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Reverse Inequalities through k-weighted Fractional Operators with Two Parameters

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Abstract. The aim of this paper is to present an approach to improve reverse Minkowski and Hölder-type inequalities using k-weighted fractional integral operators ${}_{a^+}\mathfrak{I}_w^{\mu}$ with respect to a strictly increasing continuous function μ, by introducing two parameters of integrability, p and q. For various choices of μ we get interesting special cases.

1. Introduction

Fractional calculus, in which the introduction of several distinct fractional integral operators is used in solving integeral inequalities, has proved useful in applications in such fields as physics, engineering, and computer science. The operators introduced include the Riemann-Liouville, Hadamard, Katugompola, and proportional fractional integral operators. The proportional fractional integral operator is particularly noteworthy in being a generalized fractional operator. Specific applications of generalized operators can be found, for example, in [8], and [6].

In [7], the weighted fractional integral is defined as follows. For an integrable function f on the interval $[a, b]$ and for a differentiable function μ such that $\mu'(t) \neq 0$ for all $t \in [a, b]$, it is

$${}_{a^+}I_w^{\beta} f(x) = \frac{1}{w(x)\Gamma(\beta)} \int_a^x \mu'(s)(\mu(x) - \mu(s))^{\beta-1} w(s) f(s) ds, \quad x > a,$$

where w is a weighted (positive measurable) function.

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Researchers have expanded and improved Minkowski's inverse inequality by applying it to fractional integral operators. This has resulted in the development of new mathematical tools that have enhanced our ability to solve problems in various fields. See [9], [10], [13].

In [1], the author presented the following generalization of the reverse Minkowski's inequality, for any measurable functions $f, g > 0$ on (a, b) and $p \geq 1$, if $0 < c < m \leq \frac{\alpha f(x)}{g(x)} \leq M$ for all $x \in [a, b]$, then

$$\begin{aligned} \frac{M+\alpha}{\alpha(M-c)} \left(\int_a^b (\alpha f(x) - cg(x))^p dx \right)^{\frac{1}{p}} &\leq \left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} + \left(\int_a^b g^p(x) dx \right)^{\frac{1}{p}} \\ &\leq \frac{m+\alpha}{\alpha(m-c)} \left(\int_a^b (\alpha f(x) - cg(x))^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Using $c = 1$ we get Sroysang's inequalities [12, Theorem 2.2] and if we put $\alpha = c = 1$ we obtain the Sulaiman's inequalities [11, Theorem 3.1].

In [2], the authors provide a generalization of the reverse Hölder's inequality. For $\lambda, \gamma > 0$ and $f, g, w > 0$ measurable functions on (a, b) and $p > 1$, $(\frac{1}{p} + \frac{1}{p'} = 1)$, if $0 < m \leq \frac{\alpha f^\lambda(x)}{g^\gamma(x)} \leq M$ for all $x \in [a, b]$ we have

$$\left(\int_a^b f^\lambda(x) w(x) dx \right)^{\frac{1}{p}} \left(\int_a^b g^\gamma(x) w(x) dx \right)^{\frac{1}{p'}} \leq \left(\frac{M}{m} \right)^{\frac{1}{pq}} \int_a^b f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x) w(x) dx.$$

Moreover, a new version to the reverse Hölder's inequality with two parameters was has been presented on time scales in [3].

Motivated by the above literature, the present paper introduces a new definition of weighted fractional operators of a function with respect to another function. In Section 2, we present new versions of the reverse Minkowski-type inequality using k-weighted fractional operators with two parameters. Section 3 establishes the reverse Hölder-type inequality in fractional calculus using k-weighted fractional operators with two parameters. At the end of the paper, we conclude with a brief summary of the findings.

2. k-weighted Fractional Operators

Let $[a, b] \subseteq (0, +\infty)$, where $a < b$. In this section, we present a definition of the k-weighted fractional integrals of a function f with respect to the function μ and we prove that they are bounded in a specified space.

Definition 2.1. Let $\beta > 0$, $k > 0$ and μ be a positive, strictly increasing differentiable function such that $\mu'(s) \neq 0$ for all $s \in [a,b]$. The left and right sided k-weighted fractional integral of a function f with respect to the function μ on $[a,b]$ are defined respectively as follows.

$$ {}_{a^+}\mathfrak{I}_w^{\mu} f(x) = \frac{1}{w(x)k\,\Gamma_k(\beta)} \int_a^x \mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1} w(s) f(s) ds, \quad x > a. \tag{2.1} $$

$$ {}_{b^-}\mathfrak{I}_w^{\mu} f(x) = \frac{1}{w(x)k\,\Gamma_k(\beta)} \int_x^b \mu'(s)(\mu(s)-\mu(x))^{\frac{\beta}{k}-1} w(s) f(s) ds, \quad x < b, \tag{2.2} $$

where w is a weighted function and the k-gamma function defined by

$$ \Gamma_k(\beta) = \int_0^{\infty} t^{\beta-1} e^{-\frac{t^k}{k}} dt. $$

When $f(s) = 1$, we denote

$$ {}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) = \frac{1}{w(x)k\,\Gamma_k(\beta)} \int_a^x \mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1} w(s)\, ds, \quad x > a, $$

and

$$ {}_{b^-}\mathfrak{I}_w^{\mu} \mathbf{1}(x) = \frac{1}{w(x)k\,\Gamma_k(\beta)} \int_x^b \mu'(s)(\mu(s)-\mu(x))^{\frac{\beta}{k}-1} w(s)\, ds, \quad x < b. $$

Remark 2.1. Let $c > 0$ be a positive constant if $f(s) = c$, we get

$$ {}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{c}.(x) = c\, \left[{}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) \right]. $$

The space $L_p^W[a,b]$ of all real-valued Lebesgue measurable functions f on $[a,b]$ with norm conditions:

$$ \parallel f \parallel_p^W = \left(\int_a^b \mid f(x) \mid^p W(x) dx \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < +\infty. $$

is known as weighted Lebesgue space, where W be a weight function (measurable and positive).

