}_{u,v} = W_{u,v}\mathcal{J}$. It is proved that (see, Theorem 2.11 in [15]) $\mathcal{A}_{u,v} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is a conjugation if and only if it has either of the following forms:

(i) there exist $\alpha, \lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,

$$u(z) = \lambda \quad \text{and} \quad v(z) = \alpha z. \tag{3.1}$$

(ii) there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \partial\mathbb{D}$ such that for all $z \in \mathbb{D}$,

$$u(z) = \lambda k_\alpha(z) \quad \text{and} \quad v(z) = \frac{\overline{\alpha}}{\alpha}\frac{\alpha - z}{1 - \overline{\alpha} z}. \tag{3.2}$$

As it is noted in [15], $\mathcal{C}_1$ denotes $\mathcal{A}_{u,v}$ with u and v of the form (3.1) and $\mathcal{C}_2$ denotes $\mathcal{A}_{u,v}$ with u and v of the form (3.2). Theorems in the following can be obtained by [15, Theorem 3.1] and [15, Theorem 3.2].

Proposition 3.2. *$W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_1$-symmetric if and only if for all $z \in \mathbb{D}$,*

$$\psi(z) = \frac{d}{1 - \alpha c_0 z} \quad \textit{and} \quad \varphi(z) = c_0 + \frac{c_1 z}{1 - \alpha c_0 z}, \tag{3.3}$$

where $c_0, c_1 \in \mathbb{D}$ and $d \in \mathbb{C}$.

Proposition 3.3. *$W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_2$-symmetric if and only if for all $z \in \mathbb{D}$,*

$$\psi(z) = \frac{d(c_0^2 - \alpha c_1)}{c_0^2 - \alpha c_1 - (c_1 - c_2)z}$$

and

$$\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}, \tag{3.4}$$

where $c_0, c_1, c_2, d \in \mathbb{C}$.

4. Complex symmetric $W_{\psi,\varphi}$ when $\varphi \in \mathrm{Aut}(\mathbb{D})$

In [15], the authors found the explicit forms of the weighted composition operators $W_{\psi,\varphi}$ to be complex symmetric with respect to the standard conjugation $\mathcal{J}$ when $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Lemma 4.1. *Let $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ be the form in Theorem 3.1, where $a_0, a_1 \in \mathbb{D}$. Then $\varphi \in \mathrm{Aut}(\mathbb{D})$ if and only if either of the following statements holds*:

(i) *there exists $\gamma \in \mathbb{D} \setminus \{0\}$ such that $\varphi(z) = \frac{\overline{\gamma}}{\gamma} \frac{\gamma - z}{1-\overline{\gamma}z}$, where $\gamma = \frac{a_1+1}{a_0}, z \in \mathbb{D}$.*

(ii) *there exists $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \beta z$, where $\beta = a_1, z \in \mathbb{D}$.*

Proof. The necessity is from [15, Proposition 2.8] and the sufficiency is obviously checked. □

However, the explicit forms of weighted composition operators $W_{\psi,\varphi}$ that are complex symmetric with respect to $\mathcal{C}_1$ and $\mathcal{C}_2$ when $\varphi \in \mathrm{Aut}(\mathbb{D})$ are still unknown. Since it is essential for our study, we prove them in a similar way with [15, Proposition 2.8].

Lemma 4.2. *Let φ be the form* (3.3)*. Then $\varphi \in \mathrm{Aut}(\mathbb{D})$ if and only if either of the following statements holds*:

(i) *there exists $\gamma \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = \frac{\overline{\gamma}}{\gamma\alpha} \frac{\gamma - z}{1 - \overline{\gamma}z}, \tag{4.1}$$

where $\gamma = \frac{c_0}{\alpha c_0^2 - c_1}$, $z \in \mathbb{D}$.

(ii) *there exists $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \beta z$, where $\beta = c_1$, $z \in \mathbb{D}$.*

Proof. Since $\varphi \in \mathrm{Aut}(\mathbb{D})$, there exist $\gamma \in \mathbb{D} \setminus \{0\}$, $\beta \in \partial\mathbb{D}$ such that

$$\varphi(z) = \beta \frac{\gamma - z}{1 - \overline{\gamma}z}. \tag{4.2}$$

Equating $\varphi(z) = \frac{c_0 + (c_1 - \alpha c_0^2)z}{1 - \alpha c_0 z}$ from (3.3) and (4.2), we have that

$$c_0 + (c_1 - \alpha c_0^2 - \overline{\gamma}c_0)z - \overline{\gamma}(c_1 - \alpha c_0^2)z^2 = \beta\gamma - (\beta + \beta\alpha\gamma c_0)z + \alpha\beta c_0 z^2.$$

Comparing the constants and the coefficients of z and z^2 respectively,

$$c_0 = \beta\gamma, \tag{4.3}$$

$$c_1 - \alpha c_0^2 - \overline{\gamma}c_0 = -\beta(1 + \alpha\gamma c_0), \tag{4.4}$$

$$-\overline{\gamma}(c_1 - \alpha c_0^2) = \alpha\beta c_0. \tag{4.5}$$

If $\gamma = 0$, then the second form of this lemma is proved.

If $\gamma \in \mathbb{D} \setminus \{0\}$, we substitute (4.3) into (4.5) to obtain that $c_1 = \frac{(|\gamma|^2 - 1)\alpha\beta^2\gamma}{\overline{\gamma}}$. Then substituting c_0 and c_1 again into (4.4), we obtain that $\beta^2\alpha\gamma(|\gamma|^2 - 1) = \beta\overline{\gamma}(|\gamma|^2 - 1)$. Since $|\gamma| \neq 1$ and $\beta \neq 0$, we have that $\beta = \frac{\overline{\gamma}}{\alpha\gamma}$. Then substituting $\beta = \frac{\overline{\gamma}}{\alpha\gamma}$ into (4.3), we obtain the form

$$\varphi(z) = \frac{\overline{\gamma}}{\gamma\alpha} \frac{\gamma - z}{1 - \overline{\gamma}z}$$

without calculation. Moreover, an easy calculation shows that $\gamma = \frac{c_0}{\alpha c_0^2 - c_1}$. The converse part is obviously checked. This completes the proof. □

Lemma 4.3. *Let φ be the form* (3.4)*. Then $\varphi \in \operatorname{Aut}(\mathbb{D})$ if and only if either of the following statements holds*:

(i) *there exists $\gamma \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = \frac{|\alpha|^2 - \alpha\overline{\gamma}}{\overline{\alpha}\gamma - |\alpha|^2} \frac{\gamma - z}{1 - \overline{\gamma} z}, \ z \in \mathbb{D}, \tag{4.6}$$

where

$$\gamma = \frac{\alpha(\overline{\alpha} c_0^2 - c_1)}{|\alpha|^2 c_1 - c_2}.$$

(ii) $\varphi(z) = z, \ z \in \mathbb{D}$.

