ABSOLUTE SUMMABILITY FACTORS FOR CESÀRO AND RIESZ MEANS

MEHMET ALI SARIGÖL

ABSTRACT. In this paper we characterize the sets of summability factors $(|C, 0|_k, |R, p_n|_s)$ and $(|R, p_n|_k, |C, 0|_s)$, $1 < k \leq s < \infty$, which also extends some known results.

AMS Mathematics Subject Classification : 40C05, 40D25, 40F05, 46A45.
Key words and phrases : Absolute Cesàro and Riesz summability, summability factors, equivalence theorems, matrix transformations.

1. Introduction

One of research areas in the theory of summability is absolute summability factors and comparison of the methods, which plays an important roles in Fourier analysis and Approximation theory, and has been widely examined by many authors up to now. On this topic, Bosanquet and Das [4] therein is an important resource. First we recall related definitions. Let $\Sigma x_n$ be an infinite series with partial sum $s_n$. By $(t^\alpha_n)$ denote the n-th Cesàro means of order $\alpha$ with $\alpha > -1$ of the sequences $(s_n)$. By Flett’s notation (see [5]), the series $\Sigma x_n$ is called summable $|C, \alpha|_k$, $k \geq 1$, if

$$
\sum_{n=1}^{\infty} n^{k-1} |t^\alpha_n - t^\alpha_{n-1}|^k < \infty.
$$

Also, we note that the method $|C, 0|_k$ reduces to

$$
\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty.
$$

The sequence-to-sequence transformation

$$
V_n = \frac{1}{R_n} \sum_{n=0}^{n} r_n s_n. \quad (1.1)
$$
defines the sequence \((V_n)\) of the \((R, r_n)\) Riesz means of the sequence \((s_n)\), generated by the sequence of coefficients \((r_n)\), where \((r_n)\) be a sequence of positive real constants with \(R_n = r_0 + r_1 + \cdots + r_n \to \infty\) as \(n \to \infty\). The series \(\Sigma x_n\) is called summable \(|R, r_n|_k\), \(k \geq 1\), if (see [14])

\[
\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k < \infty.
\]

Let \(X\) and \(Y\) be two summability methods. The set of summability factors \((X, Y)\) is defined by the set of all sequences \(\lambda = (\lambda_v)\) such that \(\Sigma \lambda_v x_v\) is summable \(Y\), whenever \(\Sigma x_v\) is summable \(X\). The set \((X, Y)\) was studied by various authors. For more information, we refer to Bosanquet and Das [4] therein. Further, for the special case \(\lambda = e = (1, 1, \ldots)\), this set reduces to inclusion relation \(X \Rightarrow Y\), which means that \(X\) includes \(Y\).

Throughout the paper, \(k^*\) denotes the conjugate of \(k > 1\), i.e., \(1/k^* + 1/k = 1\).

Inclusion problems dealing with absolute Cesàro and absolute Riesz mean summabilities of infinite series were studied by many authors (see, for instance, [2-14]). Hereof, the following result was established by Bor [2].

**Theorem 1.1.** Let \(1 < k < \infty\) and

\[
\sum_{n=v}^{\infty} n^{k-1} \frac{v^k}{R_v^k R_{v-1}^{k-1}} = O\left(\frac{v^{k-1} r_v^{k-1}}{R_v^{k-1}}\right).
\]

If

\[
R_{n+1} \geq d R_n
\]

where \(d\) is a constant such that \(d > 1\), then \(|C, 0|_k \Leftrightarrow |R, r_n|_k\).

We note that (1.2) is equivalent to \(P_n = O(p_n)\). In fact, \(P_n = O(p_n)\) holds if and only if there exists a constant \(M > 0\) such that, for all \(n \geq 1\), \(P_n/p_n \leq M\), or, equivalently, \(P_n \geq d P_{n-1}\), where \(d = M^{1/k} > 1\).

It is obvious for \(k = 1\) that this result is satisfied and also gives a Tauber condition (1.3) for the summability method \(|R, r_n|\).

Further, by omitting the condition (1.2), it has recently been shown in [12] that the condition (1.3) is not only sufficient but also necessary for the conclusion of Theorem 1.1 to hold as follows.

