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최소 표현 라플라스 변환에 기초한 단계형 확률변수의 시뮬레이션에 관한 연구

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Simulation of the Phase-Type Distribution Based on the Minimal Laplace Transform

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ABSTRACT

The phase-type, PH, distribution is defined as the time to absorption into a terminal state in a continuous-time Markov chain. As the PH distribution includes family of exponential distributions, it has been widely used in stochastic models. Since the PH distribution is represented and generated by an initial probability vector and a generator matrix which is called the Markovian representation, we need to find a vector and a matrix that are consistent with given set of moments if we want simulate a PH distribution. In this paper, we propose an approach to simulate a PH distribution based on distribution function which can be obtained directly from moments. For the simulation of PH distribution of order 2, closed-form formula and streamlined procedures are given based on the Jordan decomposition and the minimal Laplace transform which is computationally more efficient than the moment matching methods for the Markovian representation. Our approach can be used more effectively than the Markovian representation in generating higher order PH distribution in queueing network simulation.

Key words : Phase-type distribution, moment matching, Jordan decomposition, Laplace transform

요 약

단계형 확률분포는 마코프 체인이 특정 상태로 흡수되는 시점까지 거쳐가는 여러 단계에서 체재하는 시간들의 합으로 정의되며 대기행렬 시스템과 신뢰성 분석 모형 등에 광범위하게 사용된다. 연속적 단계형 분포의 경우 흡수 상태로 진입하기까지 거쳐가는 각각의 단계에서의 체재 시간이 지수분포를 따르므로 연속적 단계형 분포는 다양한 지수분포들의 합 또는 볼록 결합으로 나타낼 수 있다. 단계형 분포를 생성하는 가장 일반적이면서도 직관적인 방법은 마코비안 표현방법이라 불리는 초기 확률벡터와 전이 생성행렬에 의해 주어지는 조건부 확률을 이용하는 것이다. 적률이 주어진 상황에서 단계형 변수를 생성하는 방법에 대한 기존의 연구들은 대부분 적률을 마코비안 표현방법으로 변환하는 것을 전제로 하고 있다. 본 연구에서는 적률을 마코비안 표현방법으로 변환하지 않고 확률 분포함수를 결정하여 단계형 확률변수를 생성하는 방법에 대해 살펴보고 마코프 표현을 사용하는 기존의 방법 대신에 조단 분해법과 최소 표현 라플라스 변환을 이용하여 2계 단계형 확률변수를 분포함수를 결정하는 공식과 절차를 제시한다. 이러한 접근 방법은 고차원의 단계형 확률분포를 이용하여 대기행렬의 시뮬레이션을 하는 경우에 마코비안 표현방법의 전이행렬을 결정하여 변수를 생성하는 경우보다 효율적이다.

주요어 : 단계형 확률변수, 적률에 의한 모수 결정, 조단 분해법, 라플라스 변환

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1. Introduction

A random variable with PH-type distribution can be generated by an initial probability vector α and a generator matrix $\mathbf{A}$. The pair $(\alpha, \mathbf{A})$ is called the Markovian representation and the dimension of the probability vector α and generator matrix $\mathbf{A}$ is the order of PH-type distribution. That is, if α is $1\times n$ and **A** is $n\times n$, then the order of PH-type distribution is n in which case we denote the PH-type distribution by PH (n). If there is no probability mass at zero, the Markovian representation (α, **A**) contains n^2+n-1 parameters whereas the minimal number of parameters is $2n-1$. That is, the number of independent moments for PH(2) is only three whereas the number of parameters in (α, **A**) is five. As n gets large, the difference between the number of parameters in (α, **A**) representation and the minimal number of parameters increases which makes it difficult to idenitfy the Markovian representation (α, **A**) by the moments matching; see Bobbio et al.[1], Johnson and Taaffe[2, 3, 4], O'Cinneide[6], Telek and Horváth[7, 8]. On the other hand, however, the moment matching with the Laplace transform (LT) is straightfor ward with the minimal form of LT which is available; see Kim[5].