Proof. Since $\varphi \in \operatorname{Aut}(\mathbb{D})$, there exist $\gamma \in \mathbb{D} \setminus \{0\}, \ \beta \in \partial\mathbb{D}$ such that

$$\varphi(z) = \beta \frac{\gamma - z}{1 - \overline{\gamma} z}. \tag{4.7}$$

Equating $\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2) z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2) z}$ from (3.4) and (4.7) and comparing the constants and the coefficients of z and z^2, respectively, we have that

$$\overline{\gamma}(|\alpha|^2 c_1 - c_2) = \beta\overline{\alpha}(c_1 - c_2), \tag{4.8}$$

$$\alpha(\overline{\alpha} c_0^2 - c_1) = \beta\gamma\overline{\alpha}(c_0^2 - \alpha c_1), \tag{4.9}$$

$$\overline{\gamma}\alpha(\overline{\alpha} c_0^2 - c_1) + (|\alpha|^2 c_1 - c_2) = \overline{\alpha}\beta\gamma(c_1 - c_2) + \beta\overline{\alpha}(c_0^2 - \alpha c_1). \tag{4.10}$$

If $\gamma = 0$, then we get $c_1 = c_2$, $c_0^2 = \frac{c_1}{\overline{\alpha}}$ since $\alpha, \beta \neq 0$ and

$$|\alpha|^2 c_1 - c_2 = \beta\overline{\alpha}(c_0^2 - \alpha c_1). \tag{4.11}$$

Putting $c_1 = c_2$, $c_0^2 = \frac{c_1}{\overline{\alpha}}$ into (4.11), we have that $\beta = -1$. It follows that $\varphi(z) = \frac{\alpha(\overline{\alpha}\frac{c_1}{\overline{\alpha}} - c_1) - (|\alpha|^2 - 1) c_1 z}{\overline{\alpha}(\frac{c_1}{\overline{\alpha}} - \alpha c_1)} = z$, which implies the second part of this lemma.

If $\gamma \in \mathbb{D} \setminus \{0\}$, then combining (4.8), (4.9) and (4.10), we have that $\frac{|\alpha|^2 c_1 - c_2}{\overline{\alpha} c_0^2 - c_1} = \frac{\alpha}{\gamma}$ and $\frac{|\alpha|^2 c_1 - c_2}{c_0^2 - \alpha c_1} = \overline{\alpha}\beta$, which implies that

$$c_0^2 = \frac{1}{\overline{\alpha}} \frac{|\alpha|^2 \beta\gamma - \alpha}{\beta\gamma - \alpha} c_1, \tag{4.12}$$

$$c_2 = \left(1 - \frac{\alpha\overline{\gamma}}{\overline{\alpha}} \frac{|\alpha|^2 - 1}{\beta\gamma - \alpha}\right) c_1. \tag{4.13}$$

Putting c_0^2 and c_1 into $\frac{|\alpha|^2 c_1 - c_2}{c_0^2 - \alpha c_1} = \overline{\alpha}\beta$, we get

$$\beta = \frac{|\alpha|^2 - \alpha\overline{\gamma}}{\overline{\alpha}\gamma - |\alpha|^2}. \tag{4.14}$$

Then we have

$$\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2) z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2) z}$$

$$= \frac{\alpha(\overline{\alpha}\frac{1}{\overline{\alpha}}\frac{|\alpha|^2\beta\gamma-\alpha}{\beta\gamma-\alpha}c_1 - c_1) - (|\alpha|^2 c_1 - (1-\frac{\alpha\overline{\gamma}}{\overline{\alpha}}\frac{|\alpha|^2-1}{\beta\gamma-\alpha})c_1)z}{\overline{\alpha}(\frac{1}{\overline{\alpha}}\frac{|\alpha|^2\beta\gamma-\alpha}{\beta\gamma-\alpha}c_1 - \alpha c_1) - \overline{\alpha}(c_1 - (1-\frac{\alpha\overline{\gamma}}{\overline{\alpha}}\frac{|\alpha|^2-1}{\beta\gamma-\alpha})c_1)z}$$
$$= \beta\frac{\gamma - z}{1-\overline{\gamma}z}.$$

Moreover, an easy calculation shows that $\gamma = \frac{\alpha(\overline{\alpha}c_0^2 - c_1)}{|\alpha|^2 c_1 - c_2}$. The converse part is obviously checked. This completes the proof. □

5. Normal $\mathcal{J}$-symmetric $W_{\psi,\varphi}$

This first proposition in the following gives the explicit forms of weighted composition operators which is normal and complex symmetric with respect to the standard conjugation $\mathcal{J}$, which are obtained from Corollary 3.7 in [13].

Proposition 5.1. *Suppose that $\varphi \in S(\mathbb{D})$. Then $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{J}$-symmetric and normal if and only if $\psi(z) = \frac{b}{1-a_o z}$, $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ and*

$$\operatorname{Im} a_0 - |a_0|^2 \operatorname{Im} a_0 + \operatorname{Im} \overline{a_0} a_1 = 0, \tag{5.1}$$

where $a_0, a_1 \in \mathbb{D},\ b \in \mathbb{C},\ z \in \mathbb{D}$.

Remark 5.2. Note that the proposition above can also be proved by Proposition 2.2 with the equivalent condition (2.2).

In the following, we investigate weighted composition operators to be normal and complex symmetric with respect to the standard conjugation $\mathcal{J}$ on $H^2(\mathbb{D})$ when $\varphi \in \operatorname{Aut}(\mathbb{D})$.