**Theorem 1.2.** Let \(1 < k \leq s < \infty\). Then, \(|C, 0|_k \Rightarrow |R, r_n|_s\) if and only if

\[
\left(\sum_{v=1}^{m} \frac{R_v^k}{R_v^{k-1}} \right)^{1/k^*} \left(\sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} r_n}{R_n R_{n-1}}\right)^{s} \right)^{1/s} = O(1).
\]

**Theorem 1.3.** Let \(1 < k \leq s < \infty\). Then, \(|R, r_n|_k \Rightarrow |C, 0|_s\) if and only if

\[
\left(\sum_{v=m-1}^{m} \frac{1}{v} \left(\frac{R_{v-1} R_v}{r_v}\right)^{k^*} \right)^{1/k^*} \left(\sum_{n=m}^{m+1} \frac{n^{s-1}}{R_n^s}\right)^{1/s} = O(1).
\]
Theorem 1.4. Let \( 1 \leq k < \infty \) Then, \(|C,0|_k \Leftrightarrow |R,r_n|_k\) if and only if the condition (1.3) is satisfied.

2. Main Results

The purpose of this paper is to characterize the summability sets \(|C,0|_k, |R, r_n|_s\) and \(|R, r_n|_k, |C, 0|_s\), for the case \(1 < k \leq s < \infty\), which also extend some well known results.

The set of all sequences consisting \(k\)-absolutely convergent series is denoted by \(\ell_k\).

A factorable matrix \(T\) is defined by

\[
t_{nv} = \begin{cases} b_n a_v, & 0 \leq v \leq n, \\ 0, & v > n. \end{cases}
\]

where \((b_n)\) and \((a_n)\) are sequences of real or complex numbers.

Now we prove the following theorems.

Theorem 2.1. Let \(1 < k \leq s < \infty\) and \(\lambda = (\lambda_v)\) be a sequence of numbers. Then, \(\lambda \in (|C,0|_k, |R,p_n|_s)\) if and only if

\[
\left( \sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v} |\lambda_v|^{k^*} \right)^{1/k^*} \left( \sum_{n=m}^{\infty} \left( \frac{n^{1/s} P_n}{P_n + p_{n-1}} \right)^{s} \right)^{1/s} = O(1). \tag{2.1}
\]

Theorem 2.2. Let \(1 < k \leq s < \infty\) and \(\lambda = (\lambda_n)\) be a sequence of numbers. Then, \(\lambda \in (|R,p_n|_k, |C,0|_s)\) if and only if

\[
\left( \sum_{v=m-1}^{m} \frac{1}{v} \left( P_{v-1} P_v \right)^{k^*} \right)^{1/k^*} \left( \sum_{n=m}^{m+1} \left| \frac{n^{1/s} \lambda_n}{P_n} \right|^{s} \right)^{1/s} = O(1). \tag{2.2}
\]

It may be noticed that Theorem 2.1-2.2 are, in the special case \(u = e\), reduced to Theorems 1.2-1.3, respectively.

Also, if we take \(r = \lambda = e\), then \(|R, r_n|_k = |C,1|_k\) and \(R_n = n + 1\). Further, since

\[
\sum_{n=m}^{\infty} \frac{1}{n(n+1)^{s}} = O\left( \frac{1}{m^{s}} \right),
\]

condition (2.1) holds but not condition (2.2). Therefore, the following result of Flett \[4\] is immediately deduced.

Corollary 2.3. Let \(1 < k \leq s < \infty\). Then, \(|C,0|_k \Rightarrow |C,1|_s\), but \(|C,1|_k \nleftrightarrow |C,0|_s\).
**Proof of Theorem 2.1.** We first note from a result of Bennett [1] that a factorable matrix $T$ defines a bounded linear operator $L_T : \ell_k \to \ell_s$ such that $L_T(x) = T(x)$ for all $x \in \ell_k$ iff

$$\left( \sum_{v=0}^{n} |a_v|^{k^*} \right)^{1/k^*} \left( \sum_{n=m}^{\infty} |b_n|^s \right)^{1/s} = O(1), \quad (2.3)$$

where $k^*$ is the conjugate of indices $k$. Now, let $\sigma^0_n$ and $T_n$ be Cesàro $(C,0)$ and Riesz means $(R, r_n)$ of the series $\sum x_n$ and $\sum \lambda_n x_n$, respectively. Then, by (2.1),

$$\sigma^0_n = \sum_{v=0}^{n} x_v$$

$$V_n = \frac{1}{R_n} \sum_{v=0}^{n} r_v \sum_{r=0}^{v} \lambda_r x_r$$

and so $\Delta V_0 = \lambda_0 x_0$.

