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GENERALIZED ALTERNATING SIGN MATRICES AND SIGNED PERMUTATION MATRICES

Richard A. Brualdi and Hwa Kyung Kim

Abstract. We continue the investigations in [6] extending the Bruhat order on $n \times n$ alternating sign matrices to our more general setting. We show that the resulting partially ordered set is a graded lattice with a well-define rank function. Many illustrative examples are given.

1. Introduction

An $n \times n$ *alternating sign matrix* (abbreviated to ASM) is a $(0, \pm 1)$ matrix such that, ignoring 0's, the $+1$'s and -1's in each row and each column alternate beginning and ending with $+1$. The origins and many properties of ASMs can be found in [1, 3, 9–11].

In [6] a generalization of ASMs was defined as follows: Let $u = (u_1, u_2, \ldots, u_n)$, $u' = (u'_1, u'_2, \ldots, u'_n)$, $v = (v_1, v_2, \ldots, v_m)$, and $v' = (v'_1, v'_2, \ldots, v'_m)$ be vectors of ± 1's. If A is an $m \times n$ matrix, we define $A(u, u'|v, v')$ to be the $(m+2) \times (n+2)$ matrix (1) with rows indexed by $0, 1, \ldots, m+1$ and columns indexed by $0, 1, \ldots, n+1$.

$$(1) \qquad A(u,u'|v,v') = \begin{array}{c||ccccc||c} 0 & u_1 & u_2 & \cdots & u_{n-1} & u_n & 0 \\ \hline\hline v_1 & & & & & & v'_1 \\ v_2 & & & & & & v'_2 \\ \vdots & & & A & & & \vdots \\ v_{m-1} & & & & & & v'_{m-1} \\ v_m & & & & & & v'_m \\ \hline\hline 0 & u'_1 & u'_2 & \cdots & u'_{n-1} & u'_n & 0 \end{array}.$$

We then write $A = A(u, u'|v, v')[1, 2, \ldots, m|1, 2, \ldots, n]$ to denote that A is the middle $m \times n$ submatrix of $A(u, u'|v, v')$, and also write $A(u, u'|v, v') \to A$.

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A $(u,u'|v,v')$-*ASM* is an $m\times n$ $(0,\pm1)$-matrix A as in (1) such that, ignoring 0's, the $+1$'s and -1's in rows $0,1,2,\ldots,m+1$ and columns $0,1,2,\ldots,n+1$ of the $(0,\pm1)$-matrix $A(u,u'|v,v')$ alternate. Thus the condition that the first and last nonzero entry in each row and column of an ASM is $+1$ is relaxed, where now the first and last nonzero of each row is determined by v and v', respectively, and the first and last nonzero of each column is determined by u and u', respectively. Note also that, unlike for ASMs, the first and last rows and columns of a $(u,u'|v,v')$-ASM depend on u,u',v,v' and so may contain more than one nonzero entry. We denote the set of $(u,u'|v,v')$-ASMs by $\mathcal{A}_{m,n}(u,u'|v,v')$. If $u=u'$ and $v=v'$ and $m=n$, we often abbreviate these notations to: (u,v)-ASM and $\mathcal{A}_n(u,v)$, respectively. If $v=u$, we also use the abbreviations (u)-ASM and $\mathcal{A}_n(u)$. Observe that if $m=n$ and $u=v=(-1,-1,\ldots,-1)$, then a (u)-ASM is an ordinary ASM, and $\mathcal{A}_n(u)$ is the usual set $\mathcal{A}_n$ of $n\times n$ ASMs.

Example 1.1. Let $m=2$ and $n=3$, and let $u=(1,-1,1)$, $u'=(-1,1,-1)$, $v=(1,-1)$ and $v=(1,-1)$. Then a $(u,u'|v,v')$-ASM is given below:

$$\left[\begin{array}{r||rrr||r} 0 & 1 & -1 & 1 & 0\\ \hline\hline 1 & -1 & 1 & -1 & 1\\ -1 & 1 & -1 & 1 & -1\\ \hline\hline 0 & -1 & 1 & -1 & 0\end{array}\right] \to \left[\begin{array}{rrr} -1 & 1 & -1\\ 1 & -1 & 1\end{array}\right].$$

In [6], necessary and sufficient conditions are given for the nonemptiness of $\mathcal{A}_{m,n}(u,u'|v,v')$. In this paper we primarily consider the case where $m=n$ and $u=u'$ and $v=v'$, that is, the set $\mathcal{A}_n(u,v)$. In this case, a 180 degree rotation gives a bijection from $\mathcal{A}_n(u,v)$ to $\mathcal{A}_n(\overline{u},\overline{v})$ where $\overline{u}=(u_n,\ldots,u_1)$ and $\overline{v}=(v_n,\ldots,v_1)$. We also sometimes restrict our attention to the case where $u=v$, as in ordinary ASMs; in this case, a 90 degree clockwise rotation give a bijection from $\mathcal{A}_n(u)$ to $\mathcal{A}_n(\overline{u})$. Also $\mathcal{A}_n(u)$ is invariant under a reflection over the main diagonal.

The nonemptiness of $\mathcal{A}_n(u,v)$ is easily decided.

Lemma 1.2. *Let $u=(u_1,u_2,\ldots,u_n)$ and $v=(v_1,v_2,\ldots,v_n)$ be vectors of ±1's. Then $\mathcal{A}_n(u,v)\neq\emptyset$ if and only if u and v contain the same number of $+1$'s.*

Proof. Suppose that u contains p $(+1)$'s and $(n-p)$ (-1)'s and v contains q $(+1)$'s and $(n-q)$ (-1)'s. Let $A\in\mathcal{A}_n(u,v)$. Then p columns of A sum to -1 and $(n-p)$ columns sum to $+1$. Similarly, q rows of A sum to -1 and $(n-q)$ rows sum to $+1$. Hence

$$-p+(n-p)=-q+(n-q) \text{ implying that } p=q.$$

Conversely, if $p=q$, let $I_{u,v}$ be the $(0,\pm1)$-matrix whose $p\times p$ submatrix in those columns for which $u_i=+1$ and those rows for which $v_i=+1$ equals $-I_p$, and whose $(n-p)\times(n-p)$ submatrix in those columns for which $u_i=-1$

and those rows for which $v_i = -1$ equals I_{n-p}, with all other entries equal to 0. Thus $I_{u,v}$ is in $\mathcal{A}_n(u,v)$. $\square$

Note that in the above proof, we can replace I_p by any $p \times p$ permutation matrix and I_{n-p} by any $(n-p) \times (n-p)$ permutation matrix, and these $p!(n-p)!$ signed permutation matrices (definition in the next example) are the only signed permutation matrices in $\mathcal{A}_n(u,v)$.

Example 1.3. Here and elsewhere we usually use '+' in place of $+1$ and '$-$' in place of -1. Also, when we block the cells of a matrix, we usually do not put in 0's with an empty cell signifying a 0. Let $u = (-,-,+,+,-)$ and $v = (-,+,-,+,-)$. Then the following matrix A is in $\mathcal{A}_n(u,v)$.

$$
\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & + & - & 0 \\
\hline\hline
- & + & & & & & - \\
\hline
+ & & & & - & & + \\
\hline
- & & & & & + & - \\
\hline
+ & & & - & & & + \\
\hline
- & & + & & & & - \\
\hline\hline
0 & - & - & + & + & - & 0
\end{array}
\rightarrow A = \left[\begin{array}{c|c|c|c|c}
+ & & & & \\
\hline
 & & & - & \\
\hline
 & & & & + \\
\hline
 & & - & & \\
\hline
 & + & & &
\end{array}\right].
$$

The matrix A has a special property in that there is exactly one nonzero in each row and in each column, as in the proof of Lemma 1.2. Thus A is a *signed permutation matrix*, that is, a permutation matrix in which some of the 1's have been replaced with -1's. In order that $\mathcal{A}_n(u,v)$ contain a signed permutation matrix it is necessary and sufficient, as in this example, that u and v have the same number p of $+1$'s and so the same number $(n-p)$ of -1's. $\square$

Let u and v be vectors of ± 1's each containing p $(+1)$'s and $(n-p)$ (-1)'s. Let $A \in \mathcal{A}_n(u,v)$. Then the $+1$'s determine a $p \times p$ submatrix A_+ of A and the -1's determine the complementary $(n-p)\times(n-p)$ submatrix A_- of A. Let A_+ be $-P$ where P is a $p \times p$ permutation matrix, and let A_- be the $(n-p)\times(n-p)$ matrix Q where Q is a permutation matrix, and where all other entries of A equal 0. Then A is a signed permutation matrix in $\mathcal{A}_n(u,v)$, and every signed permutation matrix in $\mathcal{A}_n(u,v)$ arises in this way. Let L_p be the $p \times p$ *anti-identity* $(0,1)$ *matrix* (1's on the antidiagonal). If $P = L_p$ (so $A_+ = -L_p$) and $Q = I_{n-p}$ (so $A_- = I_{n-p}$), then the resulting signed permutation matrix is denoted by $I_n(u,v)$ and is called the (u,v)-*identity matrix.* If $P = I_p$ (so $A_+ = -I_p$) and $Q = L_{n-p}$ (so $A_- = L_{n-p}$) matrix, then the resulting signed permutation matrix is denoted by $L_n(u,v)$ and is called the (u,v)-*anti-identity matrix.*

Example 1.4. Continuing with Example 1.3, we get the (u, v)-identity matrix as given in

$$\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & + & - & 0 \\
\hline\hline
- & + & & & & & - \\
\hline
+ & & & & - & & + \\
\hline
- & & + & & & & - \\
\hline
+ & & & - & & & + \\
\hline
- & & & & & + & - \\
\hline\hline
0 & - & - & + & + & - & 0
\end{array}
\rightarrow
\left[\begin{array}{c|c|c|c|c}
+ & & & & \\
\hline
 & & & - & \\
\hline
 & + & & & \\
\hline
 & & - & & \\
\hline
 & & & & +
\end{array}\right] = I_5(u, v),$$

and the (u, v)-anti-identity matrix as given in

$$\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & + & - & 0 \\
\hline\hline
- & & & & & + & - \\
\hline
+ & & & - & & & + \\
\hline
- & & + & & & & - \\
\hline
+ & & & & - & & + \\
\hline
- & + & & & & & - \\
\hline\hline
0 & - & - & + & + & - & 0
\end{array}
\rightarrow
\left[\begin{array}{c|c|c|c|c}
 & & & & + \\
\hline
 & & - & & \\
\hline
 & + & & & \\
\hline
 & & & - & \\
\hline
+ & & & &
\end{array}\right] = L_5(u, v).$$