Corollary 5.3. *Suppose that φ satisfies the hypothesis in Proposition 5.1. Then $\varphi \in \operatorname{Aut}(\mathbb{D})$ if and only if it has either of the following forms*:

(i) *there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \frac{\overline{\alpha}}{\alpha}\frac{\alpha - z}{1-\overline{\alpha}z},\ z \in \mathbb{D}$.*

(ii) *there exists $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \beta z,\ z \in \mathbb{D}$.*

Proof. By Lemma 4.1, we are only supposed to check (5.1). If there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \frac{\overline{\alpha}}{\alpha}\frac{\alpha-z}{1-\overline{\alpha}z}$, where $a_0 = \alpha\beta$ and $a_1 = \frac{\beta^2\alpha(|\alpha|^2-1)}{\overline{\alpha}}$. Substituting a_0 and a_1 into (5.1), we have that

$$\operatorname{Im} \alpha\beta - |\alpha\beta|^2 \operatorname{Im} \alpha\beta + \operatorname{Im} \overline{\alpha\beta}\frac{\alpha\beta^2(|\alpha|^2-1)}{\overline{\alpha}} = 0.$$

Furthermore, the second part is trivial to be checked. This completes the proof. □

Observe that Proposition 2.2 doesn't require that φ has an interior fixed point. (In fact, some special φ with interior fixed point and δ with $|\delta| \leq 1$ are included in Proposition 2.2, e.g. $p = 0$, $\delta \neq 1$ or $p \neq 0$, $\delta = 1$ or $p \neq 0$, $\delta \neq 1$, $|\delta| = 1$.)

In the following, we give three examples according to Proposition 5.1 when φ has an interior fixed point, φ is of hyperbolic type and parabolic type, respectively.

Example 5.4. Suppose that $\varphi \in S(\mathbb{D})$ has an interior fixed point $p \in \mathbb{D}$, φ is nonconstant. Then $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{J}$-symmetric and normal if and only if

$$\psi(z) = \gamma\frac{1-p^2}{1-p^2\delta + p(\delta-1)z},\ \varphi(z) = \frac{p(1-\delta)+(\delta-p^2)z}{1-p^2\delta+p(\delta-1)z},$$

where $\gamma = \psi(p) \in \mathbb{C},\ p \in (-1,1),\ z \in \mathbb{D}$.

Proof. Suppose that φ, ψ have the forms in Proposition 2.1, that is,

$$\psi(z) = \gamma\frac{1-|p|^2}{1-|p|^2\delta+\overline{p}(\delta-1)z}, \tag{5.2}$$

$$\varphi(z) = \frac{p(1-\delta)+(\delta-|p|^2)z}{1-|p|^2\delta+\overline{p}(\delta-1)z}. \tag{5.3}$$

Since $W_{\psi,\varphi}$ is $\mathcal{J}$-symmetric, ψ, φ should coordinate with the form in Proposition 3.1, that is,

$$\psi(z) = \frac{\psi(0)}{1-a_0 z}, \tag{5.4}$$

$$\varphi(z) = \frac{(a_1-a_0^2)z+a_0}{1-a_0 z}. \tag{5.5}$$

Equating (5.2) and (5.4), then comparing the constants and the coefficients of z and z^2, respectively, we have that

$$\psi(0) = \frac{\gamma(1-|p|^2)}{1-|p|^2\delta},\ a_0 = \frac{\overline{p}(1-\delta)}{1-|p|^2\delta}. \tag{5.6}$$

In fact, substituting $\psi(0)$ and a_0 of the forms above into (5.4), we get (5.2).

In the sequence, equating (5.3) and (5.5), then comparing the constants and the coefficients of z and z^2, respectively, we have that

$$a_0 = \frac{p(1-\delta)}{1-|p|^2\delta}, \tag{5.7}$$

$$a_1 = a_0^2 + \frac{a_0(|p|^2-\delta)}{\overline{p}(\delta-1)}, \tag{5.8}$$

$$(a_1-a_0^2)(1-|p|^2\delta)+\overline{p}(\delta-1)a_0 = (\delta-|p|^2)-p(1-\delta)a_0. \tag{5.9}$$

Comparing (5.6) and (5.7), we get $p \in \mathbb{R}$. Furthermore, substituting (5.7) and (5.8) into (5.9), we get $(p-\overline{p})\delta + (p-\overline{p})\delta|p|^4 - (p-\overline{p})2\delta|p|^2 = 0$, which is trivial. In fact, substituting $a_0 = \frac{p(1-\delta)}{1-p^2\delta}$, $a_1 = \delta\frac{(p^2-1)^2}{(1-p^2\delta)^2}$ into (5.5), we also get (5.3). This completes the proof. $\square$

Corollary 5.5. *Suppose that φ satisfies the hypothesis in Example 5.4. Then $\varphi \in \operatorname{Aut}(\mathbb{D})$ if and only if it has either of the following forms*:

(i) $\varphi(z) = \beta\frac{\alpha-z}{1-\overline{\alpha}z}$, *where* $\alpha = \frac{p(1-\overline{\delta})}{1-p^2\overline{\delta}} \in \mathbb{D}$, $\beta = \frac{p^2-\delta}{1-p^2\delta} \in \partial\mathbb{D}$ *and* $\delta \in \partial\mathbb{D}$, $z \in \mathbb{D}$.

(ii) $\varphi(z) = \delta z$, *where* $\delta \in \partial\mathbb{D}$, $z \in \mathbb{D}$.