1. Put $W \equiv 1$, the space $L_p^W[a,b]$ reduces to the classical space $L_p[a,b]$.
2. Choose $W(x) = w^p(x)\,\mu'(x)$, we get

$$ L_{X_w^p}[a,b] = \left\{ f : \parallel f \parallel_{X_w^p} = \left(\int_a^b \mid w(x) f(x) \mid^p \mu'(x) dx \right)^{\frac{1}{p}} < \infty \right\}. \tag{2.3} $$

In the following Theorem we show that the k-weighted fractional integrals are clearly defined.

Theorem 2.1. The fractional integrals (2.1), (2.2) are defined for all functions $f \in L_{X_w^1}[a,b]$ and we have

$$\text{(2.4)} \qquad {}_{a^+}\mathfrak{I}_w^{\mu} f(x) \in L_{X_w^1}[a,b], \qquad {}_{b^-}\mathfrak{I}_w^{\mu} f(x) \in L_{X_w^1}[a,b].$$

Moreover

$$\text{(2.5)} \qquad \left\| {}_{a^+}\mathfrak{I}_w^{\mu} f(x) \right\|_{X_w^1} \le C \left\| f(x) \right\|_{X_w^1}, \qquad \left\| {}_{b^-}\mathfrak{I}_w^{\mu} f(x) \right\|_{X_w^1} \le C \left\| f(x) \right\|_{X_w^1},$$

where

$$C = \frac{(\mu(b) - \mu(a))^{\frac{\beta}{k}}}{\Gamma_k(\beta + k)}.$$

Proof. For all $\frac{\beta}{k} > 0$, by using Fubini's Theorem, we get

$$\left\| {}_{a^+}\mathfrak{I}_w^{\mu} f(x) \right\|_{X_w^1} = \int_a^b w(x) \left| {}_{a^+}\mathfrak{I}_w^{\mu} f(x) \right| \mu'(x) dx$$

$$\le \frac{1}{k\,\Gamma_k(\beta)} \int_a^b \int_a^x |w(s) f(s)|\, \mu'(s) (\mu(x) - \mu(s))^{\frac{\beta}{k} - 1} \mu'(x)\, ds\, dx$$

$$= \frac{1}{k\,\Gamma_k(\beta)} \int_a^b |w(s) f(s)| \left(\int_s^b (\mu(x) - \mu(s))^{\frac{\beta}{k} - 1} \mu'(x)\, dx \right) \mu'(s)\, ds$$

$$= \frac{1}{\beta\,\Gamma_k(\beta)} \int_a^b |w(s) f(s)|\, (\mu(b) - \mu(s))^{\frac{\beta}{k}}\, \mu'(s)\, ds$$

$$\le \frac{(\mu(b) - \mu(a))^{\frac{\beta}{k}}}{\Gamma_k(\beta + k)} \int_a^b |w(s) f(s)|\, \mu'(s)\, ds$$

$$= C \left\| f(x) \right\|_{X_w^1}.$$

Similarly

$$\int_a^b w(x) \left| {}_{b^-}\mathfrak{I}_w^{\mu} f(x) \right| \mu'(x) dx \le \frac{(\mu(b) - \mu(a))^{\frac{\beta}{k}}}{\Gamma_k(\beta + k)} \left\| f(x) \right\|_{X_w^1}.$$

This gives us our desired formulas (2.5) and (2.4). □

Setting $\mu(\tau) = \tau$, then ${}_{a+}\mathfrak{I}_w^{\mu} f(x)$ and ${}_{b-}\mathfrak{I}_w^{\mu} f(x)$ reduce to the k-weighted fractional integral of Riemann-Liouville operator of order $\beta \ge 0$.

$$\mathcal{RL}_1^k f(x) = \frac{1}{w(x) k\, \Gamma_k(\beta)} \int_a^x (x - s)^{\frac{\beta}{k} - 1} w(s) f(s) ds, \quad x > a.$$

$$\mathcal{RL}_2^k f(x) = \frac{1}{w(x)k\,\Gamma_k(\beta)} \int_x^b (s-x)^{\frac{\beta}{k}-1} w(s)f(s)ds, \quad x<b,.$$

Setting $\mu(\tau) = \ln\tau$, then ${}_{a+}\mathfrak{I}_w^{\mu}f(x)$ and ${}_{b-}\mathfrak{I}_w^{\mu}f(x)$ reduce to the k-weighted fractional integral of Hadamard operator of order $\beta \geq 0$.

$$\mathcal{H}_1 f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left(\ln\frac{x}{s}\right)^{\frac{\beta}{k}-1} f(s)\frac{ds}{s}, \quad x>a>1.$$

$$\mathcal{H}_2 f(x) = \frac{1}{k\Gamma_k(\beta)} \int_x^b \left(\ln\frac{s}{x}\right)^{\frac{\beta}{k}-1} f(s)\frac{ds}{s}, \quad x<b.$$