For any $m \times n$ matrix $A = [a_{ij}]$, the *sum-matrix* (also called *corner matrix* in [11]) $\Sigma(A) = [\sigma_{ij}]$ of A is the $m \times n$ matrix where

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{k \le i} \sum_{l \le j} a_{kl} \quad (1 \le i \le m, 1 \le j \le n),$$

the sum of the entries in the leading $i \times j$ submatrix of A. Define $\sigma_{ij} = 0$ if $i = 0$ or $j = 0$. Then the matrix A is uniquely determined by its sum-matrix $\Sigma(A)$, namely,

$$a_{ij} = \sigma_{ij} - \sigma_{i,j-1} - \sigma_{i-1,j} + \sigma_{i-1,j-1} \quad (1 \le i, j \le n).$$

The sum-matrix $\Sigma(A) = [\sigma_{ij}]$ of a matrix $A \in \mathcal{A}_n(u, v)$ has the following properties which are easily verified:

- The entries in row i and column i of $\Sigma(A)$ are taken from the set $\{0, \pm 1, \pm 2, \ldots, \pm i\}$.
- Consecutive entries in a row or column differ in absolute value by at most 1.
- $\sigma_{in} = -(v_1 + v_2 + \cdots + v_i)$ for $1 \le i \le m$.
- $\sigma_{mj} = -(u_1 + u_2 + \cdots + u_j)$ for $1 \le j \le n$.

Example 1.5. Let $u = v = (1, 1, 1, 1)$ so that $-I_4 \in \mathcal{A}_4(u, v)$. Then

$$\Sigma(-I_4) = \left[\begin{array}{c|c|c|c}
-1 & -1 & -1 & -1 \\
\hline
-1 & -2 & -2 & -2 \\
\hline
-1 & -2 & -3 & -3 \\
\hline
-1 & -2 & -3 & -4
\end{array}\right].$$

Now let $u = (1, -1, -1, 1)$ and $v = (-1, -1, 1, 1)$. Then

$$A = \left[\begin{array}{c|c|c|c} & 1 & & \\ \hline & & 1 & \\ \hline -1 & & & \\ \hline & & & -1 \end{array}\right] \in \mathcal{A}_4(u, v) \text{ and } \Sigma(A) = \left[\begin{array}{c|c|c|c} 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 2 & 2 \\ \hline -1 & 0 & 1 & 1 \\ \hline -1 & 0 & 1 & 0 \end{array}\right].$$

Finally, let $u = v = (1, -1, 1, -1, 1)$, a palindromic vector. Then the matrices

$$A = \left[\begin{array}{c|c|c|c|c} -1 & 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 & 1 \\ \hline -1 & 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 & 1 \\ \hline -1 & 1 & -1 & 1 & -1 \end{array}\right] \in \mathcal{A}_5(u, v)$$

and

$$\Sigma(A) = \left[\begin{array}{c|c|c|c|c} -1 & 0 & -1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 & -1 \end{array}\right]$$

are both *palindromic matrices*, that is read the same from left-to-right and right-to-left, and from top-to-bottom and bottom-to-top, equivalently, each of the rows and each of the columns is palindromic. $\Box$

We now briefly summarize the contents of this paper. In the next section we generalize the Bruhat order on the set $\mathcal{S}_n$ of permutations of order n (equivalently, the set $\mathcal{P}_n$ of $n \times n$ permutation matrices) and the set $\mathcal{A}_n$ of $n \times n$ ASMs to the set $\mathcal{A}_n(u, v)$. As with $\mathcal{A}_n$ we obtain a ranked lattice. In the following section we investigate the rank of $\mathcal{A}_n(u, v)$ and show that $\mathcal{A}_n$ has the largest rank and determine the smallest rank for a given n. We illustrate our work with many examples.

2. Bruhat order

There is a partial order, called the *Bruhat order* and denoted by $\preceq_B$, defined on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$, equivalently, the set $\mathcal{P}_n$ of $n \times n$ permutation matrices, which has also been extended to the set $\mathcal{A}_n$ of $n \times n$ ASMs [9]. We briefly describe this partial order in its various equivalent forms.

(i) For $\pi, \tau \in \mathcal{S}_n$, $\pi \preceq_B \tau$ provided π can be obtained from τ by a sequence of transpositions each of which reduces the number of inversions, not necessarily the set of inversions. There is such a sequence of transpositions each of which reduces the number of inversions by exactly one (but not, in general, by removing one inversion). The identity $\iota_n = (1, 2, \ldots, n)$ (the identity matrix I_n using $\mathcal{P}_n$) is the unique minimal permutation in the Bruhat order on $\mathcal{S}_n$; the anti-identity $\zeta_n = (n, \ldots, 2, 1)$ (the anti-identity matrix L_n using $\mathcal{P}_n$) is the unique maximal permutation.

In terms of permutation matrices, for $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if P can be obtained from Q by a sequence of *interchanges* involving 2×2 submatrices (not necessarily with consecutive rows and consecutive columns):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) Another characterization of the Bruhat order is: For $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if $\Sigma(P) \geq \Sigma(Q)$ (entrywise).

(iii) The Bruhat order extends to $\mathcal{A}_n$ by defining for $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$. Then $(\mathcal{A}_n, \preceq_B)$ is a graded lattice extending the partially ordered set $(\mathcal{P}_n, \preceq_B)$, and indeed is the (unique up to isomorphism) smallest lattice extending $(\mathcal{P}_n, \preceq_B)$ (the *Dedekind-MacNeille completion* of $(\mathcal{P}_n, \preceq_B)$) [9]. The minimal element of $(\mathcal{A}_n \preceq_B)$ is I_n, and the maximal element is L_n.

(iv) For $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ if and only if there is a sequence of ASMs, $X_1 = A_1, X_2, \ldots, X_p = A_2$, such that for $t = 1, 2, \ldots, p-1$, X_t can be obtained from X_{t+1} by an *interchange* which adds to a 2×2 submatrix $X_{t+1}[i, j|k, l]$ $(i < j, k < l)$ of X_{t+1} the 2×2 matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

In order that the resulting matrices be ASMs, the 2×2 submatrix of X_{t+1} must equal

$$\left[\begin{array}{c|c} 0 \text{ or } -1 & 0 \text{ or } 1 \\ \hline 0 \text{ or } 1 & 0 \text{ or } -1 \end{array}\right].$$

An interchange is the analogue for ASMs of a transposition of a permutation matrix.

(v) Let $P, Q \in \mathcal{P}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{P}_n$ is defined by $P \preceq_b Q$ provided P can be obtained from Q by a sequence of *adjacent interchanges*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

applied to 2×2 submatrices with consecutive rows and consecutive columns. Since interchanges reduce the number of inversions of the corresponding permutations by exactly 1, $(\mathcal{P}_n, \preceq_b)$ is a subpartially ordered set of $(\mathcal{P}_n, \preceq_B)$ and, in fact, $(\mathcal{P}_n, \preceq_b)$ is a lattice. This lattice can be equivalently described in terms of the corresponding permutations in $\mathcal{S}_n$ as follows: For $\pi_1, \pi_2 \in \mathcal{S}_n$, $\pi_1 \preceq_b \pi_2$ if and only if the set $\mathrm{inv}(\pi_1)$ of inversions of π_1 is a subset of the set $\mathrm{inv}(\pi_2)$ of inversions of π_2. In general, $\mathrm{inv}(\pi_1) \cap \mathrm{inv}\pi_2$ is the set of inversions of a unique permutation $\pi_1 \wedge \pi_2$, the *meet* of π_1 and π_2, and $\mathrm{inv}(\pi_1) \cup \mathrm{inv}\pi_2$ is the set of inversions of a unique permutation $\pi_1 \vee \pi_2$, the *join* of π_1 and π_2 [2].

(vi) Let $A_1, A_2 \in \mathcal{A}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{A}_n$ can be defined by writing $A_1 \preceq_b A_2$ provided A_1 can be obtained from A_2 by a sequence of *adjacent interchanges* each of which adds to a 2×2 submatrix $A_2[i, i+1|j, j+1]$ of A_2 with consecutive rows and columns, the $n \times n$ matrix $T_{i,j}$ which is all 0's except for its 2×2 submatrix determined by rows i and $i+1$ and columns j and $j+1$ which equals

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and where the result is also an ASM. Unlike for $\mathcal{P}_n$, the Bruhat order and weak Bruhat order coincide on $\mathcal{A}_n$. (A different definition of weak Bruhat order on $\mathcal{A}_n$ is given in [8]; with this definition, the weak Bruhat order differs from the Bruhat order on $\mathcal{A}_n$ for $n \geq 3$ and indeed is not a lattice.)

These and other properties of ASMs can be found in several places including [2, 4, 5, 7, 9].

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be vectors of ± 1's where u and v have the same number of $+1$'s and the same number of -1's. We now consider extending the Bruhat order on $\mathcal{A}_n$ to the class $\mathcal{A}_n(u, v)$ of (u, v)-ASMs. There are three possibilities for defining $A_1 \preceq_B A_2$ for $A_1, A_2 \in \mathcal{A}_n(u, v)$, namely:

(a) $\Sigma(A_1) \geq \Sigma(A_2)$ (entrywise). (The immediate generalization of the Bruhat order on $\mathcal{A}_n$.)

(b) A_1 can be obtained from A_2 by a sequence of interchanges. (Known to be equivalent to (a) for $\mathcal{A}_n$.)

(c) A_1 can be obtained from A_2 by a sequence of adjacent interchanges. (The weak-Bruhat order condition which is known to be equivalent to (b) for $\mathcal{A}_n$.)

We shall show that these three possibilities are also equivalent for $\mathcal{A}_n(u, v)$. Thus each determines the Bruhat order on $\mathcal{A}_n(u, v)$, and it will be shown that this Bruhat order on $\mathcal{A}_n(u, v)$ is a lattice, indeed a graded lattice with a well-defined rank function. We clearly have that (c) implies (b), and it is easy to check that (b) implies (a). Thus we have only to show that (a) implies (c) to obtain the equivalence.

We prove the following theorem using (c) above.