Proof. By Lemma 4.1, if there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \frac{\overline{\alpha}}{\alpha}\frac{\alpha - z}{1-\overline{\alpha}z}$, where $a_0 = \alpha\beta$ and $a_1 = \frac{\beta^2\alpha(|\alpha|^2-1)}{\overline{\alpha}}$. By the proof of Example 5.4, equating $a_0 = \alpha\beta$ and $a_0 = \frac{p(1-\delta)}{1-p^2\delta}$, we obtain that $\alpha = \frac{p(1-\delta)}{1-p^2\overline{\delta}}$ since $\beta = \frac{\overline{\alpha}}{\alpha}$. Likewise, equating the two forms of a_1, and substituting $\alpha = \frac{p(1-\delta)}{1-p^2\overline{\delta}}$ into it, we obtain that $\frac{1-\overline{\delta}}{1-p^2\overline{\delta}} = \frac{1-\delta}{p^2-\delta}$. Hence, $\beta = \frac{\overline{\alpha}}{\alpha} = \frac{p^2-\delta}{1-p^2\delta}$. Since $\beta \in \partial\mathbb{D}$, we get $\delta \in \partial\mathbb{D}$. In fact, substituting α, β into $\varphi(z) = \frac{\overline{\alpha}}{\alpha}\frac{\alpha-z}{1-\overline{\alpha}z}$, we get (5.3) by an easy calculation. Otherwise, if there exists $\beta \in \partial\mathbb{D}$ such that $\varphi(z) = \beta z$, then obviously $\beta = \delta$. This completes the proof. ☐

Example 5.6. There is no $\mathcal{J}$-symmetric and normal weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ if φ is a hyperbolic automorphism linear-fractional self-map with Denjoy-Wolff point $1 \in \partial\mathbb{D}$.

Proof. The result can be similarly proved by Example 6.6. ☐

Example 5.7. There is no $\mathcal{J}$-symmetric and normal weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ if φ is a hyperbolic non-automorphism linear-fractional self-map.

Proof. The result can be directly obtained by the remark below Proposition 13 in [1]. ☐

Example 5.8. The weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{J}$-symmetric and normal when φ is a parabolic linear-fractional self-map if and only if $\psi(z) = \frac{d}{1-a_0z}$ and φ has either of the following forms:

(i) $\varphi(z) = \frac{(1-2a_0)z+a_0}{1-a_0z}$, $\operatorname{Im} a_0 = |a_0|^2$, $a_0, z \in \mathbb{D}$.

(ii) $\varphi(z) = \frac{(1+2a_0)z+a_0}{1-a_0z}$, $\operatorname{Im} a_0 = -|a_0|^2$, $a_0, z \in \mathbb{D}$.

Proof. Since φ is a hyperbolic automorphism linear-fractional self-map, then we have either of the following assertions:

(i) φ has the Denjoy-Wolff point $\zeta_1 = 1$ and $a_1 = a_0 - 1$. Hence, $\varphi(z) = \frac{(1-2a_0)z+a_0}{1-a_0z}$ and (5.1) turns to be $\operatorname{Im} a_0 = |a_0|^2$.

(ii) φ has the Denjoy-Wolff point $\zeta_2 = -1$ and $a_1 = a_0 + 1$. Hence, $\varphi(z) = \frac{(1+2a_0)z+a_0}{1-a_0z}$ and (5.1) turns to be $\operatorname{Im} a_0 = -|a_0|^2$. This completes the proof. ☐

6. Normal $\mathcal{C}_1$-symmetric $W_{\psi,\varphi}$

Recall that the anti-linear operator $\mathcal{A}_{u,v}$ on $H^2(\mathbb{D})$ is denoted by $\mathcal{C}_1$ if u and v have the forms (3.1).

Theorem 6.1. *Suppose that $\varphi \in S(\mathbb{D})$ is nonconstant. Then $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_1$-symmetric and normal if and only if*

$$\psi(z) = \frac{1}{1-\alpha c_o z}, \ \varphi(z) = \frac{(c_1 - \alpha c_0^2)z + c_0}{1-\alpha c_0 z}$$

and

$$(\overline{c_0} - \alpha c_0)(1 - |c_0|^2) + \alpha c_0 \overline{c_1} - \overline{c_0} c_1 = 0, \tag{6.1}$$

where $c_0, c_1 \in \mathbb{D}$, $z \in \mathbb{D}$.

Proof. Firstly, since $W_{\psi,\varphi}$ is $\mathcal{C}_1$-symmetric, by (3.3),

$$\psi(z) = \frac{d}{1 - \alpha c_0 z} \quad \text{and} \quad \varphi(z) = \frac{(c_1 - \alpha c_0^2)z + c_0}{1 - \alpha c_0 z}.$$

Assume that $\varphi(z) = \frac{az+b}{cz+d}$ and $\psi(z) = K_{\sigma(0)}(z) = \frac{d}{cz+d}$. Equating the two forms of φ, an easy calculation shows that $a = c_1 - \alpha c_0^2, b = c_0$ and $c = -\alpha c_0$. Similarly, equating the two forms of ψ, we get $d = \psi(0) = 1$. Furthermore, since $\sigma(z) = \frac{\overline{c_1 - \alpha c_0^2} z + \overline{\alpha c_0}}{1 - \overline{c_0} z}$,

$$|\varphi(0)| = |c_0| = |\sigma(0)|. \tag{6.2}$$

Another tedious calculation shows that for each $z \in \mathbb{D}$,

$$\varphi \circ \sigma(z) = \frac{(|c_1 - \alpha c_0^2| - |c_0|^2)z + (c_0 - |c_0|^2 c_0 + \overline{\alpha c_0} c_1)}{(1 - \alpha c_0 z)(1 - \overline{c_0} z)}$$

and

$$\sigma \circ \varphi(z) = \frac{(|c_1 - \alpha c_0^2| - |c_0|^2)z + (\overline{\alpha c_0} - |c_0|^2 \overline{\alpha c_0} + c_0 \overline{c_1})}{(1 - \alpha c_0 z)(1 - \overline{c_0} z)}.$$

It follows that $\sigma \circ \varphi = \varphi \circ \sigma$ if and only if

$$(\overline{c_0} - \alpha c_0)(1 - |c_0|^2) + \alpha c_0 \overline{c_1} - \overline{c_0} c_1 = 0. \tag{6.3}$$

By (2.2), the normality follows. This completes the proof. □

Example 6.2. It makes sense to give concrete examples of the normal and non-normal $\mathcal{C}_1$-symmetric weighted composition operators $W_{\psi,\varphi}$ with $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$.

Set $c_0 = \frac{3}{4}, c_1 = -\frac{1}{20}(1+i), \alpha = 1$. Then

$$\varphi(z) = \frac{(-49 - 4i)z + 60}{80 - 60z} \in S(\mathbb{D}),\ \psi(z) = \frac{4}{4 - 3z}.$$

And it is easily checked that $(\overline{c_0} - \alpha c_0)(1 - |c_0|^2) + \alpha c_0 \overline{c_1} - \overline{c_0} c_1 = \frac{3i}{40} \neq 0$. Therefore, $W_{\psi,\varphi}$ is $\mathcal{C}_1$-symmetric but non-normal.