In this paper, we consider the problem of simulating a PH(2) with probability distribution function (pdf) obtained from moments. Unlike previous studies on moment matching of PH distribution, we focus on the transformation of moments to the LT and then to the pdf. Since our approach does not involve the numerical transformation procedure from moments to the Markovian representation, computational procedure is much more streamlined than previous moment matching methods.

The paper is organized as follows. In Section 2, we review known results on the representation of PH(2)s. Then, we present our main result on the minimal LT of PH(2) in Section 3. The pdf of PH(2) by inversion of LT is given in Section 4 followed by the numerical examples given in Section 5. We conclude in Section 6 with discussions on future direction of research.

2. Preliminaries

2.1 Markovian representation (α, A)

A PH(n) is fully described by $1\times n$ vector $\alpha=(\alpha_1,...,\alpha_n)$ and $n\times n$ generator matrix **A** where the entries of α are non-negative. The vector becomes a probability vector if $\alpha_1+\cdots+\alpha_n=1$ in which case there is no probability mass at zero. In this paper, we only consider PH-type distribution with $\alpha_1+\cdots+\alpha_n=1$. Therefore, there are n^2+n-1 parameters in the Markovian representation (α, **A**) whereas the minimal number of parameters of a PH(n) with no probability mass at zero is $2n-1$. For PH(2), the transition rate matrix **A** is given in terms of four rate parameters $(\lambda_1,\lambda_{12},\lambda_{21},\lambda_2)$, i.e.

$$\mathbf{A}=\begin{bmatrix}-\lambda_1 & \lambda_{12}\\ \lambda_{21} & -\lambda_2\end{bmatrix}$$

where $\lambda_1\geq\lambda_{12}$ and $\lambda_2\geq\lambda_{21}$.

2.2 Probability density function and moments

Let X be a non-negative random variable with PH-type distribution (α, **A**) with no probability mass at zero. The pdf can be obtained as

$$f(x)=\boldsymbol{\alpha}\, e^{\boldsymbol{A}x}(-\boldsymbol{A})\,\boldsymbol{e} \qquad (2.1)$$

where e is a $n\times 1$ vector of ones. The k-th moment of X is given as

$$E(X^k)=k!\boldsymbol{\alpha}(-\mathrm{A})^{-k}\boldsymbol{e}.$$

Also, reduced moments are defined as

$$r_k=E(X^k)/k!$$

A PH(n) with no probability mass at zero is completely described by $2n-1$ moments; see Telek et al.[9] for minimal representation of PH(n) distribution. Therefore, the minimal number of parameters for a PH(2) is three. That is, three moments, e.g. (r_1,r_2,r_3), are needed for moment matching for PH(2).

2.3 Eigenvalues and Jordan representation

By the similarity transformation, the reduced moments can be determined in terms of the eigenvalues of $(-\boldsymbol{A})^{-1}$. Let $\boldsymbol{E}$ be an $n\times n$ matrix whose diagonal entries are eigenvalues of $(-\boldsymbol{A})^{-1}$. Then the matrix $(-\boldsymbol{A})^{-1}$ can be decomposed as $-\boldsymbol{A}^{-1}=\boldsymbol{\Gamma}^{-1}\boldsymbol{E}\boldsymbol{\Gamma}$ where the similarity transformation matrix Γ is normalized so that its row sums are all equal to 1, i.e. $\Gamma e = 1$; see Telek et al.[9] for details. Let $v = (v_1, ..., v_n)$ be an $1 \times n$ vector such that $v = \alpha \Gamma^{-1}$. Then, the moments can be obtained as

$$\begin{aligned} r_k &= \alpha(-A)^{-k} e = \alpha(\Gamma^{-1} E \Gamma)^k e \\ &= \alpha \Gamma^{-1} E^k \Gamma e \\ &= v E^k e. \end{aligned} \tag{2.2}$$

The pair (v, E) is called the Jordan representation which is minimal since the vector v satisfies $v_1 + \cdots + v_n = 1$. It is worthwhile to mention that $v_1, ..., v_n$ are not necessarily non-negative.