Theorem 2.1. *Let A_1 and A_2 be in $\mathcal{A}_n(u, v)$. Then there exists a unique $M \in \mathcal{A}_n(u, v)$ such that*

$$\Sigma(M) = \max\{\Sigma(A_1), \Sigma(A_2)\} \ (\textit{entrywise maximum}).$$

Moreover, A_1 and A_2 can be obtained from M by a sequence of adjacent interchanges.

Before proving this theorem we give some of its consequences.

Corollary 2.2. *The three possibilities* (a), (b), *and* (c) *are equivalent for $\mathcal{A}_n(u,v)$ and each defines the Bruhat order $\preceq_B$ on $\mathcal{A}_n(u,v)$.*

Proof. Let $A_1, A_2 \in \mathcal{A}_n(u,v)$ with $\Sigma(A_1) \geq \Sigma(A_2)$. Then $\max\{\Sigma(A_1), \Sigma(A_2)\} = \Sigma(A_1)$, and by Theorem 2.1 A_1 can be obtained from A_2 by a sequence of adjacent interchanges. The corollary now follows from the discussion preceding Theorem 2.1. $\square$

Note that this corollary implies that, as with ordinary ASMs, there is no difference between the Bruhat order and the weak Bruhat order on $\mathcal{A}(u,v)$.

Corollary 2.3. *The partially ordered set $(\mathcal{A}_n(u,v), \preceq_B)$ is a distributive lattice where for $A_1, A_2 \in \mathcal{A}_n(u,v)$, the meet and join are given by*

(i) *$A_1 \wedge A_2 = B$ where B is the matrix in $\mathcal{A}_n(u,v)$ such that $\Sigma(B) = \max\{\Sigma(A_1), \Sigma(A_2)\}$.*
(ii) *$A_1 \vee A_2 = C$ where C is the matrix in $\mathcal{A}_n(u,v)$ such that $\Sigma(C) = \min\{\Sigma(A_1), \Sigma(A_2)\}$.*

Proof. By Theorem 2.1 $\mathcal{A}_n(u,v)$ has a well-defined meet and assertion (i) follows. Assertion (ii) can be obtained from the analogue of Theorem 2.1 with minimum replacing maximum; it also follows from the fact that $\mathcal{A}_n(u,v)$ is finite and so the existence of joins follows from the existence of meets.

In order that $(\mathcal{A}_n(u,v), \preceq_B)$ be distributive, we must have

$$A_1 \wedge (A_2 \vee A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}_n(u,v)$. The distributive property for the real numbers with the usual $\leq$ order relation holds trivially. It then follows from (i) and (ii) that $(\mathcal{A}_n(u,v), \preceq_B)$ is also distributive. $\square$

Corollary 2.4. *The (u,v)-identity matrix $I_n(u,v)$ is the unique minimal element in $\mathcal{A}_n(u,v)$, and the (u,v)-anti-identity matrix $L_n(u,v)$ is the unique maximal element in $\mathcal{A}_n(u,v)$.*

Proof. For $1 \leq k \leq n$, let $u^+(1,2,\ldots,k)$ equal the number of $+1$'s in $u_1, u_2, \ldots, u_k$ and let $u^+(k+1,k+2,\ldots,n)$ be the number of $+1$'s in $u_{k+1}, u_{k+2}, \ldots, u_n$. Similar notations are used for the number of -1's, and also for v. For $1 \leq e, f \leq n$, the number of $+1$'s in the leading $e \times f$ submatrix of $I_n(u,v)$ is

$$\alpha^+ = \min\{v^-(1,2,\ldots,e), u^-(1,2,\ldots,f)\}$$

and the number of -1's in the leading $e \times f$ submatrix of $I_n(u,v)$ is

$$\alpha^- = v^+(1,2,\ldots,e) - \min\{v^+(1,2,\ldots,e), u^+(f+1,f+2,\ldots,n)\}.$$

Hence for $\Sigma(I_n(u,v)) = [\sigma_{ij}]$, $\sigma_{ef} = \alpha^+ - \alpha^-$. Clearly, the sum matrix of any matrix $A \in \mathcal{A}_n(u,v)$ not equal to $I_n(u,v)$ is entrywise less than or equal to σ_{ef}

for all e and f with strict inequality for at least one e, f. A similar calculation can be done for $L_n(u, v)$. $\square$

For a matrix $A \in \mathcal{A}_n(u, v)$, let $\rho'(A)$ equal the sum of the entries of $\Sigma(A)$, and let $\rho(A) = \rho'(I_n(u, v)) - \rho'(A)$.

Corollary 2.5. *The partially ordered set* $(\mathcal{A}_n(u, v), \preceq_B)$ *is a graded lattice where the rank of* $A \in (\mathcal{A}_n(u, v), \preceq_B)$ *equals* $\rho(A)$.

Proof. By Corollary 2.3 $(\mathcal{A}_n(u, v), \preceq_B)$ is a lattice. Let $A_1, A_2 \in \mathcal{A}_n(u, v)$ where A_2 covers A_1. Then $\Sigma(A_1) - \Sigma(A_2) \geq 0$ and $A_1 \wedge A_2 = A_1$. By Theorem 2.1, A_1 can be obtained from A_2 by a sequence of adjacent interchanges. Each interchange increases the sum of the entries of the corresponding Σ-matrix by 1 and gives a matrix $A' \in \mathcal{A}_n(u, v)$ with $A_1 \preceq_B A' \prec_B A_2$. Since A_2 covers A_1, we have that $A_1 = A'$ and $\rho'(A_1) = \rho'(A_2) + 1$. Hence A_1 can be obtained from A_2 by one adjacent interchange and $\rho(A_1) = \rho(A_2) - 1$. Hence $(\mathcal{A}_n(u, v), \preceq_B)$ is graded with rank function $\rho(\cdot)$ where $\rho(I_n(u, v)) = 0$ and $\rho(L_n(u, v)) = \rho'(I_n(u, v)) - \rho'(L_n(u, v))$. $\square$

We now prove Theorem 2.1.

Proof of Theorem 2.1. Let $A_1 = [a^1_{ij}], A_2 = [a^2_{ij}] \in \mathcal{A}_n(u, v)$, and let $D = [d_{ij}] = \Sigma(A_1) - \Sigma(A_2)$, an integral matrix with last row and last column all 0's. If $D = 0$, then $A_1 = A_2$ and there is nothing to prove. So assume that $D \neq 0$.

If $u_j = +1$ then, since the ± 1's alternate, the partial sum of column j of a matrix in $\mathcal{A}_n(u, v)$ down to a row i is either 0 or -1; if $u_j = -1$, this partial sum is either 0 or 1. Let $k \geq 1$ be the smallest integer such that row k of D is nonzero. Then the following *k-partial column sum property* holds: *The partial column sums of* A_1 *and* A_2 *down to row* $(k-1)$ *are equal.*

It thus follows that row k of D is a $(0, \pm 1)$-vector.

Let $l_1 \geq 1$ be the smallest integer such that $d_{kl_1} \neq 0$. Then $d_{kl_1} = \pm 1$, and since we may interchange A_1 and A_2, we may assume that $d_{kl_1} = +1$. Let $l_2 \geq l_1$ be the smallest integer such that $d_{kj} = 1$ for $l_1 \leq j \leq l_2$ and $d_{k,l_2+1} \neq 1$. Since $d_{kn} = 0$, such an l_2 exists. The k-partial column sum property implies that $d_{k,l_2+1} = 0$. Thus we have either

Case (i) $a^1_{kl_1} = a^2_{k,l_2+1} = +1$ and $a^2_{kl_1} = a^1_{k,l_2+1} = 0$, or

Case (ii) $a^1_{kl_1} = a^2_{k,l_2+1} = 0$ and $a^2_{kl_1} = a^1_{k,l_2+1} = -1$.

We only argue the first case (i) with the argument for the second case (ii) being very similar. So assume (i) holds.

Consider the sequences determined by columns l_1 and l_2 of A_2:

$$u_{l_1}, a^2_{1l_1}, a^2_{2l_1}, \ldots, a^2_{k-1,l_1}, (a^2_{k,l_1} + 1)$$

and

$$u_{l_2+1}, a^2_{1,l_2+1}, a^2_{2,l_2+1}, \ldots, a^2_{k-1,l_2+1}, (a^2_{k,l_2+1} - 1).$$

Since $a^1_{kl_1} = 1$ and $a^2_{k,l_1} = 0$, the $+1$'s and -1's in the first of these sequences alternate. Since $a^1_{k,l_2+1} = 0$ and $a^2_{k,l_2+1} = +1$, the $+1$'s and -1's in the second of these sequences alternate. Thus there exists a q with $l_1 \le q \le l_2$ such that in the sequences

$$u_q, a^2_{1q}, a^2_{2q}, \ldots, a^2_{k-1,q}, (a^2_{k,q}+1) \text{ and}$$
$$u_{q+1}, a^2_{1,q+1}, a^2_{2,q+1}, \ldots, a^2_{k-1,q+1}, (a^2_{k,q+1}-1)$$

the $+1$'s and -1's alternate. We choose the smallest such q.

Now consider the alternating sequences

$$v_i, a^2_{i1}, a^2_{i2}, \ldots, a^2_{iq} \text{ for } i > k. \tag{2}$$

Suppose that the last nonzero entry each of these sequences is -1 for all $i > k$. Then for each such i we compute

$$\begin{aligned} d_{iq} &= \Sigma(A_1)_{iq} - \Sigma(A_2)_{iq} \\ &= \left(\Sigma(A_1)_{i-1,q} + \sum_{j=1}^{q} a^1_{ij}\right) - \left(\Sigma(A_2)_{i-1,q} + \sum_{j=1}^{q} a^2_{ij}\right) \\ &= (\Sigma(A_1)_{i-1,q} - \Sigma(A_2)_{i-1,q}) + \left(\sum_{j=1}^{q} a^1_{ij} - \sum_{j=1}^{q} a^2_{ij}\right) \\ &= d_{i-1,q} + \sum_{j=1}^{q} a^1_{ij} - \sum_{j=1}^{q} a^2_{ij}. \end{aligned}$$

Since the last nonzero of each of the sequences (2) is -1, we have that

$$\sum_{j=1}^{q} a^1_{ij} \ge \sum_{j=1}^{q} a^2_{ij}.$$

Therefore $d_{iq} \ge d_{i-1,q}$ and so the sequence $d_{kq}, d_{k+1,q}, \ldots, d_{nq}$ is nondecreasing. This contradicts the fact that $d_{kq} = 1$ and $d_{nq} = 0$. Let $p(\ge k)$ be the smallest integer such that the last nonzero of the sequence $v_{p+1}, a^2_{p+1,1}, a^2_{p+1,2}, \ldots, a^2_{p+1,q}$ is $+1$ and let $A^1_2 = A_2 + T_{p,q}$, where $T_{p,q}$ is the matrix which is all zeros except for its 2×2 submatrix, determined by rows p and $p+1$ and columns q and $q+1$, equal to

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then $A^1_2 \in \mathcal{A}_n(u,v)$ such that $D - \big(\Sigma(A_1) - \Sigma(A^1_2)\big)$ has just one nonzero entry $+1$.