Corollary 6.3. *Suppose that* φ *satisfies the hypothesis in Theorem 6.1. Then* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *if and only if it has either of the following forms*:

(i) *there exist* $\gamma \in \mathbb{D} \setminus \{0\}$ *and* $\beta \in \partial\mathbb{D}$ *such that* $\varphi(z) = \frac{\overline{\gamma}}{\gamma\alpha} \frac{\gamma - z}{1 - \overline{\gamma} z}$.

(ii) *there exists* $\beta \in \partial\mathbb{D}$ *such that* $\varphi(z) = \beta z$.

Proof. By Lemma 4.2, we are only supposed to check (6.1), which obviously holds by an easy calculation. This completes the proof. □

In the following, we give three examples according to Theorem 6.1 when φ has an interior fixed point, φ is of hyperbolic type and parabolic type, respectively.

Example 6.4. Suppose that $\varphi \in S(\mathbb{D})$ has an interior fixed point $p \in \mathbb{D}$, φ is nonconstant and $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_1$-symmetric. Then $W_{\psi,\varphi}$ is normal if and only if it has either of the following forms:

(i) $\psi(z) = \frac{\gamma(1-|p|^2)}{1-\alpha pz}$, $\varphi(z) = \frac{p-|p|^2 z}{1-\alpha pz}$, where $\gamma = \psi(p) \in \mathbb{C}, z \in \mathbb{D}$.

(ii) $\psi(z) = \frac{\gamma(1-\alpha p^2)}{1-\alpha p^2\delta+\alpha p(\delta-1)z}$, $\varphi(z) = \frac{p(1-\delta)+(\delta-\alpha p^2)z}{1-\alpha p^2\delta+\alpha p(\delta-1)z}$, where $\gamma = \psi(p) \in \mathbb{D}$, $z \in \mathbb{D}$.

Proof. As what we do in Example 5.4, equating the two forms of ψ, we get $\psi(0) = \frac{\gamma(1-|p|^2)}{1-|p|^2\delta}$, $c_0 = \frac{\overline{p}(1-\delta)}{\alpha(1-|p|^2\delta)}$. Further equating the two forms of φ, we get

$$c_0 = \frac{p(1-\delta)}{(1-|p|^2\delta)}, \tag{6.4}$$

$$c_1 = \alpha c_0^2 + \frac{\alpha c_0(|p|^2-\delta)}{\overline{p}(\delta-1)}, \tag{6.5}$$

$$\delta - |p|^2 - p(1-\delta)\alpha c_0 = (1-|p|^2\delta)(c_1 - \alpha c_0^2) + c_0\overline{p}(\delta-1). \tag{6.6}$$

Substituting (6.4) and (6.5) into (6.6), by some tedious but trivial calculation, we have that

$$\delta(\alpha p - \overline{p})(|p|^4 - 2|p^2| + 1) = 0,$$

which only holds when $\overline{p} = \alpha p$ or $\delta = 0$ since $p \in \mathbb{D}$. If $\delta = 0$, then $\psi(z) = \frac{\gamma(1-|p|^2)}{1-\alpha pz}$, $\varphi(z) = \frac{p-|p|^2 z}{1-\alpha pz}$. Observe that under this circumstance we must have $\alpha p = \overline{p}$ to equate $\varphi(z) = \frac{p-|p|^2 z}{1-\alpha pz}$ and $\varphi(z) = \frac{(c_1-\alpha c_0^2)z+c_0}{1-\alpha c_0 z}$. Therefore, $c_0 = p, c_1 = 0$ in (3.3). If $\alpha p = \overline{p}, \delta \neq 0$, then $c_0 = \frac{p(1-\delta)}{1-\alpha p^2\delta}$, $c_1 = \alpha c_0^2 + \frac{c_0(\alpha p^2-\delta)}{p(\delta-1)}$ in (3.3). Moreover, it is easily checked that $c_0 \in \mathbb{D}$ and $c_1 \in \mathbb{D}$ if and only if $\delta \neq 1$. This completes the proof. □

Corollary 6.5. *Suppose that φ satisfies the hypothesis in Example 6.4. If $\varphi \in \mathrm{Aut}(\mathbb{D})$, then $\varphi(z) = \frac{2p-(1+\alpha p^2)z}{1+\alpha p^2-2\alpha pz}, z \in \mathbb{D}$.*

Proof. Firstly consider $\varphi(z) = \frac{p-|p|^2 z}{1-\alpha pz}$ with $c_0 = p, c_1 = 0$ in (3.3). By Lemma 4.2, if φ has the form (4.1), then $p = \beta\gamma$, $\frac{(|\gamma|^2-1)p\alpha\beta}{\overline{\gamma}} = 0$, which implies that $p = 0$ since $\alpha, \beta \neq 0, |\gamma| \neq 1$, which further implies that $\varphi = 0$.

Moreover, we consider $\varphi(z) = \frac{p(1-\delta)+(\delta-\alpha p^2)z}{1-\alpha p^2\delta+\alpha p(\delta-1)z}$ with $c_0 = \frac{p(1-\delta)}{1-\alpha p^2\delta}$, $c_1 = \alpha c_0^2 + \frac{c_0(\alpha p^2-\delta)}{p(\delta-1)}$ in (3.3). Since $\beta = \frac{\overline{\gamma}}{\alpha\gamma}$ and $c_0 = \beta\gamma$, we have that $\gamma = \frac{p(1-\overline{\delta})}{1-|p|^2\overline{\delta}}$. (We still write $|p|^2$ here instead of αp^2.) Equating $\varphi(z) = \frac{p(1-\delta)+(\delta-|p|^2)z}{1-|p|^2\delta+\overline{p}(\delta-1)z}$ and (4.1), then comparing the constants and the coefficients of z and z^2, respectively, we have that

$$\beta = \frac{|p|^2 - \delta}{1 - |p|^2\delta},$$

$$(|p|^2 - \delta)(1 - \overline{\delta}) = (1 - \delta)(1 - |p|^2\delta), \tag{6.7}$$

$$|p|^2|1 - \delta|^2(1 - \delta) = |p|^2|1 - \delta|^2(1 - \overline{\delta}). \tag{6.8}$$

Observe that (6.8) implies $\delta \in \mathbb{R}$. Hence, by (6.7), we have that $\delta = -1$ since $p \in \mathbb{D}$, which also implies that $\gamma = \frac{2p}{1+|p|^2}$. Hence, $\varphi(z) = \frac{2p-(1+|p|^2)z}{1+|p|^2-2\overline{p}z}$. This completes the proof. □

Example 6.6. There is no $\mathcal{C}_1$-symmetric and normal weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ if φ is a hyperbolic automorphism of $\mathbb{D}$ with Denjoy-Wolff point $1 \in \partial\mathbb{D}$.