Setting $\mu(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ where $\rho>0$, then ${}_{a+}\mathfrak{I}_w^{\mu}f(x)$ and ${}_{b-}\mathfrak{I}_w^{\mu}f(x)$ reduce to the k-weighted fractional integral of Katugompola operator of order $\beta \geq 0$.

$$\mathcal{K}_1 f(x) = \frac{(\rho+1)^{1-\frac{\beta}{k}}}{w(x)k\,\Gamma_k(\beta)} \int_a^x \left(x^{\rho+1}-s^{\rho+1}\right)^{\frac{\beta}{k}-1} w(s)s^{\rho}f(s)ds, \quad x>a.$$

$$\mathcal{K}_2 f(x) = \frac{(\rho+1)^{1-\frac{\beta}{k}}}{w(x)\,k\,\Gamma_k(\beta)} \int_x^b \left(s^{\rho+1}-x^{\rho+1}\right)^{\frac{\beta}{k}-1} w(s)s^{\rho}f(s)ds, \quad x<b,.$$

Setting $\mu(\tau) = \frac{(\tau-a)^{\theta}}{\theta}$ for $\theta>0$ (respectively $\mu(\tau) = \frac{(b-\tau)^{\theta}}{\theta}$), then ${}_{a+}\mathfrak{I}_w^{\mu}f(x)$ (respectively ${}_{b-}\mathfrak{I}_w^{\mu}f(x)$) are reduce to the k-weighted fractional integral of conformable operator of order $\beta \geq 0$.

$$\mathcal{C}_1 f(x) = \frac{\theta^{1-\frac{\beta}{k}}}{w(x)k\,\Gamma_k(\beta)} \int_a^x \left((x-a)^{\theta}-(s-a)^{\theta}\right)^{\frac{\beta}{k}-1} w(s)\frac{f(s)}{(s-a)^{1-\theta}}ds, \quad x>a.$$

$$\mathcal{C}_2 f(x) = \frac{\theta^{1-\frac{\beta}{k}}}{w(x)\,k\,\Gamma_k(\beta)} \int_x^b \left((b-x)^{\theta}-(b-s)^{\theta}\right)^{\frac{\beta}{k}-1} w(s)\frac{f(s)}{(b-s)^{1-\theta}}ds, \quad x<b,.$$

For example see [5]. The most important feature of the k-weighted fractional integrals ${}_{a+}\mathfrak{I}_w^{\mu}f(x)$ and ${}_{b-}\mathfrak{I}_w^{\mu}f(x)$ is that they give certain types of the k-weighted fractional depends on the choice of the function μ. We present the following Lemma [4], [3], that is used to prove our results.

Lemma 2.1. Let $1<q\leq p<\infty$ and f, W be non-negative measurable functions on $[a,\, b]$. We suppose that $0<\int_a^b f^r(s)W(s)ds<\infty$ for $r>1$, then

$$\int_a^b f^q(s)W(s)ds \leq \left(\int_a^b W(s)ds\right)^{\frac{p-q}{p}} \left(\int_a^b f^p(s)W(s)ds\right)^{\frac{q}{p}} \tag{2.6}$$

and

$$\int_a^b f^{q'}(s)W(s)ds \geq \left(\int_a^b W(s)ds\right)^{\frac{p'-q'}{p'}} \left(\int_a^b f^{p'}(s)W(s)ds\right)^{\frac{q'}{p'}}.$$

Proof. If $p=q$ we get equality and for $p\neq q$ using the Hölder's integral inequality with $\dfrac{p}{q}>1$, we have

$$\begin{aligned}\int_a^b f^q(s)W(s)ds &= \int_a^b \left(W^{\frac{p-q}{p}}(s)\right)\left(f^q(s)W^{\frac{q}{p}}(s)\right)ds\\ &\leq \left(\int_a^b W(s)ds\right)^{\frac{p-q}{p}}\left(\int_a^b f^p(s)W(s)ds\right)^{\frac{q}{p}}.\end{aligned}$$

Since $1<q\leq p<\infty \Longrightarrow 1<p'\leq q'<\infty$, thus the proof of the second inequality is similar to the first one. □

Corollary 2.1. Let $1<q\leq p<\infty$, f be non-negative measurable function on $[a,\, x]$ and μ be a positive strictly increasing differentiable on $[a,b]$ and ${}_{a^+}\mathfrak{I}^{\mu}_{w}$ is the operator defined by (2.1), then

$$\left({}_{a^+}\mathfrak{I}^{\mu}_{w} f^q(x)\right)^{\frac{1}{q}} \leq \left[{}_{a^+}\mathfrak{I}^{\mu}_{w}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}^{\mu}_{w} f^p(x)\right)^{\frac{1}{p}}, \tag{2.7}$$

and

$$\left({}_{b^-}\mathfrak{I}^{\mu}_{w} f^q(x)\right)^{\frac{1}{q}} \leq \left[{}_{b^-}\mathfrak{I}^{\mu}_{w}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{b^-}\mathfrak{I}^{\mu}_{w} f^p(x)\right)^{\frac{1}{p}}.$$

Proof. Using the inequality (2.6) by taking $W(s)=\dfrac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)$, we obtain that

$$\begin{aligned}&\int_0^x \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)f^q(s)ds\\ &\leq \left(\int_0^x \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)ds\right)^{\frac{p-q}{p}}\\ &\times\left(\int_0^x \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)f^p(s)ds\right)^{\frac{q}{p}},\end{aligned}$$

this gives the desired results. □

3. Reverse Minkowski Type Inequalities via k-weighted Fractional Integrals

Let $0 \leq a < b < +\infty$, w be a weight function and f, g be a positive measurable functions on $[a,\, b]$, suppose that $f^p,\ g^p \in L_{X_w^1}[a, b]$, where

$$L_{X_w^1}[a,b] = \left\{ f : \parallel f \parallel_{X_w^1} = \left(\int_a^b \mid w(x) f(x) \mid \mu'(x) dx \right) < \infty \right\}. \tag{3.1}$$