3. LT and its inversion of PH(2) distribution

3.1 Minimal LT in terms of characteristic polynomial

The pdf and moments of a PH(n) distribution with (α, $\mathbf{A}$) can be obtained from the LT which is given as

$$\tilde{f}(s) = \alpha(s\boldsymbol{I} - \boldsymbol{A})^{-1}(-\boldsymbol{A})e \tag{3.1}$$

where $\boldsymbol{I}$ is an $n \times n$ identity matrix. Note that

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathrm{Adj}\,(s\boldsymbol{I} - \boldsymbol{A})}{|\, s\boldsymbol{I} - \boldsymbol{A} \,|}.$$

where Adj(·) is an $n \times n$ adjoint matrix and |·| is the determinant of a matrix. Let (a_0, a_1) be the coefficient of the characteristic polynomial of $\boldsymbol{A}$, i.e.

$$|\, s\boldsymbol{I} - \boldsymbol{A} \,| = s^2 + a_1 s + a_0.$$

where $a_0 = |-\boldsymbol{A}|$ and $a_1 = \mathrm{Tr}(-\boldsymbol{A})$. Following the argument in Kim[5], if we let $b_1 = \alpha(-\boldsymbol{A})e$, then the LT of a PH(2) in Eq. (3.1) can be written in terms of 3 parameters, (a_0, a_1, b_1), as follows

$$\tilde{f}(s) = \frac{b_1 s + a_0}{s^2 + a_1 s + a_0}. \tag{3.2}$$

Since a PH(2) is completely described by three moments (r_1, r_2, r_3) which are uniquely determined by the LT in Eq. (3.2) as

$$\begin{aligned} r_k &= \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}\tilde{f}(s)\right]_{s=0} \\ &= \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}\left(\frac{b_1 s + a_0}{s^2 + a_1 s + a_0}\right)\right]_{s=0} \end{aligned}$$

there is one-to-one correspondence between (r_1, r_2, r_3) and (a_0, a_1, b_1). That is,

$$\begin{aligned} (r_1, r_2) &= \left(\frac{a_1 - b_1}{a_0}, \frac{a_1(a_1 - b_1)}{a_0^2} - \frac{a_1}{a_0}\right), \\ r_3 &= \frac{a_1^2(a_1 - b_1)}{a_0^3} + \frac{b_1 - 2a_1}{a_0^2} \end{aligned}$$

and

$$\begin{aligned} (a_0, a_1) &= \left(\frac{r_1^2 - r_2}{r_2^2 - r_1 r_3}, \frac{r_1 r_2 - r_3}{r_2^2 - r_1 r_3}\right), \\ b_1 &= \frac{2r_1 r_2 - r_1^3 - r_3}{r_2^2 - r_1 r_3} \end{aligned} \tag{3.3}$$

by which the LT can be determined based on moments. So, we can get the pdf of PH(2) from the moments and the LT but without (α, $\mathbf{A}$).

In this paper, we assume that the numerator and denominator of the LT in Eq. (3.2) are coprime, i.e.,

$$b_1 \neq \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}. \tag{3.4}$$

Otherwise, Eq. (3.2) reduces to the LT of a exponential distribution.

3.2 Minimal LT in terms of eigenvalues

Let (v, $\boldsymbol{E}$) be the Jordan representation of a PH(n). The pdf and moments of a PH(n) distribution can be obtained from the LT given as

$$\tilde{f}(s) = \boldsymbol{v}(s\boldsymbol{I} - \boldsymbol{E})^{-1}(-\boldsymbol{E})\boldsymbol{e}. \qquad (3.5)$$

The inversion of the LT in Eq. (3.5) results in much simpler form than the inversion of the LT given in (3.1). For PH(2), let (ν_1, ν_2) be the eigenvalues of $(-\boldsymbol{A})^{-1}$ with $\nu_1 \geq \nu_2$. Also let $\boldsymbol{v} = (v_1, 1 - v_1)$. That is, $(-1/\nu_1, -1/\nu_2)$ are the roots of the characteristic polynomial equation given as $s^2 + a_1 s + a_0 = 0$. Below, we consider two different cases for the inversion of the LT in (3.5) to obtain a pdf given in terms of (ν_1, ν_2) and v_1.