Proof. Since φ is a hyperbolic automorphism linear-fractional self-map with Denjoy-Wolff point $1 \in \partial\mathbb{D}$, we can assume that

$$\varphi(z) = \frac{(r + 1 - t)z + r + t - 1}{(r - t - 1)z + r + t + 1},$$

where $r = \frac{1}{\varphi'(1)} > 1, t \in \mathbb{C}$. Equating $\varphi(z) = \frac{(r+1-t)z+r+t-1}{(r-t-1)z+r+t+1}$ and (4.1), then comparing the constants and the coefficients of z and z^2 respectively, we have that

$$\gamma = \frac{r - t - 1}{\alpha(r - t + 1)} = \overline{\frac{\alpha(r + t - 1)}{r + t + 1}},$$

which implies that $r+t = 1$. Hence, $\gamma = 0$, which is impossible. This completes the proof. □

Example 6.7. There is no $\mathcal{C}_1$-symmetric and normal weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ if φ is a hyperbolic non-automorphism of $\mathbb{D}$.

Proof. The result can be directly obtained by the remark below Proposition 13 in [1]. □

Example 6.8. The weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_1$-symmetric and normal when φ is a parabolic linear-fractional self-map with the Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$ if and only if $\psi(z) = \frac{\zeta^2}{\zeta^2-c_0z}$, $\varphi(z) = \frac{(\zeta^2c_1-c_0^2)z+\zeta^2c_0}{\zeta^2-c_0z}$ and

$$(\zeta^2\overline{c_0} - c_0)(1 - |c_0|^2) + c_0\overline{c_1} - \zeta^2\overline{c_0}c_1 = 0,$$

where $c_0, c_1 \in \mathbb{D}, z \in \mathbb{D}$.

Proof. Assume that φ is a parabolic linear fractional self-map of $\mathbb{D}$. If $W_{\psi,\varphi}$ is $\mathcal{C}_1$-symmetric and normal, Theorem 6.1 implies that $\psi(z) = \frac{1}{1-\alpha c_o z}, \varphi(z) = \frac{(c_1-\alpha c_0^2)z+c_0}{1-\alpha c_0 z}$, where $c_0, c_1 \in \mathbb{D}$ satisfying $(\overline{c_0} - \alpha c_0)(1 - |c_0|^2) + \alpha c_0\overline{c_1} - \overline{c_0}c_1 = 0$.

Since φ is parabolic, it has the Denjoy-Wolff point $\zeta = \frac{1+\alpha c_0^2 - c_1}{2\alpha c_0}$ and $(c_1 - \alpha c_0^2 - 1)^2 = 4\alpha c_0^2$. Therefore, $\alpha = \frac{1}{\zeta^2}$ and we obtain the desired result.

The converse holds by applying Theorem 6.1 with $\alpha = \frac{1}{\zeta^2}$. This completes the proof. $\square$

7. Normal $\mathcal{C}_2$-symmetric $W_{\psi,\varphi}$

Recall that the anti-linear operator $\mathcal{A}_{u,v}$ on $H^2(\mathbb{D})$ is denoted by $\mathcal{C}_2$ if u and v have the forms (3.2).

Just as we calculate in Section 5, we should firstly equate the two forms of ψ and φ. Suppose that $\varphi(z) = \frac{az+b}{cz+d}$ and $\psi(z) = \frac{d}{cz+d}$.

Equating $\psi(z) = \frac{d}{cz+d}$ and $\psi(z) = \frac{\psi(0)(c_0^2 - \alpha c_1)}{c_0^2 - \alpha c_1 - (c_1 - c_2)z}$ in (3.4), then comparing the constants and the coefficients of z and z^2, respectively, we have that $\psi(0) = 1$ and $c = \frac{d(c_2 - c_1)}{c_0^2 - \alpha c_1}$. Therefore,

$$\psi(z) = \frac{c_0^2 - \alpha c_1}{c_0^2 - \alpha c_1 - (c_1 - c_2)z} = \frac{d}{cz+d}.$$

Further equating $\varphi(z) = \frac{az+b}{cz+d}$ and $\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}$ in (3.4), then comparing the constants and the coefficients of z and z^2, respectively, we have that

$$a = \frac{-d(|\alpha|^2 c_1 - c_2)}{\overline{\alpha}(c_0^2 - \alpha c_1)}, \ b = \frac{d(|\alpha|^2 c_0^2 - \alpha c_1)}{\overline{\alpha}(c_0^2 - \alpha c_1)}, \ c = \frac{d(c_2 - c_1)}{c_0^2 - \alpha c_1}.$$

Substituting the expressions of a, b, c above into $\varphi(z) = \frac{az+b}{cz+d}$, we obtain that

$$\varphi(z) = \frac{az+b}{cz+d} = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}.$$

Theorem 7.1. *Suppose that $\varphi \in S(\mathbb{D})$ is nonconstant. Then $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_2$-symmetric and normal if and only if*

$$\psi(z) = \frac{c_0^2 - \alpha c_1}{c_0^2 - \alpha c_1 - (c_1 - c_2)z}$$

and

$$\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}$$

and either of the followings holds:

(i) $|c_1 - c_2| = |\overline{\alpha} c_0^2 - c_1| = |c_0^2 - \alpha c_1| \neq \frac{||\alpha|^2 c_1 - c_2|}{|\alpha|}$.