Theorem 3.1. *Let* $f,\, g > 0,\ 1 \leq q \leq p < +\infty,\ \alpha > 0$ *and*

$$0 < c < m \leq \frac{\alpha f(s)}{g(s)} \leq M, \quad \textit{for} \ \ \textit{all}\ s \in [a, x], \tag{3.2}$$

then

$$\begin{aligned} &\frac{M+\alpha}{\alpha(M-c)} \left({}_{a^+}\mathfrak{I}_w^{\mu} (\alpha f(x) - c g(x))^q dx \right)^{\frac{1}{q}} \leq \\ &\left({}_{a^+}\mathfrak{I}_w^{\mu} f^q(x) dx \right)^{\frac{1}{q}} + \left(\left[{}_{a^+}\mathfrak{I}_w^{\mu} \boldsymbol{1}(x) \right]^{\frac{p-q}{q}} \ {}_{a^+}\mathfrak{I}_w^{\mu} g^p(x) dx \right)^{\frac{1}{p}} \\ &\leq \frac{m+\alpha}{\alpha(m-c)} \left(\left[{}_{a^+}\mathfrak{I}_w^{\mu} \boldsymbol{1}(x) \right]^{\frac{p-q}{q}} \ {}_{a^+}\mathfrak{I}_w^{\mu} (\alpha f(x) - c g(x))^p dx \right)^{\frac{1}{p}}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\frac{M+\alpha}{\alpha(M-c)} \left({}_{b^-}\mathfrak{I}_w^{\mu} (\alpha f(x) - c g(x))^q dx \right)^{\frac{1}{q}} \leq \\ &\left({}_{b^-}\mathfrak{I}_w^{\mu} f^q(x) dx \right)^{\frac{1}{q}} + \left(\left[{}_{b^-}\mathfrak{I}_w^{\mu} \boldsymbol{1}(x) \right]^{\frac{p-q}{q}} \ {}_{b^-}\mathfrak{I}_w^{\mu} g^p(x) dx \right)^{\frac{1}{p}} \\ &\leq \frac{m+\alpha}{\alpha(m-c)} \left(\left[{}_{b^-}\mathfrak{I}_w^{\mu} \boldsymbol{1}(x) \right]^{\frac{p-q}{q}} \ {}_{b^-}\mathfrak{I}_w^{\mu} (\alpha f(x) - c g(x))^p dx \right)^{\frac{1}{p}}. \end{aligned} \tag{3.4}$$

Proof. From the assumption (3.2) we get

$$0 < \frac{1}{c} - \frac{1}{m} \leq \frac{1}{c} - \frac{g(s)}{\alpha f(s)} \leq \frac{1}{c} - \frac{1}{M},$$

hence

$$\frac{c\, M}{M-c} \leq \frac{c\, \alpha\, f(s)}{\alpha f(s) - c\, g(s)} \leq \frac{c\, m}{m-c},$$

that yields

$$\frac{M}{\alpha(M-c)} (\alpha f(s) - c\, g(s)) \leq f(s) \leq \frac{m}{\alpha(m-c)} (\alpha f(s) - c\, g(s)),$$

taking the q^{th} power of the above inequality and multiplying by the positive quotient

$$\frac{1}{w(x) k\, \Gamma_k(\beta)} \mu'(s) (\mu(x) - \mu(s))^{\frac{\beta}{k}-1} w(s)$$

we obtain

$$\left(\frac{M}{\alpha(M-c)}\right)^q \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)(\alpha f(s)-cg(s))^q$$

$$\leq \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)f^q(s)$$

$$\leq \left(\frac{m}{\alpha(m-c)}\right)^q \frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)(\alpha f(s)-cg(s))^q,$$

integrating with respect to s over $[a,x]$, we get

$$\begin{aligned}\frac{M}{\alpha(M-c)}\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha f(x)-c\,g(x))^q dx\right)^{\frac{1}{q}} &\leq \left({}_{a^+}\mathfrak{I}^{\mu}_{w} f^q(x)dx\right)^{\frac{1}{q}}\\ &\leq \frac{m}{\alpha(m-c)}\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha f(x)-cg(x))^q dx\right)^{\frac{1}{q}},\end{aligned} \tag{3.5}$$

applying the inequality (2.7) on the right-hand side of (3.5), we get

$$\begin{aligned}\frac{M}{\alpha(M-c)}\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha f(x)-cg(x))^q dx\right)^{\frac{1}{q}} &\leq \left({}_{a^+}\mathfrak{I}^{\mu}_{w} f^q(x)dx\right)^{\frac{1}{q}}\\ \leq \frac{m}{\alpha(m-c)}\left[{}_{a^+}\mathfrak{I}^{\mu}_{w}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}&\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha f(x)-cg(x))^p dx\right)^{\frac{1}{p}}.\end{aligned} \tag{3.6}$$