As already indicated, Case (ii) proceeds in a very similar way. Thus continuing, we obtain for some k_1 and k_2, that there are adjacent interchanges $A_1 \to A^1_1 \to A^2_1 \to \cdots \to A^{k_1}_1$ and $A_2 \to A^1_2 \to A^2_2 \to \cdots \to A^{k_2}_2$, with $A^{k_1}_1 = A^{k_2}_2$ and $\Sigma(A^{k_1}_1) = \Sigma(A^{k_2}_2) = \max\{\Sigma(A_1), \Sigma(A_2)\}$. $\square$

Example 2.6. We give an example to illustrate the proof of Theorem 2.1. Let A_1 and A_2 be determined, respectively, by:

$$
\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & + & & & & & - \\ \hline
- & & & & + & & - \\ \hline
- & & & & & + & - \\ \hline
+ & & & - & & & + \\ \hline
- & & + & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}
\quad \text{and} \quad
\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & + & & - \\ \hline
- & & & & & + & - \\ \hline
- & & + & & & & - \\ \hline
+ & & & - & & & + \\ \hline
- & + & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}.
$$

Then we have

$$
\Sigma(A_1) = \left[\begin{array}{c|c|c|c|c}
1 & 1 & 1 & 1 & 1 \\ \hline
1 & 1 & 1 & 2 & 2 \\ \hline
1 & 1 & 1 & 2 & 3 \\ \hline
1 & 1 & 0 & 1 & 2 \\ \hline
1 & 2 & 1 & 2 & 3
\end{array}\right], \quad
\Sigma(A_2) = \left[\begin{array}{c|c|c|c|c}
0 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 0 & 1 & 2 \\ \hline
0 & 1 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 1 & 2 \\ \hline
1 & 2 & 1 & 2 & 3
\end{array}\right],
$$

and

$$
D = \Sigma(A_1) - \Sigma(A_2) = \left[\begin{array}{c|c|c|c|c}
1 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 1 & 1 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

First, we find the first nonzero row in D (shaded below):

$$
D = \left[\begin{array}{c|c|c|c|c}
1 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 1 & 1 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

Then for $k = 1$ we can find $q = 2$ in A_2 such as

$$
A_2 : \begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & + & & - \\ \hline
- & & & & & + & - \\ \hline
- & & + & & & & - \\ \hline
+ & & & - & & & + \\ \hline
- & + & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}.
$$

Next we can find $p = 2$ in A_2 such as

$$\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & + & & - \\ \hline
- & & & & & + & - \\ \hline
- & & + & & & & - \\ \hline
+ & & & - & & & + \\ \hline
- & + & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}.$$

Then we have $A_2 + T_{2,2} = A_2^1 \in \mathcal{A}_n(u,v)$ such as

$$\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & + & & - \\ \hline
- & & & & & + & - \\ \hline
- & & + & & & & - \\ \hline
+ & & & - & & & + \\ \hline
- & + & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}
+
\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & & & - \\ \hline
- & & + & - & & & - \\ \hline
- & & - & + & & & - \\ \hline
+ & & & & & & + \\ \hline
- & & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}$$

$$=
\begin{array}{c||c|c|c|c|c||c}
0 & - & - & + & - & - & 0 \\ \hline\hline
- & & & & + & & - \\ \hline
- & & + & - & & + & - \\ \hline
- & & & + & & & - \\ \hline
+ & & & - & & & + \\ \hline
- & + & & & & & - \\ \hline\hline
0 & - & - & + & - & - & 0
\end{array}.$$

Thus $A_2 \to A_2^1$ where

$$D - \left(\Sigma(A_1) - \Sigma(A_2^1)\right) = \left[\begin{array}{c|c|c|c|c}
0 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0
\end{array}\right].$$

In the following example we construct the matrix M of Theorem 2.1.

Example 2.7. Let $u = v = (-1, -1, +1, -1)$ and consider the matrices A_1 and A_2 in $\mathcal{A}_4(u)$:

$$A_1 = \left[\begin{array}{c|c|c|c}
 & + & - & + \\ \hline
+ & - & + & \\ \hline
 & & - & \\ \hline
 & + & &
\end{array}\right], \quad A_2 = \left[\begin{array}{c|c|c|c}
+ & & & \\ \hline
 & & & + \\ \hline
- & + & - & \\ \hline
+ & & &
\end{array}\right].$$

Then we have

$$\Sigma(A_1) = \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 0 & 1 \\ \hline 1 & 2 & 1 & 2 \end{array}\right], \ \Sigma(A_2) = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 2 & 1 & 2 \end{array}\right],$$

and

$$\max\{\Sigma(A_1), \Sigma(A_2)\} = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 0 & 1 \\ \hline 1 & 2 & 1 & 2 \end{array}\right].$$

The matrix M such that $\Sigma(M) = \max\{\Sigma(A_1), \Sigma(A_2)\}$ is

$$M = \left[\begin{array}{c|c|c|c} + & & & \\ \hline & & & + \\ \hline & & - & \\ \hline & + & & \end{array}\right].$$

Example 2.8. We construct the Bruhat order of $(\mathcal{A}_4(u,v), \preceq_B)$ when $u = (-1,+1,+1,-1)$ and $v = (+1,-1,+1,-1)$. Its Hasse diagram is illustrated in Figure 1. The matrices labelled in Figure 1 are identified in Figure 2. The four unshaded entries in Figure 1 are the signed permutation matrices in $\mathcal{A}_n(u,v)$.

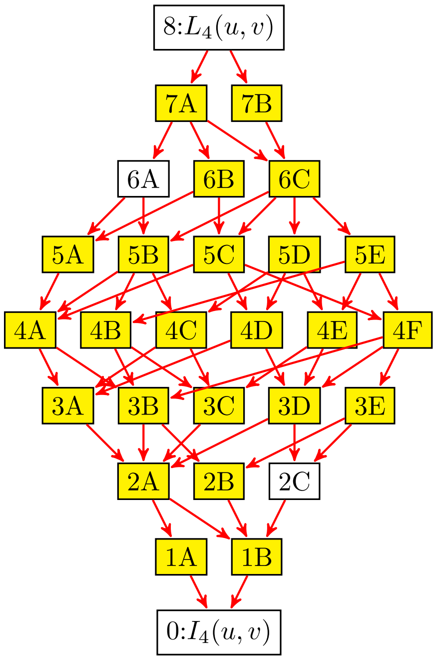

Figure 1. Hasse diagram of $(\mathcal{A}_4(u,v), \preceq_B)$ with $u = (-1,+1,+1,-1)$ and $v = (+1,-1,+1,-1)$.

The following are the matrices in the Hasse diagram:

$1A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & & & \\ \hline - & & & \\ \hline + & - & & + \end{array}\right]$$

$1B$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & + & \\ \hline & & - & \\ \hline & & & + \end{array}\right]$$

$2A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & + & \\ \hline - & + & - & \\ \hline + & - & & + \end{array}\right]$$

$2B$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & & + \\ \hline & & & - \\ \hline & & & + \end{array}\right]$$

$2C$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & & \\ \hline & & - & \\ \hline & & & + \end{array}\right]$$

$3A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & + & \\ \hline - & & & \\ \hline + & & - & + \end{array}\right]$$

$3B$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & & + \\ \hline - & + & & - \\ \hline + & - & & + \end{array}\right]$$

$3C$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline & & + & \\ \hline & & - & \\ \hline + & - & & + \end{array}\right]$$

$3D$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & & \\ \hline - & + & - & \\ \hline + & - & & + \end{array}\right]$$

$3E$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & - & + \\ \hline & & & - \\ \hline & & & + \end{array}\right]$$

$4A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & & + \\ \hline - & & + & - \\ \hline + & & - & + \end{array}\right]$$

$4B$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline & & & + \\ \hline & & & - \\ \hline + & - & & + \end{array}\right]$$

$4C$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline & & + & \\ \hline & - & & \\ \hline + & & - & + \end{array}\right]$$

$4D$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & & \\ \hline - & & & \\ \hline + & & - & + \end{array}\right]$$

$4E$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & + & & \\ \hline & & - & \\ \hline + & - & & + \end{array}\right]$$

$4F$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & - & + \\ \hline - & + & & - \\ \hline + & - & & + \end{array}\right].$$

$5A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline + & - & & + \\ \hline - & & & \\ \hline + & & & \end{array}\right]$$

$5B$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline & & & + \\ \hline & - & + & - \\ \hline + & & - & + \end{array}\right]$$

$5C$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & - & + \\ \hline - & & + & - \\ \hline + & & - & + \end{array}\right]$$

$5D$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & + & & \\ \hline & - & & \\ \hline + & & - & + \end{array}\right]$$

$5E$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & + & - & + \\ \hline & & & - \\ \hline + & - & & + \end{array}\right]$$

$6A$
$$\left[\begin{array}{c|c|c|c} & & - & \\ \hline & & & + \\ \hline & - & & \\ \hline + & & & \end{array}\right]$$

$6B$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline + & & - & + \\ \hline - & & & \\ \hline + & & & \end{array}\right]$$

$6C$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & + & - & + \\ \hline & - & + & - \\ \hline + & & - & + \end{array}\right]$$

$7A$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & + & - & + \\ \hline & - & & \\ \hline + & & & \end{array}\right]$$

$7B$
$$\left[\begin{array}{c|c|c|c} & - & & \\ \hline & & & + \\ \hline & & & - \\ \hline + & & - & + \end{array}\right]$$

Figure 2. The matrices in $(\mathcal{A}_4(u,v), \preceq_B)$ with $u = (-1,+1,+1,-1)$ and $v = (+1,-1,+1,-1)$.