3.2.1 Distinct eigenvalues

Suppose that $\nu_1 > \nu_2$ for a PH(2). Then, the matrix $\boldsymbol{E}$ is given as

$$\boldsymbol{E} = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix},$$

and the LT in Eq. (3.5) can be written in terms of 3 parameters, (ν_1, ν_2, v_1), as follows

$$\tilde{f}(s) = v_1 \frac{1/\nu_1}{s + 1/\nu_1} + (1 - v_1)\frac{1/\nu_2}{s + 1/\nu_2}, \qquad (3.6)$$

and the entry of the vector $\boldsymbol{v} = (v_1, 1 - v_1)$ can be determined by Eq (2.2), i.e. for k =1, 2, 3

$$r_k = \boldsymbol{v}\boldsymbol{E}^{\mathrm{k}}\mathrm{e} = \mathrm{v}_1 \nu_1^{\mathrm{k}} + (1 - \mathrm{v}_1)\nu_2^{\mathrm{k}}$$

from which we get

$$v_1 = \frac{r_1 - \nu_2}{\nu_1 - \nu_2} = \frac{r_1 - \nu_2^2}{\nu_1^2 - \nu_2^2} = \frac{r_2 - \nu_2^3}{\nu_1^3 - \nu_2^3}.$$

Assuming that $\nu_1 \geq \nu_2$, the one-to-one correspondence between (ν_1, ν_2, v_1) and (a_0, a_1, b_1) can be determined by equating LTs in Eqs. (3.2) and (3.6). That is,

$$(a_1, a_1, b_1) = \left(\frac{1}{\nu_1 \nu_2}, \frac{\nu_1 + \nu_2}{\nu_1 \nu_2}, \frac{\nu_1 - v_1(\nu_1 - \nu_2)}{\nu_1 \nu_2} \right),$$

and

$$(\nu_1, \nu_2) = \left(\frac{a_1 + \sqrt{a_1^2 - 4a_0}}{2a_0}, \frac{a_1 - \sqrt{a_1^2 - 4a_0}}{2a_0} \right),$$

$$v_1 = \frac{1}{2}\left(1 + \frac{(a_1 - 2b_1)\sqrt{a_1^2 - 4a_0}}{a_1^2 - 4a_0} \right) \qquad (3.7)$$

where $v_1 \neq 1$ by the restriction in Eq (3.4).

3.2.2 Identical eigenvalues

If $(-\boldsymbol{A})^{-1}$ has identical eigenvalues, i.e. $\nu_1 = \nu_2$, then the matrix **E** is given as

$$\boldsymbol{E} = \begin{bmatrix} \nu_1 & 1 \\ 0 & \nu_1 \end{bmatrix}$$

and the LT in Eq. (3.5) can be written in terms of ν_1 and v_1, as follows

$$\tilde{f}(s) = \left(1 - \frac{v_1}{\nu_1} \right) \frac{1/\nu_1}{s + 1/\nu_1}. + \frac{v_1}{\nu_1}\left(\frac{1/\nu_1}{s + 1/\nu_1} \right)^2 \qquad (3.8)$$

and the entry of the vector $\boldsymbol{v} = (v_1, 1 - v_1)$ can be determined by Eq (2.2), i.e. for i =1, 2, 3

$$r_k = \boldsymbol{v}\boldsymbol{E}^{\mathrm{k}}\mathrm{e} = \nu_1^{\mathrm{k}} + k v_1 \nu_1^{\mathrm{k}-1}$$

from which we get

$$v_1 = r_1 - \nu_1 = \frac{r_2 - \nu_1^2}{2\nu_1} = \frac{r_3 - \nu_1^3}{3\nu_1^2}.$$