Using the assumption (3.2), we obtain

$$0<m-c\leq \frac{\alpha f(s)-c\,g(s)}{g(s)}\leq M-c,$$

therefore

$$\frac{\alpha f(s)-c\,g(s)}{M-c}\leq g(s)\leq \frac{\alpha\,f(s)-c\,g(s)}{m-c},$$

for $p\geq 1$ we deduce that

$$\left(\frac{1}{M-c}\right)^p(\alpha\,f(s)-c\,g(s))^p\leq g^p(s)\leq \left(\frac{1}{m-c}\right)^p(\alpha\,f(s)-c\,g(s))^p,$$

multiplying by the positive quotient $\frac{1}{w(x)k\,\Gamma_k(\beta)}\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1}w(s)$ and integrating with respect to s over $[a,x]$, thus

$$\begin{aligned}\frac{1}{M-c}\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha\,f(x)-c\,g(x))^p\right)^{\frac{1}{p}} &\leq \left({}_{a^+}\mathfrak{I}^{\mu}_{w} g^p(x)\right)^{\frac{1}{p}}\\ &\leq \frac{1}{m-c}\left({}_{a^+}\mathfrak{I}^{\mu}_{w}(\alpha\,f(x)-c\,g(x))^p\right)^{\frac{1}{p}}.\end{aligned}$$

therefore

$$\begin{aligned}&\frac{1}{M-c}\left[{}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}_w^{\mu}(\alpha\, f(x)-c\, g(x))^p\right)^{\frac{1}{p}}\\ &\leq \left[{}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}_w^{\mu}g^p(x)\right)^{\frac{1}{p}}\\ &\leq \frac{1}{m-c}\left[{}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}_w^{\mu}(\alpha f(x)-cg(x))^p\right)^{\frac{1}{p}}.\end{aligned} \tag{3.7}$$

Applying the inequality (2.7) on the left-hand side of (3.7), we get

$$\begin{aligned}\frac{1}{M-c}\left({}_{a^+}\mathfrak{I}_w^{\mu}(\alpha\, f(x)-c\, g(x))^q\right)^{\frac{1}{q}} &\leq \left[{}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}_w^{\mu}g^p(x)\right)^{\frac{1}{p}}\\ &\leq \frac{1}{m-c}\left[{}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathfrak{I}_w^{\mu}(\alpha\, f(x)-c\, g(x))^p\right)^{\frac{1}{p}}.\end{aligned} \tag{3.8}$$

Adding the inequalities (3.6) and (3.8), we get the required inequality (3.3). The proof of the inequality (3.4) is similar to the proof of the inequality (3.3). □

We present some results which are special cases of Minkowski's reverse type inequalities via the k-weighted fractional integral (2.1) with two-parameters in the Corollaries mentioned below. Setting $\mu(\tau)=\tau$, $w(\tau)=1$ and $\beta=k=1$, then we get ${}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x)=x-a$ and

$$\mathcal{R}_{a^+}f(x)=\int_a^x f(t)dt,\quad x>a.$$

Corollary 3.1. (Reverse Minkowski type inequality via Riemann integral operator.) Let $f,\, g>0,\ 1\leq q\leq p<+\infty,\ \alpha>0$ and

$$0<c<m\leq\frac{\alpha f(s)}{g(s)}\leq M,\quad \text{for all } s\in[a,x],$$

then

$$\begin{aligned}&\frac{M+\alpha}{\alpha(M-c)}\left(\int_a^x(\alpha f(t)-cg(t))^q\, dt\right)^{\frac{1}{q}}\\ &\leq\left(\int_a^x f^q(t)\, dt\right)^{\frac{1}{q}}+\left([x-a]^{\frac{p-q}{q}}\int_a^x g^p(t)\, dt\right)^{\frac{1}{p}}\\ &\leq\frac{m+\alpha}{\alpha(m-c)}\left([x-a]^{\frac{p-q}{q}}\int_a^x(\alpha f(t)-cg(t))^p\, dt\right)^{\frac{1}{p}}.\end{aligned} \tag{3.9}$$

The inequality (3.9) is a new result via Riemann operator on $[a,x]$ with two parameters $0<q\leq p$ and for $q=p$ we get [1, Theorem 1.2].

Setting $w(\tau) = 1$ and $\mu(\tau) = \tau$ we get ${}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x) = \frac{1}{\Gamma_k(\beta+k)}(x-a)^{\frac{\beta}{k}}$ and

$$\mathcal{RL}_{a^+}^k f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x (x-t)^{\frac{\beta}{k}-1} f(t)dt, \quad x > a.$$

Corollary 3.2. (Reverse Minkowski type inequality via k-Riemann-Liouville operator.) Under the assumptions of the Corollary 3.1 , we have

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}\left(\mathcal{RL}_{a^+}^k(\alpha\, f(x) - c\, g(x))^q\right)^{\frac{1}{q}} \\
&\leq \left(\mathcal{RL}_{a^+}^k f^q(x)\right)^{\frac{1}{q}} + \left(\left[\tfrac{1}{\Gamma_k(\beta+k)}(x-a)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}} \mathcal{RL}_{a^+}^k g^p(x)\right)^{\frac{1}{p}} \qquad (3.10)\\
&\leq \frac{m+\alpha}{\alpha(m-c)}\left(\left[\frac{1}{\Gamma_k(\beta+k)}(x-a)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}} \mathcal{RL}_{a^+}^k(\alpha f(x) - cg(x))^p\right)^{\frac{1}{p}},
\end{aligned}$$

The inequality (3.10) is a new result via the k-Riemann-Liouville operator on $[a, x]$ with two parameters $0 < p \leq q$, if we take $k = 1$ we get a new Riemann-Liouville result.