(ii) $|c_1 - c_2| = |\overline{\alpha} c_0^2 - c_1| = |c_0^2 - \alpha c_1| = \frac{||\alpha|^2 c_1 - c_2|}{|\alpha|}$ *and*

$$\operatorname{Im}(\overline{A} - \overline{C})(\widetilde{A} + \widetilde{C}) = 0,$$

where $c_0, c_1, c_2, d \in \mathbb{C}, \alpha \in \mathbb{D} \setminus \{0\}, z \in \mathbb{D}$,

$$A = (|\alpha|^2 c_0^2 - \alpha c_1)(\alpha \overline{c_0}^2 - |\alpha|^2 \overline{c_1}), \ C = \alpha(\overline{c_1} - \overline{c_2})(|\alpha|^2 c_1 - c_2),$$

$$\widetilde{A} = -\alpha(|\alpha|^2\overline{c_1} - \overline{c_2})(\overline{\alpha}c_0^2 - c_1),\ \widetilde{C} = |\alpha|^2(c_0^2 - \alpha c_1)(\overline{c_1} - \overline{c_2}).$$

Proof. Since $\sigma(z) = \frac{-(|\alpha|^2\overline{c_1}-\overline{c_2})z+\alpha(\overline{c_1}-\overline{c_2})}{-\overline{\alpha}(\alpha\overline{c_0}^2-\overline{c_1})z+\alpha(\overline{c_0}^2-\overline{\alpha c_1})}$, we have that

$$|\sigma(0)| = \frac{|c_1 - c_2|}{|c_0^2 - \alpha c_1|},\ |\varphi(0)| = \frac{|\overline{\alpha}c_0^2 - c_1|}{|c_0^2 - \alpha c_1|}.$$

Hence, $|\sigma(0)| = |\varphi(0)|$ if and only if

$$|c_1 - c_2| = |\overline{\alpha}c_0^2 - c_1|.$$

Furthermore, some tedious calculations show that for each $z \in \mathbb{D}$,

$$\varphi \circ \sigma(z) = \frac{(D-B)z + (A-C)}{(\overline{C} - \overline{A})z + (E-B)},$$

and

$$\sigma \circ \varphi(z) = \frac{(D-E)z + (\widetilde{A} + \widetilde{C})}{(-\overline{\widetilde{A}} - \overline{\widetilde{C}})z + (E-B)},$$

where

$$A = (|\alpha|^2c_0^2 - \alpha c_1)(\alpha\overline{c_0}^2 - |\alpha|^2\overline{c_1}), B = |\alpha|^2|\overline{\alpha}c_0^2 - c_1|^2,$$
$$C = \alpha(\overline{c_1} - \overline{c_2})(|\alpha|^2c_1 - c_2), D = ||\alpha|^2c_1 - c_2|^2, E = |\alpha|^2|c_0^2 - \alpha c_1|^2,$$
$$\widetilde{A} = -\alpha(|\alpha|^2\overline{c_1} - \overline{c_2})(\overline{\alpha}c_0^2 - c_1), \widetilde{C} = |\alpha|^2(c_0^2 - \alpha c_1)(\overline{c_1} - \overline{c_2}).$$

It follows that $\sigma \circ \varphi = \varphi \circ \sigma$ if and only if

$$(D-B)(\overline{\widetilde{A}} + \overline{\widetilde{C}}) = (D-E)(\overline{A} - \overline{C}),$$
$$(A-C)(E-B) = (\widetilde{A} + \widetilde{C})(E-B),$$
$$(D-B)(E-B) - (A-C)(\overline{\widetilde{A}} + \overline{\widetilde{C}}) = (D-E)(E-B) + (\tilde{A} + \tilde{C})(\overline{C} - \overline{A}).$$

We assert that $B = E$ by a basic deduction. Then the conditions above turn to be

$$(D-B)(\overline{\widetilde{A}} + \overline{\widetilde{C}}) = (D-B)(\overline{A} - \overline{C}), \tag{7.1}$$

$$\operatorname{Im}(\overline{A} - \overline{C})(\widetilde{A} + \widetilde{C}) = 0. \tag{7.2}$$

If $D \neq B$, by (7.1), $A - C = \widetilde{A} + \widetilde{C}$, which implies that (7.2) always holds. If $D = B$, then (7.1) always hold. Combining what we have observed above and (2.2), the normality follows. This completes the proof. □

Example 7.2. Similarly, it also makes sense to give concrete examples of the normal and non-normal $\mathcal{C}_2$-symmetric weighted composition operators $W_{\psi,\varphi}$ with $\varphi \in S(\mathbb{D}), \psi \in H(\mathbb{D})$.

(i) Set $c_0 = \frac{1}{\sqrt{3}}, c_1 = \frac{1}{12}(1+i), c_2 = 0, \alpha = \frac{1}{2}$. Then

$$\varphi(z) = \frac{2 - 2i - (1+i)z}{7 - i - 2(1+i)z},\ \psi(z) = \frac{7-i}{7 - i - 2(1+i)z}.$$

And it is easily checked that $|c_1 - c_2| = |\overline{\alpha} c_0^2 - c_1| = |c_0^2 - \alpha c_1| \neq \frac{||\alpha|^2 c_1 - c_2|}{|\alpha|}$. Therefore, $W_{\psi,\varphi}$ is normal and $\mathcal{C}_2$-symmetric.

(ii) Set $c_0 = \sqrt{2}, c_1 = 0, c_2 = 1, \alpha = \frac{1}{2}$. Then

$$\varphi(z) = \frac{2z+1}{z+2}, \ \psi(z) = \frac{2}{z+2}.$$

And it is easily checked that the two conditions in Theorem 7.1 are not satisfied. Therefore, $W_{\psi,\varphi}$ is $\mathcal{C}_2$-symmetric but non-normal.