By equating Eqs. (3.2) and (3.8), we have

$$(a_0, a_1, b_1) = \left(\frac{1}{\nu_1^2}, \frac{2}{\nu_1}, \frac{1}{\nu_1}\left(1 - \frac{v_1}{\nu_1} \right) \right).$$

and

$$\nu_1 = \nu_2 = \frac{a_1}{2a_0} = \frac{2}{a_1}, \; v_1 = \frac{2}{a_1}\left(1 - \frac{2b_1}{a_1} \right). \qquad (3.9)$$

4. Probability function of a PH(2) by LT inversion and distribution function

In this section, we propose a new approach in generating a PH-type random variable without (α, **A**). Instead, the proposed procedure is based on the LT and the pdf obtained directly from moments. We show procedures to transform moments to LT and then to pdf based on which PH(2) random variates are generated.

4.1 Distinct eigenvalues

If the characteristic polynomial |s ***I*** - ***A***| = 0 has distinct roots, then the pdf can be obtained by inversion of the LT in (3.5), i.e.

$$f(x)=\frac{v_1}{\nu_1}e^{-x/\nu_1}+\frac{1-v_1}{\nu_2}e^{-x/\nu_2} \quad (4.1)$$

where $v_1 \neq 1$ by the restriction (3.4). Also note that $1-v_1$ is not necessarily non-negative whereas v_1 must be non-negative for the $f(x)$ in (4.1) to be a valid pdf. Depending on the sign of $1-v_1$, the pdf is either hyper-exponential or mixed generalized Erlang (MGE).

(Case: $1-v_1>0$)

If $1-v_1>0$, then we have $0<v_1<1$ and $0<v_2<1$. Therefore, the $f(x)$ in (4.1) is a hyper-exponential, i.e.

$$X\sim\begin{cases}\mathrm{Exp}(1/\nu_1) & w.p.\, v_1\\ \mathrm{Exp}(1/\nu_2) & w.p.\, 1-v_1\end{cases} \quad (4.2)$$

(Case: $1-v_1<0$)

If $1-v_1<0$, then the pdf in (4.1) is not a hyper-exponential. By a simple manipulation, however, it can be shown that it is a mixed generalized Erlang, i.e.

$$\begin{aligned}f(x)&=\left(v_1+v_2\frac{\nu_1}{\nu_2}\right)e^{-\nu_1 x}\\ &\quad+v_2\left(1-\frac{\nu_1}{\nu_2}\right)\left(\frac{e^{-x/\nu_1}}{\nu_1-\nu_2}+\frac{e^{-x/\nu_2}}{\nu_2-\nu_1}\right) \quad (4.3)\end{aligned}$$

That is,

$$X\sim\begin{cases}\mathrm{Exp}\left(\frac{1}{\nu_1}\right) & w.p.\, \frac{\nu_1}{\nu_2}+v_1\frac{\nu_2-\nu_1}{\nu_2}\\ \mathrm{Hypo}\left(\frac{1}{\nu_1},\frac{1}{\nu_2}\right) & w.p.\, (1-v_1)\frac{\nu_1}{\nu_2}\end{cases} \quad (4.4)$$

where $\mathrm{Hypo}(1/\nu_1, 1/\nu_2)$ is a hypo-exponential distribution given as the sum of two independent exponential random variables each with mean $1/\nu_1$ and $1/\nu_2$.

4.2 Identical eigenvalues

The LT in Eq. (3.6) is converted into the following pdf by inversion

$$f(x)=\left(1-\frac{v_1}{\nu_1}\right)\frac{1}{\nu_1}e^{-\frac{x}{\nu_1}}+\frac{v_1}{\nu_1}\frac{1}{\nu_1^2}xe^{-\frac{x}{\nu_1}}. \quad (4.5)$$

That is,

$$X\sim\begin{cases}\mathrm{Exp}\left(\frac{1}{\nu_1}\right) & w.p.\, 1-\frac{v_1}{\nu_1}\\ \mathrm{Erlang}\left(2,\frac{1}{\nu_1}\right) & w.p.\, \frac{v_1}{\nu_1}\end{cases} \quad (4.6)$$

where Erlang(2, ν_1) distribution is a sum of two independent and identical exponential random variables with mean $1/\nu_1$.