Setting $w(\tau) = \tau$ and $\mu(\tau) = \ln \tau$, we deduce ${}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x) = \frac{1}{\Gamma_k(\beta+k)}\left(\ln\frac{x}{a}\right)^{\frac{\beta}{k}}$ and

$$\mathcal{H}_{a^+}^k f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\beta}{k}-1} \frac{f(t)}{t}dt, \quad x > a > 1.$$

Corollary 3.3. (Reverse Minkowski type inequality via k-Hadamard operator.) Under the assumptions of the Corollary 3.1 , we have

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}\left(\mathcal{H}_{a^+}^k(\alpha\, f(x) - c\, g(x))^q\right)^{\frac{1}{q}} \\
&\leq \left(\mathcal{H}_{a^+}^k f^q(x)\right)^{\frac{1}{q}} + \left(\left[\tfrac{1}{\Gamma_k(\beta+k)}\left(\ln\tfrac{x}{a}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}} \mathcal{H}_{a^+}^k g^p(x)\right)^{\frac{1}{p}} \qquad (3.11)\\
&\leq \frac{m+\alpha}{\alpha(m-c)}\left(\left[\frac{1}{\Gamma_k(\beta+k)}\left(\ln\frac{x}{a}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}} \mathcal{H}_{a^+}^k(\alpha\, f(x) - c\, g(x))^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Inequality (3.11) is a new result via the k-Hadamard operator on $[a, x]$ with two parameters $0 < p \leq q$. If we put $k = 1$ we get a new result with the Hadamard operator.

Setting $w(\tau) = \tau$ and $\mu(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$, we get ${}_{a^+}\mathfrak{I}_w^{\mu}\mathbf{1}(x) = \frac{1}{\Gamma_k(\beta+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\beta}{k}}$ and

$$\mathcal{K}_{a^+}^k f(x) = \frac{(\rho+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\beta}{k}-1} t^{\rho} f(t)dt, \quad x > a.$$

Corollary 3.4. (Reverse Minkowski type inequality via k-Katugompola operator.) Under the assumptions of the Corollary 3.1 we have for all $\rho > -1$

(3.12)
$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}\left(\mathcal{K}^k_{a^+}(\alpha\, f(x)-c\, g(x))^q\right)^{\frac{1}{q}}\\
&\le \left(\mathcal{K}^k_{a^+}f^q(x)\right)^{\frac{1}{q}}+\left(\left[\frac{1}{\Gamma_k(\beta+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}}\mathcal{K}^k_{a^+}g^p(x)\right)^{\frac{1}{p}}\\
&\le \frac{m+\alpha}{\alpha(m-c)}\left(\left[\frac{1}{\Gamma_k(\beta+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}}\mathcal{K}^k_{a^+}(\alpha\, f(x)-c\, g(x))^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Inequality (3.12) is a new result via the k-Katugompola operator on $[a,x]$ with two parameters $0<p\le q$. If we put $k=1$ we get a new result with the Katugompola operator. Setting $w(\tau)=\tau$ and $\mu(\tau)=\frac{(\tau-a)^\theta}{\theta}$, we have

$$\begin{aligned}
&{}_{a^+}\mathfrak{I}^\mu_w\mathbf{1}(x)=\frac{1}{\Gamma_k(\beta+k)}\left(\frac{(x-a)^\theta}{\theta}\right)^{\frac{\beta}{k}}\ \text{and}\\
&\mathcal{C}^k_1 f(x)=\frac{\theta^{1-\frac{\beta}{k}}}{k\,\Gamma_k(\beta)}\int_a^x\left((x-a)^\theta-(t-a)^\theta\right)^{\frac{\beta}{k}-1}\frac{f(t)}{(t-a)^{1-\theta}}\,dt,\quad x>a.
\end{aligned}$$

Corollary 3.5. (Reverse Minkowski type inequality via fractional k-conformal integral operator.) Under the assumptions of the Corollary 3.1 , we get

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}\left(\mathcal{C}^k_1(\alpha\, f(x)-c\, g(x))^q\right)^{\frac{1}{q}}\\
&\le \left(\mathcal{C}^k_1 f^q(x)\right)^{\frac{1}{q}}+\left(\left[\frac{1}{\Gamma_k(\beta+k)}\left(\frac{(x-a)^\theta}{\theta}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}}\mathcal{C}^k_1 g^p(x)\right)^{\frac{1}{p}}\\
&\le \frac{m+\alpha}{\alpha(m-c)}\left(\left[\frac{1}{\Gamma_k(\beta+k)}\left(\frac{(x-a)^\theta}{\theta}\right)^{\frac{\beta}{k}}\right]^{\frac{p-q}{q}}\mathcal{C}^k_1(\alpha\, f(x)-c\, g(x))^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.13}$$

Inequality (3.13) is a new result via the k-conformal operator on $[a,x]$ with two parameters $0<p\le q$. If we put $k=1$ we get a result with the conformal operator.