Recall that $T_{i,j}$ is the matrix which is all zeros except for its 2×2 submatrix determined by row i and $i+1$ and columns j and $j+1$ equal to

$$\left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array}\right].$$

Let $\mathcal{T}_n$ be the set of all such $n \times n$ $T_{i,j}$. For $A_1, A_2 \in \mathcal{A}_n(u,v)$, we write $A_1 \xleftarrow{(i,j)} A_2$ provided $A_1 = A_2 + T_{i,j}$. Thus $A_1 \leftarrow A_2$ denotes that $A_1 \xleftarrow{(i,j)} A_2$ for some (i,j), that is, that A_2 covers A_1 in $(\mathcal{A}_n(u,v), \preceq_B)$.

Lemma 2.9. *For $A_1 = [a^1_{ij}] \in \mathcal{A}_n(u,v)$, if there exist C_1 and C_2 such that*

$$C_1 \xleftarrow{(p_1,q_1)} A_1 \text{ and } C_2 \xleftarrow{(p_2,q_2)} A_1$$

(so A_1 covers C_1 and C_2 in the Bruhat order),

then there exists a unique $A_2 \in \mathcal{A}_n(u,v)$ such that

$$A_2 \xleftarrow{(p_2,q_2)} C_1 \text{ and } A_2 \xleftarrow{(p_1,q_1)} C_2$$

(so C_1 and C_2 cover A_2 in the Bruhat order).

Proof. Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Without loss of generality, we may assume that $p_1 \le p_2$. Suppose $p_2 = p_1 + 1$ and $q_1 = q_2$. Since $C_1, C_2 \in \mathcal{A}_n(u,v)$, the $+1$'s and -1's in the following sequences satisfies the *alternating property*

$$v_{p_1+1}, a^1_{p_1+1,1}, a^1_{p_1+1,2}, \ldots, (a^1_{p_1+1,q_1} - 1), (a^1_{p_1+1,q_1+1} + 1),$$

and

$$v_{p_1+1}, a^1_{p_1+1,1}, a^1_{p_1+1,2}, \ldots, (a^1_{p_1+1,q_1} + 1), (a^1_{p_1+1,q_1+1} - 1).$$

It is a contradiction. A similar contradiction occurs if $p_1 = p_2$ and $|q_1 - q_2| = 1$. The remainder of the proof is divided into three cases.

Case 1 : $p_2 > p_1 + 1$ or $|q_1 - q_2| > 1$.

In this case, there is no intersection between the submatrices

$$\begin{bmatrix} a^1_{p_1,q_1} + 1 & a^1_{p_1,q_1+1} - 1 \\ a^1_{p_1+1,q_1} - 1 & a^1_{p_1+1,q_1+1} + 1 \end{bmatrix} \text{ and } \begin{bmatrix} a^1_{p_2,q_2} + 1 & a^1_{p_2,q_2+1} - 1 \\ a^1_{p_2+1,q_2} - 1 & a^1_{p_2+1,q_2+1} + 1 \end{bmatrix}.$$

Therefore $A_2 = C_1 + T_{p_2,q_2}$ satisfies the alternating property. Since

$$A_2 = C_1 + T_{p_2,q_2} = (A_1 + T_{p_1,q_1}) + T_{p_2,q_2} = (A_1 + T_{p_2,q_2}) + T_{p_1,q_1} = C_2 + T_{p_1,q_1},$$

and since $\mathcal{A}_n(u,v)$ is a lattice, A_2 is the unique matrix in $\mathcal{A}_n(u,v)$ which is covered by C_1 and C_2.

Case 2 : $p_2 = p_1 + 1$ and $q_2 = q_1 + 1$.

Since $C_1, C_2 \in \mathcal{A}_n(u,v)$, we have $a^1_{p_2,q_2} = 0$ or -1. Suppose $a^1_{p_2,q_2} = 0$. Since the sequence

$$u_{q_2}, a^1_{1,q_2}, a^1_{2,q_2}, \ldots, (a^1_{p_1,q_2} - 1), (a^1_{p_2,q_2} + 1), a^1_{p_2+1,q_2}$$

satisfy the alternating property, the last nonzero entry in the sequence

$$u_{q_2}, a^1_{1,q_2}, a^1_{2,q_2}, \ldots, a^1_{p_1,q_2}$$

should be $+1$. Then the sequence

$$u_{q_2}, a^1_{1,q_2}, a^1_{2,q_2}, \ldots, a^1_{p_1,q_2}, (a^1_{p_2,q_2} + 1), (a^1_{p_2+1,q_2} - 1)$$

does not satisfy the alternating property because of $a^1_{p_2,q_2} + 1 = 0 + 1 = +1$. Therefore we have $a^1_{p_2,q_2} = -1$. Consider the sequences:

$$u_{q_2}, a^1_{1,q_2}, a^1_{2,q_2}, \ldots, (a^1_{p_1,q_2} - 1), (a^1_{p_2,q_2} + 1 + 1), (a^1_{p_2+1,q_2} - 1)$$

and

$$v_{p_2}, a^1_{p_2,1}, a^1_{p_2,2}, \ldots, (a^1_{p_2,q_1} - 1), (a^1_{p_2,q_2} + 1 + 1), (a^1_{p_2,q_2+1} - 1).$$

These sequences satisfy the alternating property. Therefore $A_2 = C_1 + T_{p_2,q_2}$ satisfies the alternating property. Since

$$A_2 = C_1 + T_{p_2,q_2} = (A_1 + T_{p_1,q_1}) + T_{p_2,q_2} = (A_1 + T_{p_2,q_2}) + T_{p_1,q_1} = C_2 + T_{p_1,q_1},$$

we conclude as in Case 1 that there is a unique $A_2 \in \mathcal{A}_n(u,v)$ which is covered by C_1 and C_2.
Case 3 : $p_2 = p_1 + 1$ and $q_1 = q_2 + 1$. The argument here is as in Case 2. $\square$

In terms of the Hasse diagram of $(\mathcal{A}_n(u,v), \preceq_B)$, Lemma 2.9 asserts that if there are C_1 and C_2 such that

$$\begin{tikzcd} & \boxed{A_1} \arrow[dl, "(p_1,q_1)"'] \arrow[dr, "(p_2,q_2)"] & \\ \boxed{C_1} & & \boxed{C_2} \end{tikzcd}$$

then there exists a unique A_2 such that

$$\begin{tikzcd} & \boxed{A_1} \arrow[dl, "(p_1,q_1)"'] \arrow[dr, "(p_2,q_2)"] & \\ \boxed{C_1} \arrow[dr, "(p_2,q_2)"'] & & \boxed{C_2} \arrow[dl, "(p_1,q_1)"] \\ & \boxed{A_2} & \end{tikzcd}$$

Let u and v be such that $\mathcal{A}_n(u,v) \neq \emptyset$. The Hasse diagram of $(\mathcal{A}_n(u,v), \preceq_B)$ can be considered as a directed graph $\mathcal{D}_{u,v} := \mathcal{D}(\mathcal{A}_n(u,v), \preceq_B)$ whose set of vertices is $\mathcal{A}_{u,v}$ with an arc $A_1 \to A_2$ from A_1 to A_2 if and only if A_1 covers A_2. Let $d^-(A)$ and $d^+(A)$ be the indegree and the outdegree of A in $\mathcal{D}_{u,v}$. The *total-degree* $d(A)$ of A is

$$d(A) = d^-(A) + d^+(A).$$

These degrees were investigated in [7] in the case of $(\mathcal{A}_n, \preceq_B)$.
As a corollary of Lemma 2.9, we have the following.

Corollary 2.10. *Let $A \in \mathcal{A}_n(u,v)$ be such that $d^+(A) = k$. Then the number of paths of length 2 in the directed graph $\mathcal{D}_{u,v}$ that begin with A is at least $\binom{k}{2}$.*

Example 2.11. Let $u = v = (-1,-1,+1,-1)$. In Figure 3 the Hasse diagram of $(\mathcal{A}_4(u), \preceq_B)$ is illustrated. The matrices labeled in Figure 3 are identified in

Figure 4. Let

$$A = \left[\begin{array}{c|c|c|c} & + & - & + \\ \hline + & - & + & \\ \hline - & + & - & \\ \hline + & & & \end{array}\right] \in \mathcal{A}_4(u).$$

Using adjacent interchanges $T_{(i,j)}$, we see that $d^+(A) = 5$ where

$$A \xrightarrow{(1,1)} \left[\begin{array}{c|c|c|c} + & & - & + \\ \hline & & + & \\ \hline - & + & - & \\ \hline + & & & \end{array}\right] \quad (6:SC),$$

$$A \xrightarrow{(1,3)} \left[\begin{array}{c|c|c|c} & + & & \\ \hline + & - & & + \\ \hline - & + & - & \\ \hline + & & & \end{array}\right] \quad (6:LB),$$

$$A \xrightarrow{(3,1)} \left[\begin{array}{c|c|c|c} & + & - & + \\ \hline + & - & + & \\ \hline & & - & \\ \hline & + & & \end{array}\right] \quad (6:RB),$$

$$A \xrightarrow{(3,3)} \left[\begin{array}{c|c|c|c} & + & - & + \\ \hline + & - & + & \\ \hline - & + & & - \\ \hline + & & - & + \end{array}\right] \quad (6:SB),$$

$$A \xrightarrow{(2,2)} \left[\begin{array}{c|c|c|c} & + & - & + \\ \hline + & & & \\ \hline - & & & \\ \hline + & & & \end{array}\right] \quad (6:SA).$$

In Figure 3, the matrix A is the matrix labeled as (7:SA) and

$$|\{C|A \to C' \to C\}| = \binom{5}{2} = 10.$$

We make the following observations using this example: The cardinality $|\mathcal{A}_4(u)| = 52 > |\mathcal{A}_4| = 42$. The maximum length of a chain in the poset $(\mathcal{A}_n(u,v), \preceq_B)$ (that is, the rank of the poset) is the sum of all entries of $\Sigma(I_n(u,v)) - \Sigma(L_n(u,v))$. In this case with $u = v = (-1,-1,+1,-1)$, this maximum cardinality equals 9 while it equals 10 for $\mathcal{A}_4$ (see Figure 5 taken from [7] with the labeling described there).