5. Simulation of PH(2) distribution

5.1 Procedure to generate PH(2)

The results of the previous section can be put together for generating a PH(2) random variates given a set of moments. First, let (u_1, u_2, u_3) be real-valued random numbers uniformly distributed between 0 and 1 which is denoted by *U*(0,1).

Procedure: PH(2)
- Input: (r_1, r_2, r_3)
- Output: PH(2) random number x

BEGIN

Step 1. Check validity of the moments and determine if eigenvalues are distinct.

- If $r_2 = r_1^2$ and $r_3 = r_1^3$, stop. "Exponential";
- Compute (a_0, a_1, b_1) by Eq. (3.3).

$$(a_0, a_1) = \left(\frac{r_1^2 - r_2}{r_2^2 - r_1 r_3}, \frac{r_1 r_2 - r_3}{r_2^2 - r_1 r_3} \right),$$
$$b_1 = \frac{2 r_1 r_2 - r_1^3 - r_3}{r_2^2 - r_1 r_3}.$$

- If $a_1^2 - 4a_0 < 0$, stop. "Not a valid PH(2)";
- Check if eigenvalues are distinct.
  - if $a_1^2 - 4a_0 > 0$, go to step 2.
  - if $a_1^2 - 4a_0 = 0$, go to step 3.

Step 2. (Case: distinct eigenvalues)

- Determine (ν_1, ν_2, v_1) by Eq (3.7).

$$(\nu_1, \nu_2) = \left(\frac{a_1 + \sqrt{a_1^2 - 4a_0}}{2a_0}, \frac{a_1 - \sqrt{a_1^2 - 4a_0}}{2a_0} \right),$$
$$v_1 = \frac{1}{2}\left(1 + \frac{(a_1 - 2b_1)\sqrt{a_1^2 - 4a_0}}{a_1^2 - 4a_0} \right)$$

- If $v_1 < 1$, generate $(u_1, u_2) \sim U(0,1)$. Generate x as follows by Eq. (4.2) and exit.

$$x = \begin{cases} -\ln(u_2)/\nu_1 & \text{if } u_1 \le v_1 \\ -\ln(u_2)/\nu_2 & \text{if } u_1 \ge v_1 \end{cases}$$

- Otherwise, if $v_1 > 1$, then generate $(u_1, u_2, u_3) \sim U(0,1)$. Generate x as follows by Eq. (4.4) and exit.

$$x = \begin{cases} -\dfrac{\ln(u_2)}{\nu_1} & \text{if } u_1 \le \dfrac{\nu_1}{\nu_2} + v_1 \dfrac{\nu_2 - \nu_1}{\nu_2} \\ -\dfrac{\ln(u_2)}{\nu_1} - \dfrac{\ln(u_3)}{\nu_2} & \text{if } u_1 \ge \dfrac{\nu_1}{\nu_2} + v_1 \dfrac{\nu_2 - \nu_1}{\nu_2} \end{cases}$$

Step 3. (Case: identical eigenvalues)

- Determine (ν_1, v_1) by Eq (3.9).

$$\nu_1 = \nu_2 = \frac{a_1}{2a_0} = \frac{2}{a_1},$$
$$v_1 = \frac{2}{a_1}\left(1 - \frac{2b_1}{a_1} \right).$$

- Generate $(u_1, u_2, u_3) \sim U(0,1)$. Generate x as follows by Eq. (4.6) and exit.

$$x = \begin{cases} -\dfrac{\ln(u_2)}{\nu_1} & \text{if } u_1 \le 1 - \dfrac{v_1}{\nu_1} \\ -\dfrac{\ln(u_2)}{\nu_1} - \dfrac{\ln(u_3)}{\nu_1} & \text{if } u_1 \ge 1 - \dfrac{v_1}{\nu_1} \end{cases}$$

END of the procedure.