4. Reverse Hölder inequalities via k-weighted fractional integral

In the following theorem, we present and prove the reverse Hölder type inequality according to the k-weighted fractional integral with two parameters. Recall the k-weighted fractional integral defined as

$$ {}_{a^+}\mathfrak{I}_w^{\mu} f(x) = \frac{1}{w(x) k\,\Gamma_k(\beta)} \int_a^x \mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1} w(s) f(s) ds, \quad x > a. $$

Theorem 4.1. Let $0 \le a < b < +\infty$, $\lambda, \gamma > 0$, $1 < q \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Let w be a weight function and f, g be a positive measurable functions on $[a, b]$, suppose that $f^{\eta},\ g^{\eta} \in L_{X_w^1}[a, b]$ where $\eta > 0$. If

$$ 0 < m \le \frac{f^{\lambda}(s)}{g^{\gamma}(s)} \le M \quad \text{for} \quad \text{all } s \in [a, x], \tag{4.1} $$

then

$$ \begin{aligned} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{q\lambda}{p}}(x) \right)^{\frac{1}{q}} \left({}_{a^+}\mathfrak{I}_w^{\mu} g^{\frac{p'\gamma}{q'}}(x) \right)^{\frac{1}{p'}} &\le M^{\frac{1}{p\,p'}} \left(\tfrac{1}{m}\right)^{\frac{1}{q\,q'}} \left[{}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) \right]^{\frac{2}{q} - \frac{2}{p}} \\ &\left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x) \right)^{\frac{1}{p}} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x) \right)^{\frac{1}{q'}}. \end{aligned} \tag{4.2} $$

Proof. From the hypothesis (4.1) we have

$$ f^{\frac{\lambda}{p'}}(s) \le M^{\frac{1}{p'}} g^{\frac{\gamma}{p'}}(s), $$

thus

$$ f^{\lambda}(s) \le M^{\frac{1}{p'}} f^{\frac{\lambda}{p}} g^{\frac{\gamma}{p'}}(s). $$

Multiplying the above inequality by the positive quotient $\mu'(s)(\mu(x)-\mu(s))^{\frac{\beta}{k}-1} w(s)$ and integrating with respect to s over $[a, x]$, we deduce that

$$ \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\lambda}(x) \right)^{\frac{1}{p}} \le M^{\frac{1}{pp'}} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x) \right)^{\frac{1}{p}}, $$

therefore

$$ \begin{aligned} &\left[{}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) \right]^{\frac{p-q}{pq}} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\lambda}(x) \right)^{\frac{1}{p}} \\ &\qquad \le \left[{}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) \right]^{\frac{p-q}{pq}} M^{\frac{1}{pp'}} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x) \right)^{\frac{1}{p}}. \end{aligned} \tag{4.3} $$

Now replacing f by $f^{\frac{\lambda}{p}}$ in the inequality (2.7), we get

$$ \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\frac{q\lambda}{p}}(x) \right)^{\frac{1}{q}} \le \left[{}_{a^+}\mathfrak{I}_w^{\mu} \mathbf{1}(x) \right]^{\frac{p-q}{pq}} \left({}_{a^+}\mathfrak{I}_w^{\mu} f^{\lambda}(x) \right)^{\frac{1}{p}}, \tag{4.4} $$

according to inequalities (4.3) and (4.4), we deduce

$$\left({}_{a^+}\mathfrak{I}_w^\mu f^{\frac{q\lambda}{p}}(x)\right)^{\frac{1}{q}} \leq \left[{}_{a^+}\mathfrak{I}_w^\mu \mathbf{1}(x)\right]^{\frac{1}{q}-\frac{1}{p}} M^{\frac{1}{pp'}} \left({}_{a^+}\mathfrak{I}_w^\mu f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x)\right)^{\frac{1}{p}}. \tag{4.5}$$

Similarly, from the assumption (4.1) we have

$$g^{\frac{\gamma}{q}}(s) \leq m^{-\frac{1}{q}} f^{\frac{\lambda}{q}}(s),$$

multiplying by $g^{\frac{\gamma}{q'}}$

$$g^{\gamma}(s) \leq m^{-\frac{1}{q}} f^{\frac{\lambda}{q}}(s) g^{\frac{\gamma}{q'}}(s),$$

this gives us

$$\left({}_{a^+}\mathfrak{I}_w^\mu g^{\gamma}(x)\right)^{\frac{1}{q'}} \leq \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left({}_{a^+}\mathfrak{I}_w^\mu f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}},$$

yielding

$$\begin{aligned}\left[{}_{a^+}\mathfrak{I}_w^\mu \mathbf{1}(x)\right]^{\frac{q'-p'}{q'p'}} \left({}_{a^+}\mathfrak{I}_w^\mu g^{\gamma}(x)\right)^{\frac{1}{q'}} \\ \leq \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left[{}_{a^+}\mathfrak{I}_w^\mu \mathbf{1}(x)\right]^{\frac{q'-p'}{q'p'}} \left({}_{a^+}\mathfrak{I}_w^\mu f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}}.\end{aligned} \tag{4.6}$$

Again we replace f by $g^{\frac{\gamma}{q'}}$ in the inequality (2.7), we result

$$\left({}_{a^+}\mathfrak{I}_w^\mu g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \leq \left[{}_{a^+}\mathfrak{I}_w^\mu \mathbf{1}(x)\right]^{\frac{q'-p'}{q'p'}} \left({}_{a^+}\mathfrak{I}_w^\mu g^{\gamma}(x)\right)^{\frac{1}{q'}}, \tag{4.7}$$

according inequalities (4.6) and (4.7), we obtain

$$\left({}_{a^+}\mathfrak{I}_w^\mu g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \leq \left(\tfrac{1}{m}\right)^{\frac{1}{qq'}} \left[{}_{a^+}\mathfrak{I}_w^\mu \mathbf{1}(x)\right]^{\frac{1}{p'}-\frac{1}{q'}} \left({}_{a^+}\mathfrak{I}_w^\mu f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}}. \tag{4.8}$$

Finally, by multiplying the inequalities (4.5) and (4.8) we get the desired inequality (4.2). □

We now present some new inequalities with two parameters which are special cases of the inverse Hölder inequalities via the k-weighted fractional integral (2.1).