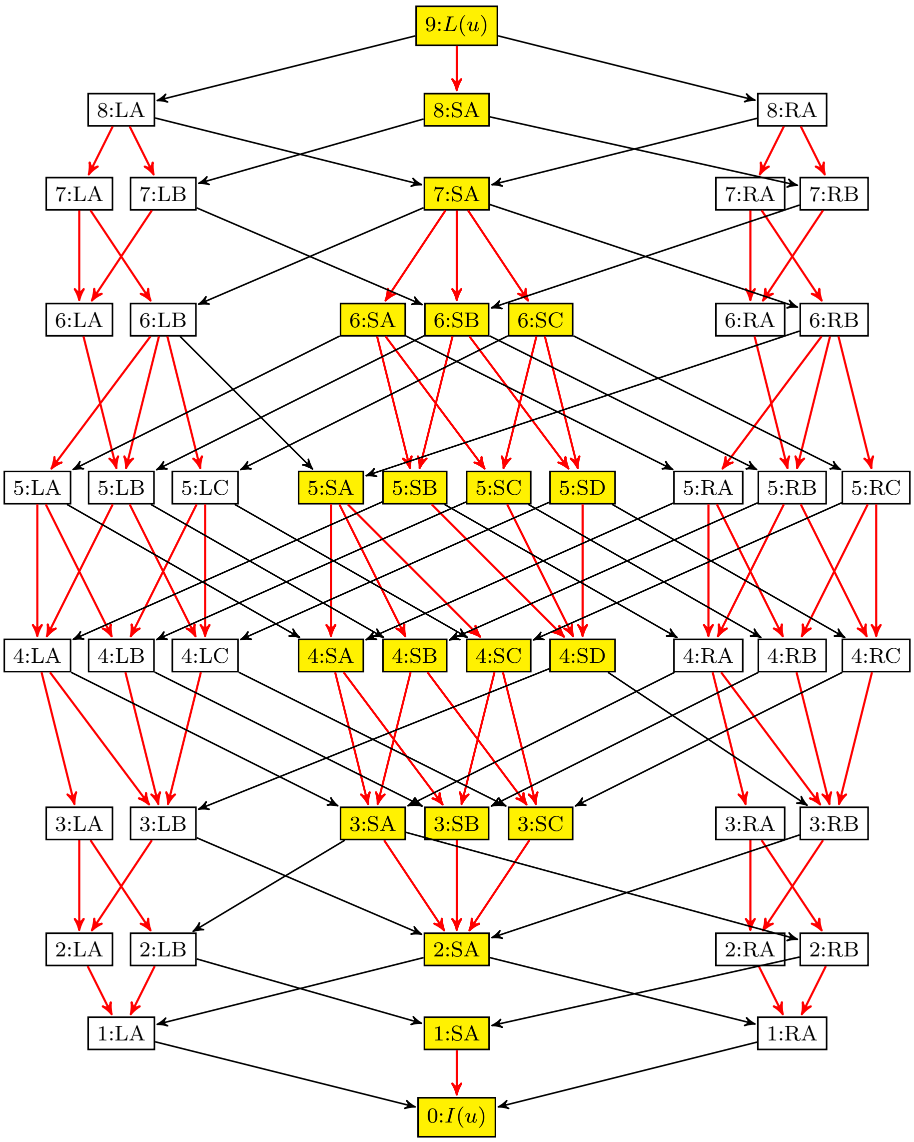

Figure 3. Hasse diagram of $(\mathcal{A}_4(u), \preceq_B)$ for $u = (-1, -1, +1, -1)$.

The matrices in the Hasse diagram drawn in Figure 3 are given in Figure 4. In Figure 3, the matrices shaded in the center are all symmetric. The matrices on the right are transposes of the corresponding matrices on the left.

$1 : LA$	$2 : LA$	$2 : LB$	$3 : LA$
$\begin{bmatrix} + & & & \\ & + & & \\ & - & & \\ & + & - & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & + & & \\ - & & & \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & & \\ & - & & \\ & + & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & & \\ - & & & \\ + & & - & + \end{bmatrix}$
$3 : LB$	$4 : LA$	$4 : LB$	$4 : LC$
$\begin{bmatrix} + & & & \\ & + & - & + \\ - & & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & - & + \\ - & & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & + & - & + \\ - & & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & & & + \\ - & + & & - \\ + & & - & + \end{bmatrix}$
$5 : LA$	$5 : LB$	$5 : LC$	$6 : LA$
$\begin{bmatrix} & + & & \\ + & & - & + \\ - & & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & - & & + \\ - & + & & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ & & & + \\ - & + & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & & \\ & & & + \\ & & & - \\ + & & - & + \end{bmatrix}$
$6 : LB$	$7 : LA$	$7 : LB$	$8 : LA$
$\begin{bmatrix} & + & & \\ + & - & & + \\ - & + & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & & \\ & & & + \\ & & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ & & + & \\ & & & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ & & + & \\ & & - & \\ + & & & \end{bmatrix}$
$0 : I(u)$	$1 : SA$	$2 : SA$	$3 : SA$
$\begin{bmatrix} + & & & \\ & + & & \\ & & - & \\ & & & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & & \\ & & - & \\ & & & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & + & - & + \\ & - & + & - \\ & + & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & - & + \\ & - & + & - \\ & + & - & + \end{bmatrix}$.
$3 : SB$	$3 : SC$	$4 : SA$	$4 : SB$
$\begin{bmatrix} + & & & \\ & + & - & + \\ & - & & \\ & + & & \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & & & + \\ & & & - \\ & + & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & & - & + \\ & - & & \\ & + & & \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & - & & + \\ & & & - \\ & + & - & + \end{bmatrix}$
$4 : SC$	$4 : SD$	$5 : SA$	$5 : SB$
$\begin{bmatrix} + & & & \\ & & & + \\ & & - & \\ & + & & \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ & + & & \\ - & & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & & \\ + & - & & + \\ & & - & \\ & + & & \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ + & & & \\ - & & + & - \\ + & & - & + \end{bmatrix}$
$5 : SC$	$5 : SD$	$6 : SA$	$6 : SB$
$\begin{bmatrix} + & & - & + \\ & + & & \\ - & & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ & & + & \\ - & + & & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ + & & & \\ - & & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ + & - & + & \\ - & + & & - \\ + & & - & + \end{bmatrix}$
$6 : SC$	$7 : SA$	$8 : SA$	$9 : L(u)$
$\begin{bmatrix} + & & - & + \\ & & + & \\ - & + & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & + & - & + \\ + & - & + & \\ - & + & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & + & & \\ & & & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & + & & \\ & & - & \\ + & & & \end{bmatrix}$

Figure 4. The matrices in $(\mathcal{A}_4(u,v), \preceq_B)$ with $u = v = (-1,-1,+1,-1)$.

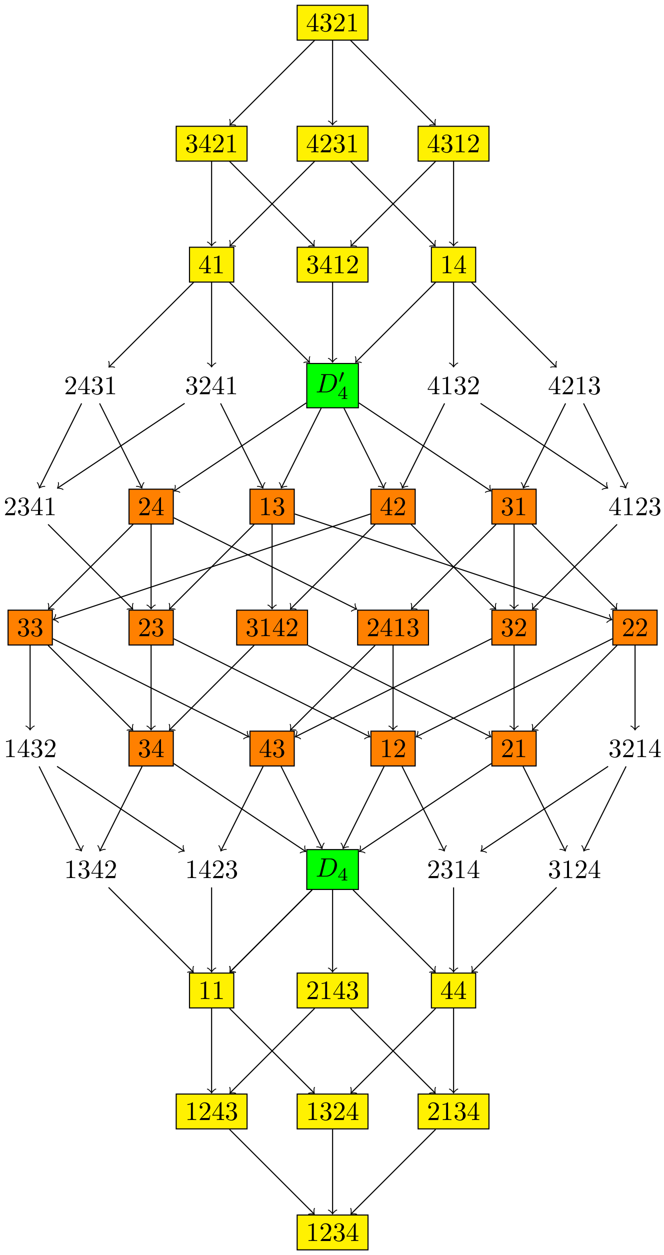

Figure 5. Hasse diagram of $(\mathcal{A}_4, \preceq_B)$.

Example 2.12. In Figure 6 we exhibit the Hasse diagram of $(\mathcal{A}_4(u), \preceq_B)$ for the palindromic vector $u = (-1, +1, +1, -1)$.