Note that the condition that $r_2 = r_1^2$ and $r_3 = r_1^3$ in Step 1 is equivalent to $b_1 = \dfrac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$.

5.2 Numerical exmples

5.2.1 Distinct eigenvalues with $1 - v_1 > 0$

Consider a PH(2) with the following set of moments (r_1, r_2, r_3) = (5/6, 7/9, 41/54) for which we have (a_0, a_1, b_1) = (3, 4, 3/2) by Eq. (3.3). We also get (ν_1, ν_2) = (1, 1/3), and v_1 = 3/4 by Eq. (3.7). The pdf in (4.1) becomes

$$f(x) = \frac{3}{4} \times e^{-x} + \frac{1}{4} \times 3e^{-3x}$$

which is hyper-exponential and can be generated as

$$X \sim \begin{cases} \mathrm{Exp}(1) & w.p.\ 3/4 \\ \mathrm{Exp}(3) & w.p.\ 1/4 \end{cases}$$

by Eq. (4.2).

5.2.2 Distinct eigenvalues with $1 - v_1 < 0$

Consider a PH(2) with the following set of moments

(r_1, r_2, r_3)= (5/9, 8/27, 25/162) for which we have (a_0, a_1, b_1) = (6, 5, 5/3) by Eq. (3.3). We also get (ν_1, ν_2) = (1/2, 1/3), and v_1= 4/3 by Eq. (3.7). The pdf in (4.1) becomes

$$f(x) = \frac{4}{3} \times 2e^{-2x} - \frac{1}{3} \times 3e^{-3x}$$

which is not a hyper-exponential, however, can be rewritten as

$$f(x) = \frac{5}{6} \times 2e^{-2x} + \frac{1}{6} \times 6\left(e^{-2x} - e^{-3x}\right).$$

by Eq. (4.3). That is, X is an MGE(2) which can be generated as

$$X \sim \begin{cases} \mathrm{Exp}(2) & w.p. 5/6 \\ \mathrm{Hypoexp}(2,3) & \mathrm{w.p.} 1/6 \end{cases}$$

by Eq. (4.4).

5.2.3 Identical eigenvalues

Consider a PH(2) with (r_1, r_2, r_3)= (5/9, 7/27, 1/9) for which we have (a_0, a_1, b_1) = (9, 6, 1) by Eq. (3.3). We also get $\nu_1 = \nu_2 = 1/3$ = (1/3, 1/3), and v_1= 2/9 by Eq. (3.9). The pdf in (4.5) becomes

$$f(x) = \frac{1}{3} \times 3e^{-3x} + \frac{2}{3} \times 3^2 x e^{-3x} .$$

That is, X is an MGE(2) which can be generated as

$$X \sim \begin{cases} \mathrm{Exp}(3) & w.p. 1/3 \\ \mathrm{Erlang}(2,3) & \mathrm{w.p.} 2/3 \end{cases}$$

by Eq. (4.6).

6. Discussions and conclusions

6.1 Discussions

The PH-type distribution is easy to generate if the initial probability vector and the generator matrix are given. In most queueing network analysis, however, the initial probability vector and the generator matrix need to be determined based on moments of arrival or departure processes. Our approach is motivated by this observation and can be used more effectively than the Markovian representation which is harder to obtain from moments for higher order PH distribution.

6.2 Conclusions and future research

We presented closed-form formula for PH(2) which can be fitted by three moments. While the procedure can be extended to higher order PH-type distribution, the number of moments required for PH(n) increases linearly, i.e. $2n-1$. Moreover, higher-order PH-type distribution may have complex eigenvalues which is associated with cycles in transitions among phases. Since the marginal distribution of stationary intervals of the Markovian arrival processes is a PH-type distribution, the results can also be generalized to generate a Markovian arrival processes.

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