Corollary 4.1. Taking $w \equiv 1$. Under the assumptions of Theorem 4.1, we result the following cases.

The reverse Hölder's inequality related to the Riemann integral:

$$\begin{aligned}\left(\int_a^x f^{\frac{q\lambda}{p}}(t)\, dt\right)^{\frac{1}{q}} \left(\int_a^x g^{\frac{p'\gamma}{q'}}(t)\, dt\right)^{\frac{1}{p'}} \\ \leq B_1 \left(\int_a^x f^{\frac{\lambda}{p}}(t)\, g^{\frac{\gamma}{p'}}(t)\, dt\right)^{\frac{1}{p}} \left(\int_a^x f^{\frac{\lambda}{q}}(t)\, g^{\frac{\gamma}{q'}}(t)\, dt\right)^{\frac{1}{q'}},\end{aligned} \tag{4.9}$$

where

$$B_1 =: B^{p,q}_{m,M}(x) = M^{\frac{1}{pp'}} \left(\frac{1}{m}\right)^{\frac{1}{qq'}} (x-a)^{\frac{2}{q}-\frac{2}{p}}.$$

The reverse Hölder's inequality related to the k-Riemann-Liouville integral:

$$\begin{aligned} &\left(\mathcal{RL}^k_{a^+} f^{\frac{q\lambda}{p}}(x)\right)^{\frac{1}{q}} \left(\mathcal{RL}^k_{a^+} g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \\ &\leq B_2 \left(\mathcal{RL}^k_{a^+} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x)\right)^{\frac{1}{p}} \left(\mathcal{RL}^k_{a^+} f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}} \end{aligned}, \tag{4.10}$$

where

$$B_2 =: B^{p,q}_{m,M}(x) = M^{\frac{1}{pp'}} \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left[\frac{1}{\Gamma_k(\beta+k)}(x-a)^{\frac{\beta}{k}}\right]^{\frac{2}{q}-\frac{2}{p}}.$$

The reverse Hölder's inequality related to the k-Hadamard integral:

$$\begin{aligned} &\left(\mathcal{H}^k_{a^+} f^{\frac{q\lambda}{p}}(x)\right)^{\frac{1}{q}} \left(\mathcal{H}^k_{a^+} g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \\ &\qquad \leq B_3 \left(\mathcal{H}^k_{a^+} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x)\right)^{\frac{1}{p}} \left(\mathcal{H}^k_{a^+} f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}} \end{aligned} \tag{4.11}$$

where

$$B_3 =: B^{p,q}_{m,M}(x) = M^{\frac{1}{pp'}} \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left[\frac{1}{\Gamma_k(\beta+k)}\left(\ln\frac{x}{a}\right)^{\frac{\beta}{k}}\right]^{\frac{2}{q}-\frac{2}{p}}.$$

The reverse Hölder's inequality associated with k-Katugompola integral:

$$\begin{aligned} &\left(\mathcal{K}^k_{a^+} f^{\frac{q\lambda}{p}}(x)\right)^{\frac{1}{q}} \left(\mathcal{K}^k_{a^+} g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \\ &\qquad \leq B_4 \left(\mathcal{K}^k_{a^+} f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x)\right)^{\frac{1}{p}} \left(\mathcal{K}^k_{a^+} f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}} \end{aligned} \tag{4.12}$$

where

$$B_4 =: B^{p,q}_{m,M}(x) = M^{\frac{1}{pp'}} \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left[\frac{1}{\Gamma_k(\beta+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\beta}{k}}\right]^{\frac{2}{q}-\frac{2}{p}}.$$

The reverse Hölder's inequality associated with k-conformal integral:

$$\begin{aligned} &\left(\mathcal{C}^k_1 f^{\frac{q\lambda}{p}}(x)\right)^{\frac{1}{q}} \left(\mathcal{C}^k_1 g^{\frac{p'\gamma}{q'}}(x)\right)^{\frac{1}{p'}} \\ &\qquad \leq B_5, \left(\mathcal{C}^k_1 f^{\frac{\lambda}{p}}(x) g^{\frac{\gamma}{p'}}(x)\right)^{\frac{1}{p}} \left(\mathcal{C}^k_1 f^{\frac{\lambda}{q}}(x) g^{\frac{\gamma}{q'}}(x)\right)^{\frac{1}{q'}}, \end{aligned} \tag{4.13}$$

where

$$B_5 =: B^{p,q}_{m,M}(x) = M^{\frac{1}{pp'}} \left(\frac{1}{m}\right)^{\frac{1}{qq'}} \left[\frac{1}{\Gamma_k(\beta+k)} \left(\frac{(x-a)^\theta}{\theta}\right)^{\frac{\beta}{k}}\right]^{\frac{2}{q}-\frac{2}{p}}.$$

Remark 4.1. Setting $q = p$ in the above inequalities (4.9), (4.10), (4.11), (4.12) and (4.13), we obtain new formulas to the reverse Hölder's inequalities in fractional calculus with one parameter $p > 1$.

5. Conclusion

We introduced new inequalities using k-weighted fractional integral operators with two parameters p and q. These inequalities are a novel extension of the reverse Minkowski and Hölder-type inequalities and include specific cases such as k-Riemann-Liouville, k-Hadamard, k-Katugompola and k-conformal fractional.

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