Corollary 7.3. *There is no $\varphi \in \mathrm{Aut}(\mathbb{D})$ satisfying the hypothesis in Theorem 7.1.*

Proof. Substituting $c_0^2 = \frac{1}{\overline{\alpha}} \frac{|\alpha|^2 \beta\gamma - \alpha}{\beta\gamma - \alpha} c_1$ and $c_2 = (1 - \frac{\alpha\overline{\gamma}}{\overline{\alpha}} \frac{|\alpha|^2 - 1}{\beta\gamma - \alpha}) c_1$ into $|\overline{\alpha} c_0^2 - c_1| = |c_0^2 - \alpha c_1|$, we have that $|\alpha|(|\gamma| - 1)(1 - |\alpha|^2) = 0$, which implies that $|\gamma| = 1$. However, this is impossible. This completes the proof. □

In the following, we give three examples according to Theorem 7.1 when φ has an interior fixed point, φ is of hyperbolic type or parabolic type, respectively.

Example 7.4. Suppose that $\varphi \in S(\mathbb{D})$ has an interior fixed point $p \in \mathbb{D}$, φ is nonconstant, then $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_2$-symmetric and normal if and only if $\psi(z) = \frac{\psi(p)(1-p^2)}{1-p^2\delta + p(\delta-1)z}$, $\varphi(z) = \frac{p(1-\delta) + (\delta - p^2)z}{1-p^2\delta + p(\delta-1)z}$, where $p \in (-1,1), z \in \mathbb{D}$.

Proof. To investigate the weighted composition operators when φ has an interior fixed point, we should equate the two relevant forms of ψ and φ.

Equating (5.2) and $\psi(z) = \frac{c_0^2 - \alpha c_1}{c_0^2 - \alpha c_1 - (c_1 - c_2)z}$, then comparing the constants and the coefficients of z and z^2 respectively, we have that

$$c_0^2 - \alpha c_1 = \frac{(1 - |p|^2\delta)(c_1 - c_2)}{\overline{p}(1-\delta)}, \tag{7.3}$$

$$\psi(p)(1 - |p|^2) = 1 - |p|^2\delta, \ \psi(p) \neq 0, \ c_0^2 \neq \alpha c_1, \ c_1 \neq c_2. \tag{7.4}$$

(In fact, if $\psi(p) = 0$ or $c_0^2 - \alpha c_1$ or $c_1 = c_2$, then ψ is trivial.)

Further equating (5.3) and $\varphi(z) = \frac{\alpha(\overline{\alpha} c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}$, then comparing the constants and the coefficients of z and z^2, respectively, we have that $c_0^2 - \alpha c_1 = I_3(c_1 - c_2)$, $\overline{\alpha} c_0^2 - c_1 = I_2 I_3 (c_1 - c_2)$, $|\alpha|^2 c_1 - c_2 = I_1(c_1 - c_2)$. Combining (7.3) and (7.4), we conclude that

$$c_0^2 - \alpha c_1 = I_3(c_1 - c_2), \tag{7.5}$$

$$\overline{\alpha} c_0^2 - c_1 = I_2 I_3 (c_1 - c_2), \tag{7.6}$$

$$|\alpha|^2 c_1 - c_2 = I_1(c_1 - c_2), \tag{7.7}$$

$$I_1 = \frac{\overline{\alpha}(p^2 - \delta)}{p(1-\delta)}, \ I_2 = \frac{\overline{\alpha} p (1-\delta)}{\alpha(1 - p^2\delta)}, \ I_3 = \frac{1 - p^2\delta}{p(1-\delta)},$$

$$p \in \mathbb{R}, \ c_0^2 \neq \alpha c_1, \ \psi(p) \neq 0, \ c_1 \neq c_2, \ \delta \neq 1, \ p \neq 0.$$

Substituting the expressions above into $\varphi(z) = \frac{\alpha(\overline{\alpha}c_0^2 - c_1) - (|\alpha|^2 c_1 - c_2)z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z}$, we obtain (5.3). This completes the proof. $\square$

Example 7.5. There is no $\mathcal{C}_2$-symmetric and normal weighted composition operator $W_{\psi,\varphi}$ on $H^2(\mathbb{D})$ if φ is a hyperbolic automorphism of $\mathbb{D}$.

Proof. The result can be directly obtained by Corollary 7.3 if φ is a hyperbolic automorphism linear-fractional self-map. And the result can be obtained by the remark below Proposition 13 in [1] if φ is a hyperbolic non-automorphism linear-fractional self-map. $\square$

Example 7.6. The weighted composition operator $W_{\psi,\varphi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is $\mathcal{C}_2$-symmetric and normal when φ is a parabolic linear-fractional self-map with the Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$ if and only if

$$\psi(z) = \frac{c_0^2 - \alpha c_1}{c_0^2 - \alpha c_1 - (c_1 - c_2)z},$$

$$\varphi(z) = \frac{\overline{\alpha}\zeta^2(c_1 - c_2) - (2\overline{\alpha}\zeta(c_1 - c_2) - \overline{\alpha}(c_0^2 - \alpha c_1))z}{\overline{\alpha}(c_0^2 - \alpha c_1) - \overline{\alpha}(c_1 - c_2)z},$$

$$(|\alpha|^2 c_1 - c_2 + \overline{\alpha}(c_0^2 - \alpha c_1))^2 = 4|\alpha|^2(c_1 - c_2)(\overline{\alpha}c_0^2 - c_1)$$

and condition (i) or (ii) in Theorem 7.1 is satisfied, where $c_0, c_1, c_2, \alpha \in \mathbb{D} \setminus \{0\}, z \in \mathbb{D}$.

Proof. Since φ is a parabolic automorphism linear-fractional self-map, then φ has the Denjoy-Wolff point $\zeta = \frac{(|\alpha|^2 c_1 - c_2) + \overline{\alpha}(c_0^2 - \alpha c_1)}{2\overline{\alpha}(c_1 - c_2)}$. Some trivial but tedious calculation shows that $\overline{\alpha}\zeta^2(c_1 - c_2) = \alpha(\overline{\alpha}c_0^2 - c_1)$. Note that

$$\zeta^2 = \frac{(|\alpha|^2 c_1 - c_2 + \overline{\alpha}(c_0^2 - \alpha c_1))^2}{4\overline{\alpha}^2(c_1 - c_2)^2} = \frac{\alpha(\overline{\alpha}c_0^2 - c_1)}{\overline{\alpha}(c_1 - c_2)}.$$

Hence, the result can be easily obtained by some tedious calculations and what we have observed above. This completes the proof. $\square$

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