$$\Delta V_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} \lambda_v x_v, \text{ for } n \geq 1.$$ 

Now, say $T'_n = n^{1/s^*} \Delta V_n$ and $\sigma'^0_n = n^{1/k^*} x_n$ for $n \geq 1$. Then, it can be written that

$$T'_n = \frac{n^{1/s^*} R_n R_{n-1}}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} \lambda_v \sigma'^0_v$$

where

$$t_{nv} = \begin{cases} 
\frac{n^{1/s^*} R_n R_{n-1} \lambda_v}{R_n R_{n-1} v^{1/k^*}} & 1 \leq v \leq n, \\
0 & v > n.
\end{cases}$$

This means that the consequence of the theorem holds iff $(T'_n) \in \ell_s$ for all $(\sigma'^0_n) \in \ell_k$, or, $T = (t_{nv}) : \ell_k \to \ell_s$ is a bounded operator. Thus, by applying (2.3) to the matrix $T$, we obtain (2.1).

**Proof of Theorem 2.2.** Let $V_n$ and $\sigma^0_n$ be means of Riesz $(R, r_n)$ and Cesàro $(C,0)$ of the series $\sum x_n$ and $\sum \lambda_n x_n$, respectively. Then, as above, $\Delta \sigma^0_n = \lambda_n x_n$, and also $\Delta V_0 = x_0$.

$$\Delta V_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} x_v, \text{ for } n \geq 1 \quad (2.4)$$

By inversion of (2.4), it can be stated that, for $n \geq 1$,

$$x_n = \frac{1}{R_{n-1}} \left( \frac{R_{n-1} R_n}{r_n} \Delta V_n - \frac{R_{n-1} R_{n-2}}{R_{n-1}} \Delta V_{n-1} \right)$$
Say $T'_n = n^{1/k^*} \Delta V_n$ and $\sigma^0_n = n^{1/s^*} \lambda_n x_n$ for $n \geq 1$. Then, it can be written that
\[
\sigma^0_n = \frac{n^{1/s^*} \lambda_n}{R_{n-1}} \left( \frac{R_{n-1} R_{n-2} T'_n}{n^{1/k^*} r_n} - \frac{R_{n-1} R_{n-2} T'_n}{(n-1)^{1/k^*} r_{n-1}} \right)
\]
\[
= \sum_{v=1}^{\infty} d_{nv} \sigma^0_v
\]
where
\[
D_{nv} = \begin{cases} 
\frac{n^{1/s^*} \lambda_n}{R_{n-1}} \left( - \frac{R_{n-1} R_{n-2} T'_n}{(n-1)^{1/k^*} r_{n-1}} \right), & v = n-1 \\
\frac{n^{1/s^*} \lambda_n}{R_{n-1}} \left( \frac{R_{n-1} R_{n-2} T'_n}{n^{1/k^*} r_n} \right), & v = n \\
0, & v > n.
\end{cases}
\]
The reminder is similarly proved.

References

2. H. Bor, A new result on the high indices theorem, Analysis 29 (2009), 403-405.

Mehmet Ali Sargol received his Ph.D. degree from Department of Mathematics, Ankara University in 1985, and has been studying in Pamukkale University in Denizli since 1994. His research interests include summability, sequence spaces, mathematical analysis, functional analysis, real analysis, Fourier analysis. He has more than 80 research papers published in the reputed international mathematics journals, and also serves as referee in many mathematical journals.
Department of Mathematics, Faculty of Art and Science, Pamukkale University, Denizli, 20160, Turkey.
E-mail: msarigol@pau.edu.tr