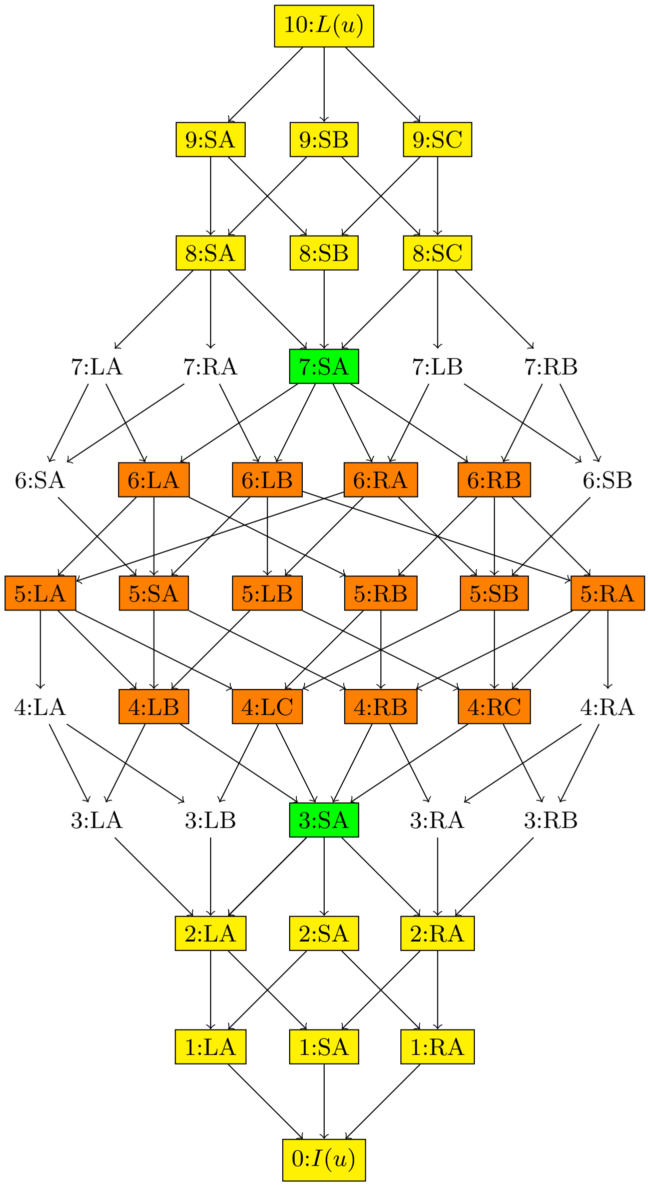

Figure 6. Hasse diagram of $(\mathcal{A}_4(u), \preceq_B)$ where $u = (-1, +1, +1, -1)$

The poset whose Hasse diagram is shown in Figure 6 is isomorphic to the poset $(\mathcal{A}_4, \preceq_B)$ whose Hasse diagram is shown in Figure 5. The coloring of the vertices in Figure 6 has been chosen to highlight this isomorphism. The isomorphism is an example of the general observation made in Example 2.11. The matrices in the Hasse diagram in Figure 6 are given in Figure 7, with the non-highlighted matrices on the right in Figure 6 equal to the transposes of those on the left.

$1:LA$	$1:SA$	$2:LA$	$2:SA$
$\begin{bmatrix} + & & & \\ & & - & \\ - & & & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & - & & \\ & & - & \\ & & & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ & - & & \\ - & + & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ & & & - \\ - & & & \\ + & - & & + \end{bmatrix}$
$3:LA$	$3:LB$	$3:SA$	$4:LA$
$\begin{bmatrix} + & & & \\ & - & & \\ - & & & \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ - & & & \\ & & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ & - & + & - \\ - & + & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & & \\ - & & & \\ & - & & \\ + & & - & + \end{bmatrix}$
$4:LB$	$4:LC$	$5:LA$	$5:LB$
$\begin{bmatrix} + & & - & + \\ & - & + & - \\ - & & & \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ - & & + & - \\ & & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ - & & + & - \\ & - & & \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & & + \\ & & & - \\ - & & & \\ + & & - & + \end{bmatrix}$
$5:SA$	$5:SB$	$6:LA$	$6:LB$
$\begin{bmatrix} + & & - & + \\ & - & & \\ - & & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & & + \\ - & + & & - \\ & & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ - & & & \\ & - & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & & + \\ & & - & \\ - & & + & - \\ + & & - & + \end{bmatrix}.$
$6:SA$	$6:SB$	$7:LA$	$7:LB$
$\begin{bmatrix} + & & - & + \\ & - & & \\ - & & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & & & - \\ & & - & \\ + & - & & + \end{bmatrix}$	$\begin{bmatrix} + & & - & + \\ - & & & \\ & - & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & & & - \\ & - & & \\ + & & - & + \end{bmatrix}$
$7:SA$	$8:SA$	$8:SB$	$8:SC$
$\begin{bmatrix} + & - & & + \\ - & + & - & \\ & - & + & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & & + \\ - & + & - & \\ & - & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} + & - & & + \\ - & & & \\ & & & - \\ + & & - & + \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & & - & \\ & - & + & - \\ + & & - & + \end{bmatrix}$
$9:SA$	$9:SB$	$9:SC$	
$\begin{bmatrix} + & - & & + \\ - & & & \\ & & - & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & & - & \\ & - & & \\ + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + \\ & - & & \\ & & & - \\ + & & - & + \end{bmatrix}$	

Figure 7. The matrices in $(\mathcal{A}_4(u), \preceq_B)$ with $u = (-1, +1, +1, -1)$.

We conclude this section with the following example and observation.

Example 2.13. Let $u = (u_1, u_2, \ldots, u_{n-1}, u_n)$ be the palindromic vector where $u_1 = u_n$ and $u_2 = \cdots = u_{n-1}$, and $v = u$. If $u_1 = u_n = -1$ and $u_2 = \cdots = u_{n-1} = -1$, then $\mathcal{A}_n(u) = \mathcal{A}_n$. If $u_1 = +1$ and $u_2 = \cdots = u_{n-1} = +1$, then $\mathcal{A}_n(u) = -\mathcal{A}_n$, the set of negatives of matrices in $\mathcal{A}_n$. The other two cases are obtained by taking $u_1 = +1$ and $u_2 = -1$, and $u_1 = -1$ and $u_2 = +1$, where the two sets of matrices obtained are negatives of each other. So we need only consider $\mathcal{A}_n(u)$ where $u = (+1, -1, \ldots, -1, +1)$. This class of matrices is invariant under a 90 degree rotation. The four corners of matrices in $\mathcal{A}_n(u)$, up to rotations, determine one of the following 2×2 matrices:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

In each case by adding

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

to this corner submatrix we obtain a bijection $A \to A'$ with $\mathcal{A}_n$. Moreover, since the only change is in the corners, it is easy to see that for $A_1, A_2 \in \mathcal{A}_n(u)$, $\Sigma(A_1) \le \Sigma(A_2)$ if and only if $\Sigma(A_1') \le \Sigma(A_2')$. Hence this bijection is a lattice isomorphism between $\mathcal{A}_n(u)$ and $\mathcal{A}_n$. □

3. Rank of the lattice $(\mathcal{A}_n(u, v), \preceq_B)$

Let $\rho(\cdot)$ denote the rank function of the graded lattice $(\mathcal{A}_n(u, v), \preceq_B)$. Since $L_n(u, v)$ is the unique maximum element of this lattice, the rank of the lattice $(\mathcal{A}_n(u, v), \preceq_B)$ equals $\rho(L_n(u, v))$. Moreover, for $A \in \mathcal{A}_n(u, v)$, we have

$$\rho(A) = \rho'(I_n(u, v)) - \rho'(A) = \rho'(I_n(u, v) - A),$$

where recall that $\rho'(\cdot)$ denotes the sum of the entries of $\Sigma(\cdot)$. We have $\rho(\mathcal{A}_n) = \binom{n+1}{3}$, see e.g. [7].

The following lemma is an elementary observation.

Lemma 3.1. *Let $D = [d_{ij}] = \Sigma(I_n - L_n)$. For $i \le j$, we have*

$$d_{ij} = \min\{i, n - j\}.$$

We first show that the largest rank of the lattices $(\mathcal{A}_n(u, v), \preceq_B)$ occurs in the classical case $(\mathcal{A}_n, \preceq_B)$.

Lemma 3.2. *Let u and v be $(-1, +1)$-vectors of order n where u and v contain the same number of $+1$'s. Then*

$$\rho(\mathcal{A}_n(u, v)) \le \rho(\mathcal{A}_n) = \binom{n+1}{3}.$$

Proof. Let $D_n(u, v) = I_n(u, v) - L_n(u, v)$ and $\Sigma(D_n(u, v)) = [d(u, v)_{ij}]$. In case, $u = v$ is an all -1 vector, we write $D_n = I_n - L_n$ and $\Sigma(D_n) = [d_{ij}]$. We claim that $\Sigma(D_n(u, v)) \le \Sigma(D_n)$ (entrywise). Each row of $D_n(u, v)$ has at most one $+1$ and at most one -1, and each row sum is 0. In $D_n(u, v)$ consider the leading $i \times j$ submatrix M_1 and its complementary $i \times (n - j)$ submatrix M_2 in the upper right corner. Without loss of generality we assume that $i \le j$. The sum of all of the entries of M_1 is $d(u, v)_{ij}$ and the sum of all entries of M_2 is $-d(u, v)_{ij}$, since each row sum of $D_n(u, v) = I_n(u, v) - L_n(u, v)$ is 0. We consider two possibilities:

(i) $i \le n - j$.
The maximum number of $+1$'s in M_1 is i, since each row contains at most one $+1$, and thus $d(u, v)_{ij} \le i$.

(ii) $i > n - j$.
The maximum number of -1's in M_2 is $(n-j)$, since each row contains at most one -1, and thus $-d(u,v)_{ij} \geq -(n-j)$. Hence we have $d(u,v)_{ij} \leq n - j$.

By Lemma 3.1, we have

$$d(u,v)_{ij} \leq d_{ij} = \min\{i, n-j\}.$$

Therefore,

$$\rho(\mathcal{A}_n(u,v)) \leq \rho(\mathcal{A}_n) = \binom{n+1}{3}. \qquad \square$$

Lemma 3.3. *Let $u = (u_1, u_2, \ldots, u_n)$ where for some k with $1 \leq k < n$, $u_1 = \cdots = u_k = -1$ and $u_{k+1} = \ldots = u_n = +1$. Then*

$$\mathcal{A}_n(u) = \{A_1 \oplus A_2 : A_1 \in \mathcal{A}_k \text{ and } -A_2 \in \mathcal{A}_{n-k}\}.$$

Therefore

$$\rho(\mathcal{A}_n(u)) = \binom{k+1}{3} + \binom{n-k+1}{3}.$$

Proof. Consider $A \in \mathcal{A}_n(u)$ and partition A as follows:

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} \text{ where } A_1 \text{ is } k \times k.$$

Then since $u_1 = \cdots = u_k = -1$, the row sums of A_{12} are all ≥ 0. Since $u_{k+1} = \cdots = u_n = +1$ the column sums of A_{12} are all ≤ 0. Suppose that $A_{12} \neq 0$, and consider the first nonzero row of A_{12} and its first nonzero x. Then $x = -1$ and hence x must be followed by a $+1$ in its row. In the column above this $+1$ there must be a -1, a contradiction. Thus $A_{12} = 0$ and similarly $A_{21} = 0$. The formula for rank now follows. $\square$

We next show that the smallest rank of the lattices $(\mathcal{A}_n(u,v), \preceq_B)$ occurs in the special cases in Lemma 3.3

Lemma 3.4. *Let u and v be $(-1,+1)$-vectors of order n where u and v contain k -1's. Let $w = (w_1, w_2, \ldots, w_n)$ where $w_1 = \cdots = w_k = -1$ and $w_{k+1} = \cdots = w_n = +1$. Then*

$$\rho(\mathcal{A}_n(u,v)) \geq \rho(\mathcal{A}_n(w)) = \binom{k+1}{3} + \binom{n-k+1}{3}.$$

Proof. For $A = [a_{ij}] \in \mathcal{A}_n(u,v)$, let $A^+ = [a^+_{ij}]$ be the $n \times n$ $(0,+1)$ matrix where $a^+_{ij} = +1$ if $a_{ij} = +1$ and $a^+_{ij} = 0$ otherwise. Let $A^- = [a^-_{ij}]$ be the $n \times n$ $(0,-1)$ matrix where $a^-_{ij} = -1$ if $a_{ij} = -1$ and $a^-_{ij} = 0$ otherwise. Then $A^+ + A^- = A$. It can be checked that

$$\rho'(I^+_n(u,v) - L^+_n(u,v)) \geq \rho'(I^+_n(w) - L^+_n(w))$$

and

$$\rho'(I^-_n(u,v) - L^-_n(u,v)) \geq \rho'(I^-_n(w) - L^-_n(w)).$$

We also have

$$\begin{aligned}
&\Sigma(I_n(u,v)) - L_n(u,v)) \\
&= \Sigma(I_n(u,v)) - \Sigma(L_n(u,v)) \\
&= \Sigma(I_n(u,v)^+ + I_n(u,v)^-) - \Sigma(L_n(u,v)^+ + L_n(u,v)^-) \\
&= \Sigma(I_n(u,v)^+ - L_n(u,v)^+) + \Sigma(I_n(u,v)^- - L_n(u,v)^-).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\rho'(I_n(u,v)) - L_n(u,v)) \\
&= \rho'(I_n(u,v)^+ - L_n(u,v)^+) + \rho'(I_n(u,v)^- - L_n(u,v)^-) \\
&\geq \rho'(I_n(w)^+ - L_n(w)^+) + \rho'(I_n(w)^- - L_n(w)^-) \\
&= \rho'(I_n(w)^+ + I_n(w)^-) - \rho'(L_n(w)^+ + L_n(w)^-) \\
&= \rho'(I_n(w)) - L_n(w)).
\end{aligned}$$

By Lemma 3.3,

$$\rho(\mathcal{A}_n(u,v)) \geq \rho(\mathcal{A}_n(w)) = \binom{k+1}{3} + \binom{n-k+1}{3}. \qquad \square$$

Theorem 3.5. *Let u and v be $(+1,-1)$ vectors of order n Then*

$$\rho(\mathcal{A}_n(u,v)) \geq \binom{\lfloor \frac{n}{2} \rfloor + 1}{3} + \binom{\lceil \frac{n}{2} \rceil + 1}{3}.$$

Proof. Let u and v be $(-1,+1)$-vectors of order n where u and v contain k -1's where $0 \leq k \leq n$. By Lemma 3.4

$$\rho(\mathcal{A}_n(u,v)) \geq \binom{k+1}{3} + \binom{n-k+1}{3}.$$

It is a simple calculus exercise to show that $\binom{k+1}{3} + \binom{n-k+1}{3}$ is a minimum when $k = \lfloor \frac{n}{2} \rfloor$ $\square$

The case of $(\mathcal{A}_n(u), \preceq_B)$ where u is palindromic is special.

Corollary 3.6. *If u is palindromic,*

$$\rho(\mathcal{A}_n(u)) = \rho(\mathcal{A}_n).$$

Proof. We have that $I_n(u,u) - L_n(u,u) = I_n - L_n$ for all palindromic u, and hence the result follows. $\square$

The lattice $(\mathcal{A}_n, \preceq_B)$ is isomorphic to the lattice $(\mathcal{A}_n(u), \preceq_B)$ when u is a palindromic vector of the form considered in Example 2.11 but this does not hold for all palindromic u as shown in Figures 8 and 9 where parts of the Hasse diagram are given for $u = (-1,-1,-1,-1,-1)$ and $u = (-1,+1,-1,+1,-1)$, respectively. There we see that $L_5(-1,+1,-1,+1,-1)$ covers a matrix which itself covers 5 other matrices while no such matrices exist for $L_5(-1,-1,-1,-1,-1)$.

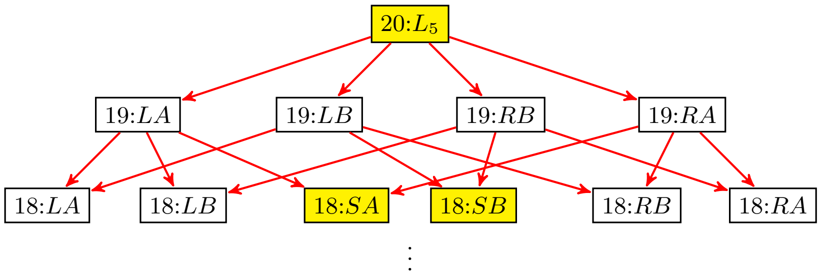

Figure 8. Hasse diagram of $(\mathcal{A}_5(u), \preceq_B)$ when $u = (-1, -1, -1, -1, -1)$.

The following are the matrices in the Hasse diagram:

$20 : L_5$	$19 : LA$	$19 : LB$
$\begin{bmatrix} & & & & + \\ & & & + & \\ & & + & & \\ & + & & & \\ + & & & & \end{bmatrix}$	$\begin{bmatrix} & & & + & \\ & & & & + \\ & & + & & \\ & + & & & \\ + & & & & \end{bmatrix}$	$\begin{bmatrix} & & & & + \\ & & + & & \\ & & & + & \\ & + & & & \\ + & & & & \end{bmatrix}$
$19 : RB$	$19 : RA$	$18 : LA$
$\begin{bmatrix} & & & & + \\ & & & + & \\ & + & & & \\ & & + & & \\ + & & & & \end{bmatrix}$	$\begin{bmatrix} & & & & + \\ & & & + & \\ & & + & & \\ + & & & & \\ & + & & & \end{bmatrix}$	$\begin{bmatrix} & & & + & \\ & & + & - & + \\ & & & + & \\ & + & & & \\ + & & & & \end{bmatrix}$
$18 : LB$	$18 : SA$	$18 : SB$
$\begin{bmatrix} & & & + & \\ & & & & + \\ & + & & & \\ & & + & & \\ + & & & & \end{bmatrix}$	$\begin{bmatrix} & & & + & \\ & & & & + \\ & & + & & \\ + & & & & \\ & + & & & \end{bmatrix}$	$\begin{bmatrix} & & & & + \\ & & + & & \\ & + & - & + & \\ & & + & & \\ + & & & & \end{bmatrix}$
$18 : RB$	$18 : RA$	
$\begin{bmatrix} & & & & + \\ & & + & & \\ & & & + & \\ + & & & & \\ & + & & & \end{bmatrix}$	$\begin{bmatrix} & & & & + \\ & & & + & \\ & + & & & \\ + & - & + & & \\ & + & & & \end{bmatrix}$	$\cdots$

.

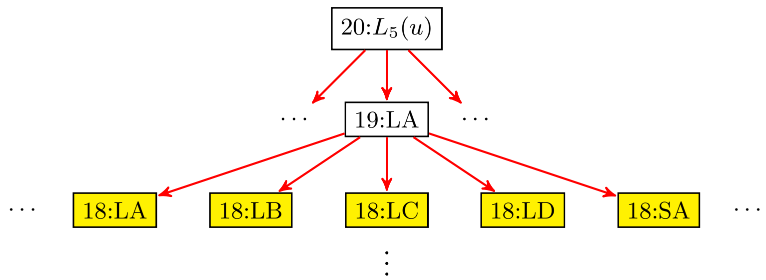

Figure 9. Hasse diagram of $(\mathcal{A}_5(u), \preceq_B)$ when $u = (-1, +1, -1, +1, -1)$.

The following are the matrices in the Hasse diagram:

$$\begin{array}{ccc}
19 : LA & 18 : LA & 18 : LB \\
\begin{bmatrix} & & & & + \\ & - & + & - & \\ & & & + & \\ & & & - & \\ + & & & & \end{bmatrix} &
\begin{bmatrix} + & - & & & + \\ - & & + & - & \\ & & & + & \\ & & & - & \\ + & & & & \end{bmatrix} &
\begin{bmatrix} & & + & - & + \\ & - & & & \\ & & & + & \\ & & & - & \\ + & & & & \end{bmatrix} \\
18 : LC & 18 : LD & 18 : SA \\
\begin{bmatrix} & & & & + \\ & - & + & & - \\ & & & & + \\ & & & - & \\ + & & & & \end{bmatrix} &
\begin{bmatrix} & & & & + \\ & - & + & - & \\ & & & + & \\ & & & & - \\ + & & & - & + \end{bmatrix} &
\begin{bmatrix} & & & & + \\ & - & + & - & \\ & + & - & + & \\ & - & + & - & \\ + & & & & \end{bmatrix}
\end{array}.$$

We conclude with a problem. By Corollary 2.3, $(\mathcal{A}_n(u,v), \preceq_B)$ is a distributive lattice. A *join-irreducible element* in a distributive lattice L is a nonzero element that cannot be expressed as the join of two elements below it. Let $\mathcal{J}$ be the set of join-irreducible elements of L. Identifying an element of L with the join-irreducible elements below it, the distributive lattice L is isomorphic to the lattice of subsets of $\mathcal{J}$ partially ordered by set-inclusion. Thus it is of interest to determine the join-irreducible elements of the lattices $(\mathcal{A}_n(u,v), \preceq_B)$.

In Example 2.8 the set of join-irreducible elements (those that have only one arrow going down) is $\mathcal{J} = \{1A, 1B, 2B, 2C, 3A, 3C, 5A, 7B\}$. It seems difficult to identify in general the join-irreducible elements of $(\mathcal{A}_n(u,v), \preceq_B)$ but it would be of interest to do so for special u and v.

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Richard A. Brualdi
Department of Mathematics
University of Wisconsin
Madison, WI 53706, USA
Email address: brualdi@math.wisc.edu

Hwa Kyung Kim
Department of Mathematics Education
Sangmyung University
Seoul 03016, Korea
Email address: indices@smu.ac.kr